Completion of metric reconstruction for a particle orbiting a Kerr black hole

Cesar Merlin

Mathematical Sciences/University of Southampton

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Outline

- Strategy for computing the gravitational SF from reconstructed MP. Adam Pound, Cesar Merlin and Leor Barack. [Phys. Rev. D 89, 024009]

- Numerical Implementation in Schwarzschild. Cesar Merlin and Abhay G. Shah [Phys. Rev. D 91, 024005]

- The completion of the solution. Cesar Merlin, Leor Barack, Amos Ori, Adam Pound and Maarten van de Meent.

Review

- The original formulation of the SF was given in the Lorenz-gauge [Poisson *et al.* (2011)]. For Kerr the field equations in the LG are not separable.
- The treatment of black-hole perturbations for Kerr is much simpler in a radiation gauge, where it is possible to reconstruct the perturbations from the perturbed Weyl scalars. [Chrzanowski-Cohen-Kegeles (1975), Wald (1978), Ori (2008), Keidl *et al.* (2010)]
- Gauge invariant quantities have been successfully computed using this idea. [Shah *et al.* (2011-12)]

Completion problem

- The solution in the RG needs to be completed by including certain completion piece. [Wald (1973)]

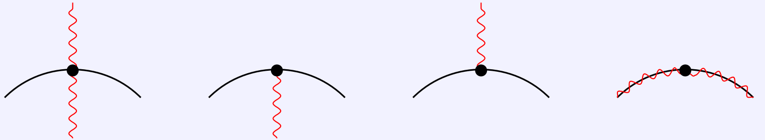
$$h_{\alpha\beta} = h_{\alpha\beta}^{(\mathrm{rec})} + h_{\alpha\beta}^{(\mathrm{comp})}.$$

- In Schwarzschild the completion is the $\ell = 0, 1$ modes of the Einstein Field Equations. [Detweiler-Poisson (2004), Barack-Lousto (2005), Price (2007)]
- To measure the mass and AM for a reconstructed pertubation one can evaluate some complicated surface-integrals. [Abbot-Desner (1982), Dolan-Barack (2011)]

- Shah *et al.* obtained the completion imposing boundary-conditions at the event horizon and at infinity, by assuming that $h^{(\mathrm{rec})}_{\alpha\beta}$ does not contain any mass and AM. [Shah *et al.* (2011-12)]
- A numerical method to determine an homogeneous piece of the 'Hertz potential' for a ring configuration was recently proposed. [Sano-Tagoshi (2014-15)]
- We want to obtain $h^{(\mathrm{comp})}_{\alpha\beta}$ rigorously.

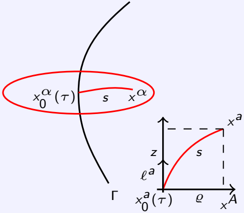
Singular structure of the radiation gauges

Pathological singularity in the radiation-gauge



Singular along ℓ^{α} — Regular at $\ell^{\alpha+}$ — Regular at $\ell^{\alpha-}$ — Discontinuous across a surface through the particle

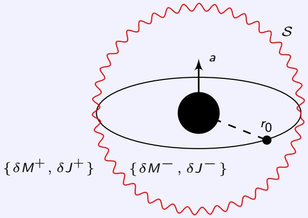
Half-string solutions	Full-string solution	No-string solution
$h^{\pm}_{\tau A} = \mp \dfrac{2m x_A}{s(s \pm z)}$	$h_{\tau A} = \dfrac{2mz x_A}{s \varrho^2}$	$h_{\tau A} = h^{+}_{\tau A}\theta(z) + h^{-}_{\tau A}\theta(-z)$
$h^{\pm}_{zA} = \pm \dfrac{2m x_A}{s(s \pm z)}$	$h_{zA} = -\dfrac{2mz x_A}{s \varrho^2}$	$h_{zA} = h^{+}_{zA}\theta(z) + h^{-}_{zA}\theta(-z)$



Completion of the solution

Wald showed (1973) that the only things we can add to the metric reconstruction are:

- Mass and angular momentum perturbations (δM and δJ).
- Perturbations to other algebraically special solutions (C-metric and Kerr-NUT).
- Gauge perturbations.



Implementation

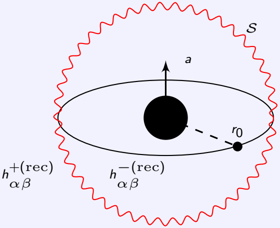
Weyl Scalar
ψ_0, ψ_4

↓

Hertz potential
Ψ

↓

Metric Perturbations
$h_{\alpha\beta}^{(\mathrm{rec})}$



Implementation

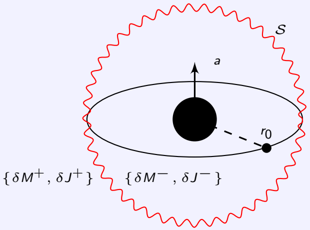
Weyl Scalar $\psi_0,\ \psi_4$

↓

Hertz potential Ψ

↓

Metric Perturbations $h^{(\mathrm{rec})}_{\alpha\beta}$



Implementation

Weyl Scalar
ψ_0, ψ_4

$\downarrow$

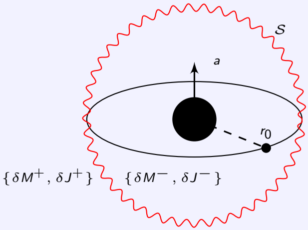
Hertz potential
Ψ

$\downarrow$

Metric Perturbations
$h^{(\mathrm{rec})}_{\alpha\beta}$

$\xrightarrow{m=0}$

Invariants
$\mathcal{I}^{(\mathrm{rec})}_{1,2}$



Completion: arbitrary δM, δJ

$\downarrow$

Invariants
$\mathcal{I}^{(\mathrm{comp})}_{1,2}$

Implementation

Weyl Scalar
ψ_0, ψ_4

↓

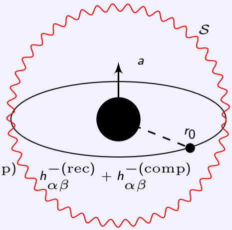
Hertz potential
Ψ

↓

Metric Perturbations
$h^{(\text{rec})}_{\alpha\beta}$

$\xrightarrow{m=0}$

Invariants
$\mathcal{I}^{(\text{rec})}_{1,2}$



$h^{+(\text{rec})}_{\alpha\beta} + h^{+(\text{comp})}_{\alpha\beta}$ $\quad$ $h^{-(\text{rec})}_{\alpha\beta} + h^{-(\text{comp})}_{\alpha\beta}$

Completion:
arbitrary δM, δJ

↓

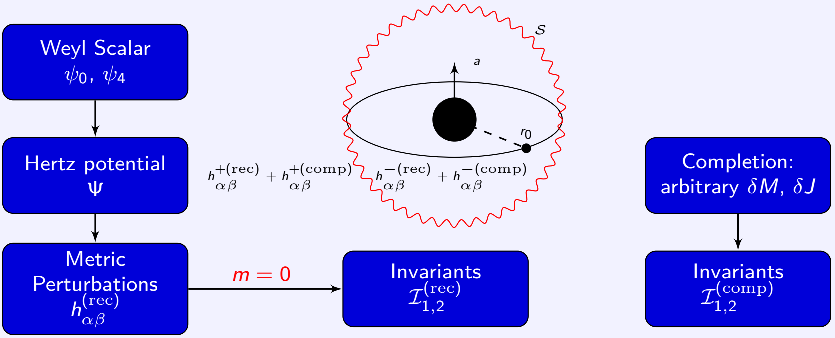
Invariants
$\mathcal{I}^{(\text{comp})}_{1,2}$

In Schwarzschild we move to a gauge in which $\mathrm{Re}(\delta\tilde{\psi}_2) = 0$,

in which $\mathcal{I}_1 \equiv \tilde{h}_{rr} = h_{rr} - 2\tilde{\xi}_{r,r} + 2\Gamma^r_{rr}\tilde{\xi}_r$, and $\mathcal{I}_2 \equiv \mathrm{Im}(\delta\psi_2)$,

with $\tilde{\xi}^r = \frac{r^4}{3M}\mathrm{Re}(\delta\psi_2)$.

Implementation



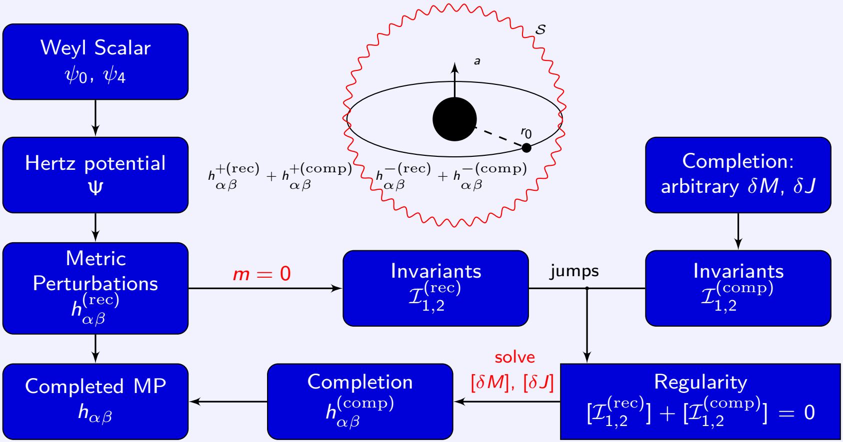
In Kerr we move to a gauge in which $\delta\tilde{\psi}_2 = 0$,

to obtain $\mathcal{I}_1 \equiv \tilde{h}_{rr}$ and $\mathcal{I}_2 \equiv \tilde{h}_{\theta\theta}$,

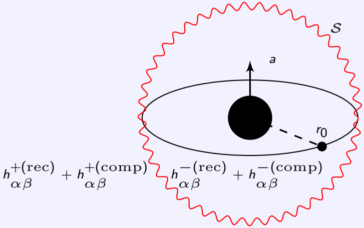
where $\tilde{h}_{ab} = h_{ab} - \tilde{\xi}_{a,b} - \tilde{\xi}_{b,a} + 2\Gamma^c_{ab}\tilde{\xi}_c$,

with $a, b \in \{r, \theta\}$, $\tilde{\xi}^r = \mathrm{Re}\left(\frac{\varrho^{-4}}{3M}\delta\psi_2\right)$ and $\tilde{\xi}^\theta = \mathrm{Im}\left(\frac{\varrho^{-4}}{3aM\sin\theta}\delta\psi_2\right)$.

Implementation



Implementation



Weyl Scalar ψ_0, ψ_4

↓

Hertz potential Ψ

↓

Metric Perturbations $h^{(\mathrm{rec})}_{\alpha\beta}$ $\xrightarrow{m=0}$ Invariants $\mathcal{I}^{(\mathrm{rec})}_{1,2}$

Completion: arbitrary δM, δJ → Invariants $\mathcal{I}^{(\mathrm{comp})}_{1,2}$

jumps → Regularity $[\mathcal{I}^{(\mathrm{rec})}_{1,2}] + [\mathcal{I}^{(\mathrm{comp})}_{1,2}] = 0$

Regularity $\xrightarrow{\text{solve } [\delta M], [\delta J]}$ Completion $h^{(\mathrm{comp})}_{\alpha\beta}$ → Completed MP $h_{\alpha\beta}$

Metric Perturbations → Completed MP $h_{\alpha\beta}$

Completed MP $h_{\alpha\beta}$ → (LL mode-sum; spatial averaging) → Self-force $F^{\alpha\ell}_{self}$

Completion for circular orbits

Our method for circular orbits:

- Obtain the $m = 0$ sector of the reconstructed part of the MP, using the CCK procedure mode by mode. This is done on each of the two sides of the sphere $\mathcal{S}$ with $r = r_0$.
- Construct two auxiliary invariants $\mathcal{I}_{1,2}^{(\mathrm{rec})}(r, \theta)$ on each side of $\mathcal{S}$, and obtain the jump across $r = r_0$ on each mode of the invariants.
- Sum all the ℓ modes of the jumps (analytically).
- Construct the completion pieces in Boyer-Lindquist coordinates:

$$\boxed{h_{\alpha\beta}^{(\mathrm{comp})\pm} = \left(\left. \frac{\partial g_{\alpha\beta}^{\mathrm{Kerr}}(M,J)}{\partial M} \right|_{J=const.} \right) \delta M^{\pm} + \left(\left. \frac{\partial g_{\alpha\beta}^{\mathrm{Kerr}}(M,J)}{\partial J} \right|_{M=const.} \right) \delta J^{\pm}.}$$

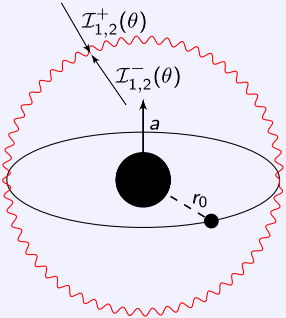
- Impose continuity for $[\mathcal{I}_{1,2}^{(\mathrm{rec})}](\theta) + [\mathcal{I}_{1,2}^{(\mathrm{comp})}](\theta)$ across $\mathcal{S}$ off-the-particle.
- Solve for the missing amplitudes $[\delta M]$ and $[\delta J]$.

Example: Circular equatorial orbits in Schwarzschild

The jump of the $m = 0$ part of the reconstructed solution for the two invariants in Schwarzschild is

$$[\mathcal{I}_1^{(\mathrm{rec})}](\theta) = \sum_{\ell=2}^{\infty} \left[\frac{8\mathrm{m}\mathcal{E}\pi}{3Mf_0^2} Y_\ell(\theta)\bar{Y}_\ell(\theta_0) + \frac{4\mathrm{m}\mathcal{E}\pi(r_0 - M)}{3Mr_0 f_0^3} Y_\ell(\theta)\bar{Y}_\ell''(\theta_0) \right],$$

$$\left[\mathcal{I}_2^{(\mathrm{rec})}\right](\theta) = \sum_{\ell=2}^{\infty} \frac{4\pi\mathrm{m}\mathcal{L}}{r_0^4} Y_\ell(\theta)\bar{Y}_\ell'(\theta_0),$$



which we sum analytically as distributions by adding and subtracting the missing $\ell = 0, 1$ pieces:

$$[\mathcal{I}_1^{(\mathrm{rec})}](\theta) = \frac{8\mathrm{m}\mathcal{E}\pi}{3Mf_0^2}\delta(\theta-\theta_0) - \frac{4\mathrm{m}\mathcal{E}\pi(r_0-M)}{3Mr_0 f_0^3}\delta'(\theta-\theta_0) - \frac{2\mathrm{m}\mathcal{E}}{3Mf_0^2},$$
$$\left[\mathcal{I}_2^{(\mathrm{rec})}\right](\theta) = -\frac{4\pi\mathrm{m}\mathcal{L}}{r_0^4}\delta'(\theta-\theta_0) + \frac{3\mathrm{m}\mathcal{L}}{r_0^4}\cos\theta.$$

From the completion pieces we obtain a piece of the invariants:

$$\left[\mathcal{I}_1^{(\mathrm{comp})}\right] = \frac{2[\delta M]}{3Mf_0^2}, \qquad \text{and} \qquad \left[\mathcal{I}_2^{(\mathrm{comp})}\right](\theta) = -\frac{3[\delta J]\cos\theta}{r_0^4}.$$

We now impose the regularity at $\theta \neq \pi/2$ to obtain

$$[\mathcal{I}_1](\theta \neq \pi/2) = -\frac{2\mathrm{m}\mathcal{E}}{3Mf_0^2} + \frac{2[\delta M]}{3Mf_0^2} = 0,$$
$$[\mathcal{I}_2](\theta \neq \pi/2) = \frac{3\mathrm{m}\mathcal{L}}{r_0^4}\cos\theta - \frac{3[\delta J]}{r_0^4}\cos\theta = 0.$$

We obtain $[\delta M] \equiv \mathrm{m}\mathcal{E}$, and $[\delta J] \equiv \mathrm{m}\mathcal{L}$. The jump of the mass and angular momentum perturbations in the invariants agrees with the specific energy and angular momentum of the particle.

Second example: Circular equatorial orbits in Kerr

The reconstructed piece (with $m = 0$ at $\theta \neq \pi/2$) gives

$$\begin{aligned}
\left[\mathcal{I}_1^{(\mathrm{rec})}\right](\theta) = &-\frac{2\dot{t}\mu\Sigma_0}{3Mr_0^3\Delta_0^2}\Big\{2a^6M\Omega^2 + ar_0^3(4Mr_0 - 6M^2 - 3r_0^2)\Omega - 3a^3r_0(r_0^2f_0 - 2M^2)\Omega - 4a^5M\Omega \\
&-a^4\left[3M^2r_0\Omega^2 + 2r_0^3\Omega^2 + M(r_0^2\Omega^2 - 2)\right] + a^2r_0\Big[3M^2(r_0^2\Omega^2 - 1) \\
&+r_0^2(5 + r_0^2\Omega^2) + Mr_0(2r_0^2\Omega^2 - 5)\Big] + r_0^3\left[3M^2 + r_0^2 + r_0^4\Omega^2 - 3Mr_0(1 + r_0^2\Omega^2)\right]\Big\}\,, \\
\left[\mathcal{I}_2^{(\mathrm{rec})}\right](\theta) = &\frac{\dot{t}\mu\Sigma_0}{3aMr_0^3}\Big\{3\Big[2a^4M\Omega + 2r_0^5\Omega + 2a^2r_0(r_0^2f_0 - 3M^2)\Omega - a^5M\Omega^2 + a^3(r_0^3\Omega^2 - M + 3M^2r_0\Omega^2 + Mr_0^2\Omega^2) \\
&-ar_0(r_0^4\Omega^2 - 3M^2 - 3Mr_0 + 3r_0^2 + 3Mr_0^3\Omega^2)\Big] + a\Big[a^4M\Omega^2 - 2a^3M\Omega + 2aMr_0(2r_0 - 3M)\Omega \\
&+a^2(M + 3M^2r_0\Omega^2 - Mr_0^2\Omega^2 - r_0^3\Omega^2) + r_0\left(3M^2 + r_0^2 + r_0^4\Omega^2 - 3Mr_0(1 + r_0^2\Omega^2)\right)\Big]\cos(2\theta)\Big\}\csc^2\theta\,.
\end{aligned}$$

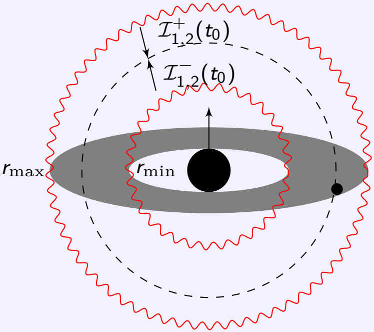
The completion part of the invariants in the Kerr case is just

$$\begin{aligned}
\left[\mathcal{I}_1^{(\mathrm{comp})}\right](\theta) &= \frac{\Sigma}{3M\Delta^2}\left\{(5a^2 + r^2)[\delta M] - 3a[\delta J]\right\}\,, \\
\left[\mathcal{I}_2^{(\mathrm{comp})}\right](\theta) &= -\frac{\Sigma}{6aM}\left\{\left(a\cos(2\theta) - 9a\right)[\delta M] + 6[\delta J]\right\}\csc^2\theta.
\end{aligned}$$

With the regularity conditions we determine that the amplitudes are $[\delta M] = \mathrm{m}\mathcal{E}$, and $[\delta J] = \mathrm{m}\mathcal{L}$.

Completion for eccentric orbits

- Obtain the jump on the reconstructed part of the two auxiliary invariants $[\mathcal{I}_{1,2}]\,(r_0(t_0),\theta)$ for a momentary ring in the libration region $r_{\rm min} < r_0(t_0) < r_{\rm max}$.
- Sum all the ℓ modes for the partial ring.
- The completion pieces are constructed in the same way as in the circular case.
- Impose continuity for $[\mathcal{I}^{\rm (rec)}_{1,2}](t_0,\theta) + [\mathcal{I}^{\rm (comp)}_{1,2}](t_0,\theta)$, at each t_0.
- Solve for the jump of missing amplitudes $[\delta M](t_0)$ and $[\delta J](t_0)$ of the completion pieces.
- Integrate over t_0 covering an orbital period.



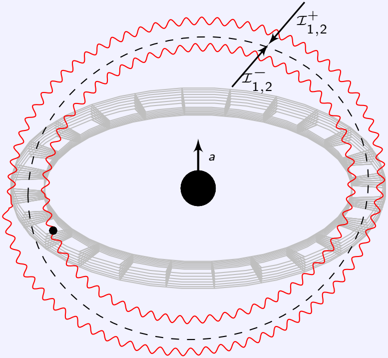
Results

After analytically performing the integrals we find that the missing amplitudes $[\delta\tilde{M}]$ and $[\delta\tilde{J}]$ correspond to the specific energy $\mathcal{E}$ and angular momentum $\mathcal{L}$ of the particle, just like in the circular-orbit case:

$$\boxed{[\delta\tilde{M}] = \mathrm{m}\mathcal{E}, \quad \text{and} \quad [\delta\tilde{J}] = \mathrm{m}\mathcal{L}.}$$

More general orbits in Kerr

- At each point in the region $r_{\min} \leq r_0 \leq r_{\max}$, $\theta_{\min} \leq \theta_0 \leq \theta_{\max}$ we find the jump on the gauge invariants corresponding to a infinitesimal ring.
- We impose continuity (at the level of the Green's function) for each ring and integrate throughout the source region.



Summary and future work

- We have identified three types of RG gauges. We have two methods to calculate the SF starting from RG MP, using new versions of the mode-sum method.
- To include the completion piece, we derived new gauge invariant quantities. By imposing continuity off the particle we extract the amplitude of the jump in the mass and angular momentum perturbations.
- We have extended our analysis to obtain the completion for equatorial-eccentric orbits. This requires to evaluate a complicated integral in the libration region. Inclined orbits will require extra considerations.
- Our completion piece is required to calculate the GSF for eccentric orbits in Kerr [van de Meent and Shah, (2015)].