Radiation-Reaction Force on a Small Charged Body to Second Order

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Outline

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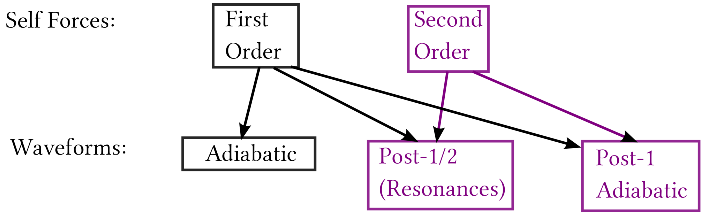
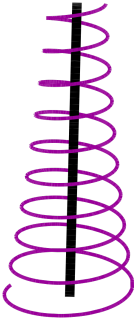
Relevance of Second Order Self Force

- Extreme mass ratio inspirals give a prominent and long-lived GW signal ((e)LISA band)
- Small object probes the field of its companion





Current State of Formal Derivations - Linear fields

		Scalar/E&M	
		1st Order	2nd Order
Flat, Forced	Limiting Process	Gralla, Harte, Wald - 2009	* Our computation *
	Regularized point force	Abraham, Lorentz, Dirac - 1930s Quin 2000; etc..	X
Curved, vacuum	Effective Field Thy	←Galley[1] 2010→	
	Regularized point force	Several, eg. Barack, Ori 2000	Rosenthal[1] 2005 Burko 2002
Fully general	Limiting Process	←Harte[2] 2008→	

[1] Nonlinear scalar

[2] Not a perturbative calculation

Current State of Formal Derivations - Gravity

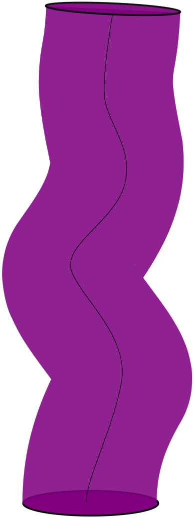
	Gravity	
	1st Order	2nd Order
Nonvacuum	Zimmerman, Poisson 2014 Linz, Freedman, Wiseman 2014	X
Limiting Process	Gralla, Wald 2008	Gralla 2012
Matched Asymptotics	Mino , Sasaki,Tanaka 1997 Quinn,Wald 1997	Adam Pound 2012
Gauge specific techniques		Rosenthal 2006

How Our Computation Fits In

- New Result : second-order electromagnetic self-force for general body
- Extension of Gralla, Harte, and Wald's first order E&M computation
- Rigorous derivation from limiting process (no regularization of singularities)
- Useful for physical understanding
- Method similar to matched asymptotic expansions (D'Eath 1996, Detweiler 2001, Poisson 2004)

Outline

Context



- Flat spacetime
- Compact body with:

$$F^{\alpha\beta} = F^{\alpha\beta}_{\text{self}} + F^{\alpha\beta}_{\text{ext}}$$

$$\nabla_\mu F^{\mu\nu}_{\text{self}} = \underbrace{j^\nu}_{\substack{\text{compact}\\ \text{support}}} \qquad \Bigg| \qquad \nabla_\mu F^{\mu\nu}_{\text{ext}} = 0$$

$$T^{\alpha\beta} = \underbrace{\boxed{\overbrace{T^{\alpha\beta}_{\text{matter}}}^{\text{compact support}} + T^{\alpha\beta}_{\text{self-self}}}}_{\text{used for body params}} + T^{\alpha\beta}_{\text{self-ext}} + T^{\alpha\beta}_{\text{ext-ext}}$$

Precise Axiom Statement [Gralla, Harte, Wald]

Axiom

There exists a one-parameter family of fields consisting of the Maxwell tensor $F_{\mu\nu}(\lambda, x^\mu)$, the charge current density $j^\mu(\lambda, x^\mu)$, and the stress energy tensor $T^M_{\mu\nu}(\lambda, x^\mu)$, which satisfy the Maxwell equations, charge current conservation and stress-energy conservation equations.

$$\begin{aligned}\nabla^\nu F_{\mu\nu}(\lambda, x^\mu) =& 4\pi j_\nu(\lambda, x^\mu)\\ \nabla_\mu j^\mu(\lambda, x^\mu) =& 0\\ \nabla_\mu T^{\mu\nu}(\lambda, x^\mu) =& 0\end{aligned}$$

These fields have support on the open interval $0 < \lambda < \lambda_0$, for some λ_0. In particular, the fields need not have a solution at $\lambda = 0$.

Precise Axiom Statement [Gralla, Harte, Wald]

Axiom

All of the fields $F_{\mu\nu}$, j^μ, and $T^M_{\mu\nu}$ are smooth in λ away from $\lambda = 0$.

Axiom

There exists functions $z^i(\lambda, t)$, $\tilde{j}^\mu(\lambda, t, X^i)$, and $\tilde{T}^{\mu\nu}_M(\lambda, t, X^i)$ such that for some global Lorentz frame coordinates (t, x^i):

$$j^\mu(\lambda, t, x^i) = \lambda^{-2}\tilde{j}^\mu\left(\lambda, t, \frac{x^i - z^i(\lambda, t)}{\lambda}\right) \tag{1}$$

$$T^{\mu\nu}_M(\lambda, t, x^i) = \lambda^{-2}\tilde{T}^{\mu\nu}_M\left(\lambda, t, \frac{x^i - z^i(\lambda, t)}{\lambda}\right) \tag{2}$$

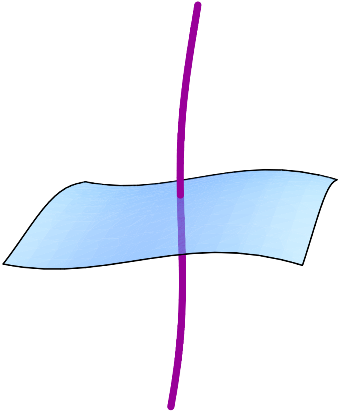
and $\tilde{j}^\mu$ and $\tilde{T}^{\mu\nu}_M$ are jointly smooth in their arguments and have compact spatial support.

Outline

Hypersurface Choice

- Definitions of extended body params (mass, spin, etc) usually use integrals over spatial hypersurfaces
- Self-energy becomes problematic at higher orders
  - Spacelike integral has dependence on past worldline
  - Radiation must be handled carefully
- We use integrals over null cones [Harte]
  - Depends exclusively on a small region of the worldline
  - Dependence on choice of hypersurface due to non-conserved stress energy

$$P^\alpha = \int_\Sigma (T^{\alpha\beta}_{\text{matter}} + T^{\alpha\beta}_{\text{self-self}})\xi_\beta d^3\Sigma_\alpha$$



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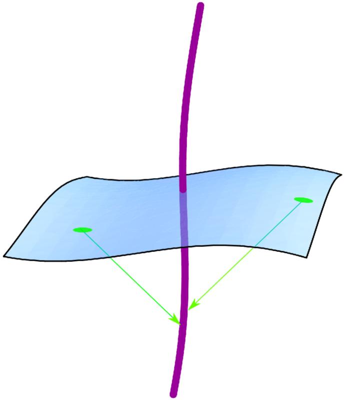

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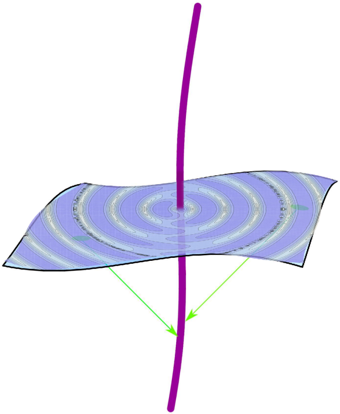

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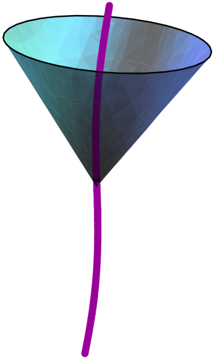

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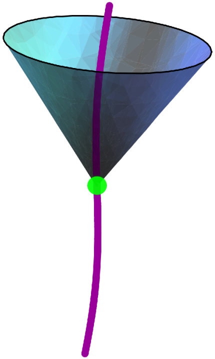
$$P^{\alpha} = \int_{\Sigma} (T^{\alpha\beta}_{\text{matter}} + T^{\alpha\beta}_{\text{self-self}}) \xi_{\beta} d^3\Sigma_{\alpha}$$



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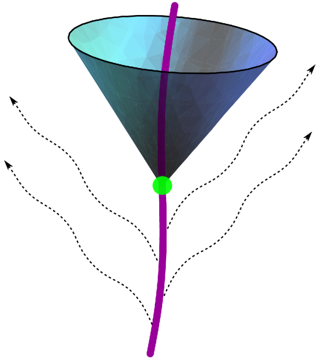


(Worldline Dependent) Bare Multipoles

- For this presentation, I'll use 'bare' Multipoles to indicate

$$q = \int_\Sigma d\Sigma_{\bar\eta} j^{\bar\eta}$$

$$J^\mu(\tau) = \int_\Sigma d\Sigma_{\bar\eta} u^{\bar\eta} j^\mu$$

$$Q^{\mu\nu}(\tau) = \int_\Sigma d\Sigma_{\bar\eta} u^{\bar\eta} j^\mu \sigma(z(\tau), x)^\nu$$

$$P^\mu(\tau) = \int_\Sigma d\Sigma_{\bar\eta} T^{\mu\bar\eta}$$

$$S^{\mu\nu}(\tau) = \int_\Sigma d\Sigma_{\bar\eta} T^{\bar\eta[\mu}\sigma(z(\tau), x)^{\nu]}$$

$$\hat m = \sqrt{\tilde P_\mu \tilde P^\mu}$$

$$Q^{\mu\nu\lambda}(\tau) = \int_\Sigma d\Sigma_{\bar\eta} u^{\bar\eta} j^\mu \sigma(z(\tau), x)^\nu \sigma(z(\tau), x)^\lambda$$

- Free indices are taken to indicate an implied parallel propagator to $z(\tau)$, u^β is the parallel transported velocity
- In general, these depend on a choice of hypersurface Σ and worldline - our results use a null surface

A Representative Worldline

- A concept of a central worldline is required define bulk motion and multipoles
- Fixed by a spin-supplementary condition ('center of mass')
  - We take

$$\tilde{S}^{\alpha\beta}_{(\Sigma)}\tilde{P}_{(\Sigma)\beta} = 0$$

  - Defines (by a nontrivial equation) a central worldline $z(\tau)$
- Spin supplementary is very similar to a gauge choice
  - Changes form of equations of motion, but not the physical result

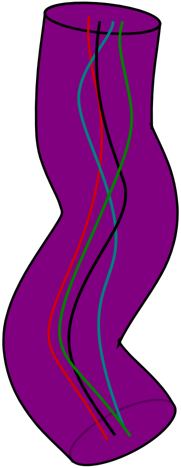


Figure: Many different worldlines; different EOMs, same physics

Our Perturbation Theory Notation

- Source fields are a one-parameter family $T^{\mu\nu}(\lambda, x^\mu)$, and $j^\mu(\lambda, x^\mu)$
- Derivation expands worldline (via acceleration) and body multipoles perturbatively, e.g. :

$$
\begin{aligned}
a^\mu =& a^{(0)\mu} + \lambda a^{(1)\mu} + \frac{\lambda^2}{2} a^{(2)\mu} \\
\tilde{Q}^{\mu\nu} =& \tilde{Q}^{(0)\mu\nu} + \lambda \tilde{Q}^{(1)\mu\nu} + \frac{\lambda^2}{2} \tilde{Q}^{(2)\mu\nu}
\end{aligned} \tag{3}
$$

- Maxwell field equations, conservation of stress energy, conservation of charge current expanded order by order to derive $D_\tau P^{(n)\mu}$ and $a^{(n)\mu}$
- We require also a projector for many portions, defined as $\mathbb{P}^\eta{}_\kappa \equiv (g^\eta{}_\kappa + u^\eta u_\kappa)$

Outline

A Reminder of the First Order Form

- The form of the acceleration given in [GHW] is

$$(a^\sigma \hat{m})^{(1)} = D_\tau(a^{(0)}{}_\mu \tilde{S}^{\sigma\mu}) + F^{(\mathrm{ext})\sigma\mu} u_\mu \tilde{q}^{(1)} + \mathbb{P}^\sigma{}_\rho \Bigg(\tfrac{2}{3}\big(\tilde{q}^{(0)}\big)^2 (D_\tau a^{(0)\rho})$$
$$+ \tfrac{1}{2} F^{(\mathrm{ext})}{}_{\mu\nu}{}^{;\rho} \tilde{Q}^{(0)\nu\mu} - 2D_\tau(u_\nu F^{(\mathrm{ext})[\nu|}{}_\mu \tilde{Q}^{(0)\mu|\rho]}) \Bigg)$$

- Can be re-written as null cone integrated, renormalized momentum:

$$\tilde{P}^{(1)\eta}\mathbb{P}_\eta{}^\kappa = \tilde{S}^{\mu\kappa} a^{(0)}{}_\mu - \tfrac{2}{3} a^{(0)\kappa} \tilde{q}^{(0)2} + 2\mathbb{P}^\kappa{}_\eta F^{(\mathrm{ext})[\eta|}{}_\nu \tilde{Q}^{(0)|\mu]\nu} u_\mu$$

- And momentum derivative

$$D_\tau(\tilde{P}^{(1)\eta})\mathbb{P}_\eta{}^\kappa = F^{(\mathrm{ext})\kappa\mu} u_\mu \tilde{q}^{(1)} + \mathbb{P}^\kappa{}_\eta F^{(\mathrm{ext})\eta}{}_{\mu;\nu} \tilde{Q}^{(0)\mu\nu}$$

- Projected equation of motion for a generalized killing momentum from [Harte]

First Order Mass and Spin Evolution

- Timelike component of Momentum Equation of motion

$$D_\tau(\tilde{P}^{(1)\mu})u_\mu = \tfrac{2}{3}(\tilde{q}^{(0)})^2 a^{(0)}{}_\mu a^{(0)\mu} + u_\lambda F^{(\mathrm{ext})\lambda}{}_{\mu;\nu}\tilde{Q}^{(0)\mu\nu}$$

- This implies a rest mass $\hat{m} \equiv \sqrt{\tilde{P}_\mu \tilde{P}^\mu}$ evolution:

$$D_\tau \hat{m}^{(1)} = \tfrac{1}{2} D_\tau F^{(\mathrm{ext})}{}_{\rho\nu}\tilde{Q}^{(0)\rho\nu} + 2\tilde{Q}_\rho^{(0)\nu} F_{\nu\mu}^{(\mathrm{ext})} a^{(0)[\mu} u^{\rho]}$$

- With spin evolution also consistent with [GHW]

$$D_\tau \tilde{S}^{\eta\nu}\mathbb{P}_\eta{}^\kappa \mathbb{P}_\nu{}^\lambda = 2F^{(\mathrm{ext})[\nu|\mu}\mathbb{P}_\nu{}^\lambda \tilde{Q}^{(1)}{}_\mu{}^{|\eta]}\mathbb{P}_\eta{}^\kappa$$

Second Order Laws of Motion - Momentum Value

$$\begin{aligned}
\tfrac{1}{2}\tilde{P}^{(2)\eta}\mathbb{P}_{\eta}{}^{\kappa} = \mathbb{P}^{\kappa}{}_{\eta}\Bigg[&\left(\tilde{S}^{(0)\mu\eta}a^{(0)}{}_{\mu}\right)^{(1)} - \left(\tfrac{2}{3}a^{(0)\eta}\tilde{q}^{(0)2}\right)^{(1)} + 2F^{(\mathrm{ext})[\eta|}{}_{\mu}\tilde{Q}^{(1)|\nu]\mu}u_{\nu} \\
&+\frac{\hat{m}^{(1)}\tilde{S}^{\eta\nu}a^{(0)}{}_{\nu}}{\hat{m}^{(0)}} - \frac{\tilde{S}^{\eta\mu}D_{\tau}\tilde{P}^{(1)}{}_{\mu}}{\hat{m}^{(0)}} + 2\frac{F^{(\mathrm{ext})[\eta|}{}_{\mu}\tilde{P}^{(1)\nu}\mathbb{P}_{\nu\rho}\tilde{Q}^{(0)\mu|\rho]}}{\hat{m}^{(0)}} \\
&+\tfrac{2}{3}\tilde{q}^{(0)}\tilde{Q}^{(0)\mu\eta}{}_{;\nu;\lambda}u_{\mu}u^{\nu}u^{\lambda} - \tfrac{5}{3}\tilde{q}^{(0)}a^{(0)}{}_{\mu}\tilde{Q}^{(0)\eta\mu}{}_{;\nu}u^{\nu} + \tfrac{1}{3}\tilde{q}^{(0)}a^{(0)}{}_{\mu}a^{(0)\eta}\tilde{Q}^{(0)\mu\nu}u_{\nu} \\
&+\tfrac{1}{3}D_{\tau}a^{(0)}{}_{\mu}\tilde{q}^{(0)}\tilde{Q}^{(0)\eta\mu} - F^{(\mathrm{ext})\eta\nu}u_{\nu}a^{(0)}{}_{\sigma}\tilde{Q}^{(0)\mu\sigma}{}_{\mu} - F^{(\mathrm{ext})\eta\nu}a^{(0)}{}_{\nu}\tilde{Q}^{(0)\mu\sigma}{}_{\mu}u_{\sigma} \\
&-\tfrac{1}{3}D_{\tau}\left(F^{(\mathrm{ext})\eta}{}_{\mu}\tilde{Q}^{(0)\mu\nu\lambda}\right)u_{\nu}u_{\lambda} - \tfrac{1}{6}D_{\tau}(F^{(\mathrm{ext})\eta\mu}u_{\mu}\tilde{Q}^{(0)\lambda\sigma\rho})u_{\lambda}u_{\sigma}u_{\rho} \\
&-\tfrac{1}{3}F^{(\mathrm{ext})}{}_{\mu}{}^{\nu;\eta}\tilde{Q}^{(0)\mu\lambda\rho}u_{\nu}u_{\lambda}u_{\rho} - D_{\tau}\left(F^{(\mathrm{ext})\eta}{}_{\mu}\tilde{Q}^{(0)\nu\mu\rho}\right)\mathbb{P}_{\nu\rho} \\
&+F^{(\mathrm{ext})\eta\sigma}{}_{;\rho}\mathbb{P}_{\mu\nu}\tilde{Q}^{(0)\mu\nu\lambda}u_{\lambda}u_{\sigma}u^{\rho} - F^{(\mathrm{ext})}{}_{\nu}{}^{\sigma}{}_{;\mu}u_{\sigma}\tilde{Q}^{(0)\nu\eta\mu} \\
&-\tfrac{16}{5}F^{(\mathrm{ext})\eta}{}_{\mu}a^{(0)}{}_{\lambda}\mathbb{P}^{\mu}{}_{\nu}\tilde{Q}^{(0)\sigma\lambda\nu}u_{\sigma} + \tfrac{13}{6}F^{(\mathrm{ext})\eta\nu}u_{\nu}a^{(0)}{}_{\mu}\tilde{Q}^{(0)\mu\sigma\rho}u_{\sigma}u_{\rho} \\
&-3F^{(\mathrm{ext})\eta}{}_{\mu}a^{(0)}{}_{\nu}\tilde{Q}^{(0)\nu\mu\lambda}u_{\lambda} + \tfrac{1}{3}F^{(\mathrm{ext})}{}_{\mu}{}^{\nu}u_{\nu}a^{(0)\eta}\tilde{Q}^{(0)\mu\lambda\sigma}u_{\lambda}u_{\sigma} \\
&-4F^{(\mathrm{ext})\eta\mu}a^{(0)}{}_{\mu}\tilde{Q}^{(0)\nu\lambda\sigma}u_{\nu}u_{\lambda}u_{\sigma} - \tfrac{2}{3}F^{(\mathrm{ext})\eta\nu}a^{(0)}{}_{\mu}\mathbb{P}_{\nu\lambda}\tilde{Q}^{(0)\lambda\mu\sigma}u_{\sigma} \\
&-F^{(\mathrm{ext})\nu}{}_{\mu}a^{(0)}{}_{\nu}\tilde{Q}^{(0)\mu\eta\lambda}u_{\lambda} + \tfrac{1}{5}F^{(\mathrm{ext})}{}_{\mu}{}^{\sigma}a^{(0)}{}_{\nu}\tilde{Q}^{(0)\mu\eta\nu}u_{\sigma}\Bigg]
\end{aligned}$$

- This gives the relation between the various body parameters and the total momentum of the object - increasingly complicated at high order

Second Order Laws of Motion - Force

$$
\begin{aligned}
\frac{1}{2} D_\tau \tilde{P}^{(2)\eta} \mathbb{P}_\eta{}^\kappa = \mathbb{P}^\kappa{}_\eta \Big[& \frac{1}{2} \tilde{q}^{(2)} F^{(\mathrm{ext})\eta\mu} u_\mu + F^{(\mathrm{ext})\eta}{}_{\mu;\nu} \tilde{Q}^{(1)\mu\nu} \\
& + \frac{4}{3} \tilde{q}^{(0)} D_\tau \left(a^{(0)\eta} a^{(0)}{}_\nu \right) \tilde{Q}^{(0)\mu\nu} u_\mu + \frac{8}{3} \tilde{q}^{(0)} D_\tau a^{(0)}{}_\mu a^{(0)[\eta|} \tilde{Q}^{(0)\nu|\mu]} u_\nu \\
& + \frac{2}{3} \tilde{q}^{(0)} D_\tau^2 a^{(0)}{}_\mu \mathbb{P}^\mu{}_\nu \tilde{Q}^{(0)\eta\nu} + \frac{1}{6} F^{(\mathrm{ext})\eta}{}_{\mu;\nu;\lambda} u^\nu u^\lambda \tilde{Q}^{(0)\mu\rho\sigma} u_\rho u_\sigma \\
& + \frac{1}{6} a^{(0)}{}_\nu u_\mu F^{(\mathrm{ext})\nu\mu;\eta} \tilde{Q}^{(0)\lambda\rho\sigma} u_\lambda u_\rho u_\sigma - \frac{2}{3} \tilde{q}^{(0)} a^{(0)}{}_\nu a^{(0)\mu} a^{(0)}{}_\mu \tilde{Q}^{(0)\eta\nu} \\
& - \frac{1}{5} a^{(0)}{}_\mu F^{(\mathrm{ext})\eta\nu}{}_{;\rho} u^\rho \mathbb{P}_{\nu\lambda} \tilde{Q}^{(0)\sigma\mu\lambda} u_\sigma + \frac{1}{2} F^{(\mathrm{ext})\eta}{}_{\mu;\nu;\rho} \mathbb{P}^\rho{}_\sigma \mathbb{P}^\nu{}_\lambda \tilde{Q}^{(0)\mu\lambda\sigma} \\
& + \frac{4}{5} a^{(0)}{}_\nu F^{(\mathrm{ext})\eta}{}_{\mu;\lambda} \mathbb{P}^\lambda{}_\sigma \tilde{Q}^{(0)\mu\nu\sigma} + a^{(0)\nu} F^{(\mathrm{ext})\eta\sigma}{}_{;\nu} \mathbb{P}_{\lambda\sigma} \tilde{Q}^{(0)\mu\lambda}{}_\mu \\
& + \frac{a^{(0)\eta}}{\hat{m}^{(0)}} \Big(\tilde{S}^{(0)\nu}{}_\mu \tilde{S}^{(0)\lambda\mu} a^{(0)}{}_\nu a^{(0)}{}_\lambda + 4 F^{(\mathrm{ext})[\nu|}{}_\lambda \tilde{S}^{(0)\mu}{}_\nu a^{(0)}{}_\mu \tilde{Q}^{(0)|\sigma]\lambda} u_\sigma \\
& + \frac{8}{3} F^{(\mathrm{ext})}{}_\nu{}^{[\mu} a^{(0)}{}_\mu \tilde{Q}^{\lambda]\nu} u_\lambda + 4 F^{(\mathrm{ext})[\mu|\nu} F^{(\mathrm{ext})}{}_{[\lambda|\nu} \tilde{Q}^{|\sigma]}{}_\mu \tilde{Q}_{|\rho]}{}^\lambda u_\sigma u^\rho \Big) \Big]
\end{aligned}
$$

- Momentum derivative nontrivially related to acceleration - full dependence on external field derived from reduction of order
- ‘Internal structure’ dependence
- Mixing between multipole and self force

Second Order Laws of Motion - Timelike Component

$$
\begin{aligned}
\tfrac{1}{2}D_\tau \tilde{P}^{(2)\mu}u_\mu = {} & \tilde{P}^{(1)\mu}a^{(1)}{}_\mu + \left(\tfrac{2}{3}(\tilde{q}^{(0)})^2 a^{(0)}{}_\mu a^{(0)\mu}\right)^{(1)} + u_\lambda F^{(\mathrm{ext})\lambda}{}_{\mu;\nu}\tilde{Q}^{(1)\mu\nu} \\
& + \tfrac{4}{3}\tilde{q}^{(0)} D_\tau a^{(0)}{}_\mu \mathbb{P}^\mu{}_\nu a^{(0)}{}_\lambda \tilde{Q}^{(0)\nu\lambda} + \tfrac{1}{6}a^{(0)}{}_\mu F^{(\mathrm{ext})\mu\nu}{}_{;\lambda}u_\nu u^\lambda \tilde{Q}^{(0)\sigma\rho\alpha}u_\sigma u_\rho u_\alpha \\
& - \tfrac{1}{5}a^{(0)}{}_\mu F^{(\mathrm{ext})\nu\lambda}{}_{;\sigma}u_\lambda u^\sigma \tilde{Q}^{(0)\rho\mu}{}_\nu u_\rho - \tfrac{1}{6}F^{(\mathrm{ext})}{}_\mu{}^\nu{}_{;\lambda;\sigma}u_\nu u^\lambda u^\sigma \tilde{Q}^{(0)\mu\rho\alpha}u_\rho u_\alpha \\
& - \tfrac{1}{2}F^{(\mathrm{ext})}{}_\mu{}^\nu{}_{;\lambda;\rho}u_\nu \mathbb{P}^\lambda{}_\sigma \mathbb{P}^\rho{}_\alpha \tilde{Q}^{(0)\mu\sigma\alpha} - a^{(0)\mu}F^{(\mathrm{ext})}{}_\nu{}^\lambda{}_{;\mu}u_\lambda \tilde{Q}^{(0)\nu\sigma\rho}u_\sigma u_\rho \\
& - \tfrac{4}{5}a^{(0)}{}_\mu F^{(\mathrm{ext})}{}_\nu{}^\sigma{}_{;\lambda}u_\sigma \tilde{Q}^{(0)\nu\mu\lambda} - a^{(0)\mu}F^{(\mathrm{ext})}{}_\nu{}^\lambda{}_{;\mu}u_\lambda \tilde{Q}^{(0)\sigma\rho\nu}\mathbb{P}_{\sigma\rho} \\
& + \tfrac{4}{3}\tilde{q}^{(0)}\left(D_\tau^2 a^{(0)}{}_\mu + a^{(0)}{}_\nu a^{(0)\nu}a^{(0)}{}_\mu\right)\mathbb{P}^\mu{}_\sigma \tilde{Q}^{(0)\sigma\lambda}u_\lambda \\
& - \tfrac{1}{\hat{m}}D_\tau\Bigg(\tilde{S}^\nu{}_\mu \tilde{S}^{\lambda\mu}a^{(0)}{}_\nu a^{(0)}{}_\lambda - 4F^{(\mathrm{ext})[\nu|}{}_\lambda \tilde{S}^\mu{}_\nu a^{(0)}{}_\mu \tilde{Q}^{(0)|\sigma]\lambda}u_\sigma \\
& - \tfrac{8}{3}F^{(\mathrm{ext})}{}_\nu{}^{[\mu}a^{(0)}{}_\mu \tilde{Q}^{\lambda]\nu}u_\lambda - 4F^{(\mathrm{ext})[\mu|\nu}F^{(\mathrm{ext})}{}_{[\lambda|\nu}\tilde{Q}^{|\sigma]}{}_\mu \tilde{Q}_{|\rho]}{}^\lambda\, u_\sigma u^\rho\Bigg)
\end{aligned}
$$

- Non-vanishing for pure monopole, though rest mass conserved for monopole
- Again, internal structure and mixing of multipole and self force

Second Order Rest Mass and Spin evolution

$$D_\tau \tilde{S}^{(1)\eta\sigma}\mathbb{P}_\eta{}^\kappa\mathbb{P}_\sigma{}^\rho = \mathbb{P}^\kappa{}_\eta\mathbb{P}^\rho{}_\sigma\Big[\tfrac{2}{3}\tilde{q}^{(0)}D_\tau a^{(0)[\sigma|}\tilde{Q}^{(0)\mu|\eta]}u_\mu + \tfrac{1}{3}\tilde{q}^{(0)}a^{(0)}{}_\mu a^{(0)[\sigma|}\tilde{Q}^{(0)\mu|\eta]}$$
$$+F^{(\text{ext})[\eta|}{}_\mu\tilde{Q}^{(1)\mu|\sigma]} + \tfrac{1}{3}F^{(\text{ext})[\eta}{}_\mu{}^{;\sigma]}\tilde{Q}^{(0)\mu\nu\lambda}u_\nu u_\lambda$$
$$+F^{(\text{ext})[\eta|}{}_{\mu;\nu}\mathbb{P}^\nu{}_\lambda\tilde{Q}^{(0)\mu|\sigma]\lambda} + \tfrac{2}{3}F^{(\text{ext})[\eta}{}_\mu a^{(0)\sigma]}\tilde{Q}^{(0)\mu\nu\lambda}u_\nu u_\lambda$$
$$+F^{(\text{ext})}{}_\mu{}^\nu u_\nu a^{(0)[\sigma|}\tilde{Q}^{(0)\mu|\eta]\lambda}u_\lambda + \tfrac{4}{5}F^{(\text{ext})[\eta|}{}_\mu a^{(0)}{}_\nu\tilde{Q}^{(0)\mu|\sigma]\nu}\Big]$$

- acceleration-dependence : self torque arises at second order

$$\hat{m}^{(2)} = -\tilde{P}^{(2)\mu}u_\mu - \tfrac{1}{\hat{m}}\Big(\tfrac{2}{9}(\tilde{q}^{(0)})^4 a^{(0)}{}_\mu a^{(0)\mu} + \tilde{S}^\nu{}_\mu\tilde{S}^{\lambda\mu}a^{(0)}{}_\nu a^{(0)}{}_\lambda$$
$$+4F^{(\text{ext})[\nu|}{}_\lambda\tilde{S}^\mu{}_\nu a^{(0)}{}_\mu\tilde{Q}^{(0)|\sigma]\lambda}u_\sigma + \tfrac{8}{3}F^{(\text{ext})}{}_\nu{}^{[\mu}a^{(0)}{}_\mu\tilde{Q}^{\lambda]\nu}u_\lambda$$
$$+4F^{(\text{ext})[\mu|\nu}F^{(\text{ext})}{}_{[\lambda|\nu}\tilde{Q}^{|\sigma]}{}_\mu\tilde{Q}_{|\rho]}{}^\lambda u_\sigma u^\rho\Big)$$

- With these terms taken into account, monopole rest mass is conserved.

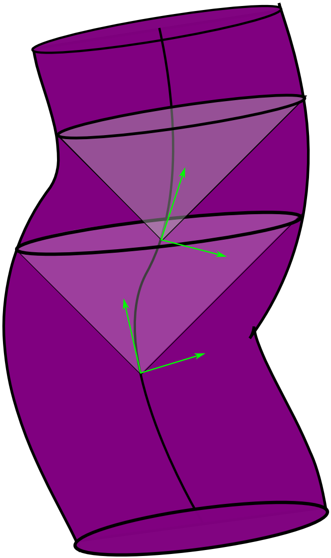
Outline

Worldline-Following Basis

- We define a 4-vector P^μ at each worldline point, then choose a tetrad basis using worldline velocity to compute above components
- Related to a momentum in a stationary basis by a time-dependent boost

$$P_\xi{}^{\hat{a}} = \int d\Sigma_\alpha T^{\alpha\beta} \xi^{\hat{a}}{}_\beta$$

$$\tilde{P}^{\hat{a}} = \Lambda(t)^{\hat{a}}{}_{\hat{b}} P_\xi{}^{\hat{b}}$$



The Self Field and the Body Parameters

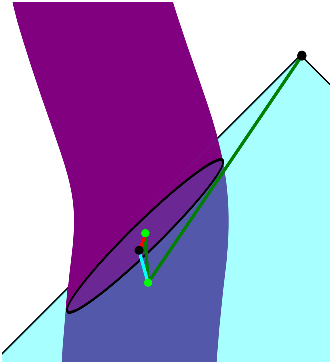
- We anticipated that the dependence of the self force may be complicated and involve degrees of freedom not usually associated with multipoles ('internal structure')
- However, we also anticipated that these values, like the static case, will be encoded in the external field degrees of freedom
- This anticipation has proved a useful check on our results and catch errors
- From this, we can work out which combinations of bare multipoles give the true field dependence, and express the self force in terms of the field degrees of freedom
- This defines the multipole renormalizations $\tilde{Q}'s$ in our equations of motion

External field on Co-accelerating Null Cones

- Field extracted from source function via Green Functions:

$$A^{\mu}(x,\tau) = \int d^4x' \delta(\sigma(x,x')) g^{\mu}{}_{\nu}(x,x') j^{\nu}(x')$$

- Re-write delta as $\delta(\tau)/\sigma_{,\tau}$ [Similar to Quinn 2000]
- Expand τ values near $\tau = \tau_{-}$
- Expand σ , $\sigma_{,\alpha}$, $g_{\alpha\beta}$ near $x = x'$
- $\sigma(x,x') = 0$ fixes the null cone surface $\tau'(x',x)$



Charge Renormalization from External Field

- No guarantee that the bare multipoles directly correspond to external field multipoles
- We start with the bare multipoles - Field expansion shows that they're alright to leading order

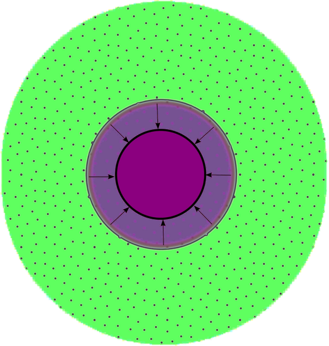
$$A^{(0)\mu}{}_{l=1}(\tau) = \frac{\hat{n}^\nu}{r^2} \int d^3x' r' \hat{n}'_\nu j^{(0)\mu}(\tau, x') = \frac{\hat{n}^\nu}{r^2} Q^{(0)\mu}{}_\nu$$

- At subleading order, the field moments become more complicated

$$A^{(1)\mu}{}_{l=1}(\tau) u_\mu = \frac{\hat{n}^\nu}{r^2}\Big(- Q^{(0)\alpha}{}_\nu + Q^{(0)\mu}{}_\nu{}^\lambda{}_{,\sigma} u_\mu u_\lambda u^\sigma + \tfrac{2}{5} a^{(0)}{}_\nu Q^{(0)\mu\lambda\sigma} u_\mu u_\lambda u_\sigma \\ - \frac{4}{5} a^{(0)}{}_\mu Q^\sigma{}_\nu{}^\mu u_\sigma - a^{(0)}{}_\mu Q^{(0)\mu}{}_\nu{}^\sigma u_\sigma \Big)$$

Method of Derivation

- Assume a one-parameter family of solutions labeled by ϵ such that mass, charge and size of body are all proportional to ϵ as $\epsilon \to 0$ [Gralla, Harte and Wald]
- Scale parameter is the ratio of external field scale to body size
- This gives two limits
  - $\epsilon \to 0$ at fixed "far zone" coordinates, body shrinks
  - $\epsilon \to 0$ at fixed "near zone" coordinates, background stretches, body remains constant



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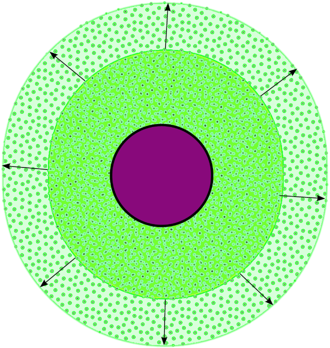

Deriving Laws of Motion

- Analytic derivation method - very similar to that of Gralla, Harte and Wald:
  - Expand acceleration dependent stress-energy conservation and higher multipole equations order-by-order

$$\int d\Sigma_\mu \nabla_\mu T^{\hat{n}\mu} = \int d\Sigma_\mu u^\mu F^{(\text{ext})\hat{n}\nu} j_\nu$$

$$\int d\Sigma_\mu \hat{n}^{\hat{i}_1 \ldots \hat{i}_n} r^m \nabla_\mu T^{\hat{n}\mu} = \int d\Sigma_\mu u^\mu F^{(\text{ext})\hat{n}\nu} \hat{n}^{\hat{i}_1 \ldots \hat{i}_n} r^m j_\nu$$

  - Impose spin supplementary condition
  - Impose $T^{\mu\nu} = T^{\mu\nu}_{\text{self-self}}$ for boundary integrals
  - invert to find acceleration

Process of Deriving the Laws of Motion

- Automated analytic derivation performed using xAct mathematica package
- Consists of around 900 mathematica cells, with several custom functions for dealing with the frame manipulations we need for working in retarded basis tetrads
- Cascaded order-by-order equation substitution is largely automated
- Without cached results, the code takes approximately 10-12 hours to run from start to end on a personal system
  - No paralellization used
  - Likely many optimizations can be made to pare that down
- Caching at several steps brings execution time down under an hour on subsequent runs.

Conclusions

- Rigorous limiting process-based techniques can be used to examine higher-order self forces
- For general (non-symmetric) bodies, the electromagnetic dipole effects are comperable to the first order self force, the quadrupole effects are comparable to the second-order self force
- These methods derive subtle effects that would be overlooked by naively adding point particle self force and multipole coupling terms
- Equations of motion require careful choice of extended body parameter definitions