Octupolar Gauge Invariants in Schwarzschild Spacetimes

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Capra 16, Kyoto, 2015

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arXiv:1505.04447

Talk Overview

- Introduction to Gauge Invariants
    - Recent Gauge Invariants: Spin Precession, Tidal Tensor
    - Deriving New Invariants: Octupoles
    - Expressions in Terms of the Perturbed Metric
- Computational Methods
    - Frequency Domain: Regge-Wheeler Gauge
    - Frequency Domain: Lorenz Gauge
    - Post Newtonian Expansion:MST
- Results
    - Compare Computational Methods
    - PN behaviour of Octupoles
    - Light Ring Divergences

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Spin Precession I

- For circular orbits, have a killing vector $k_a|_\gamma = u_a$ and axial vector ω^a such that

$$\omega^a = \frac{1}{2}\epsilon^a{}_{bcd}k^b\nabla^c k^d$$

- Define Lie- and Parallel-transported tetrads:

$$e_1^a = \cos(\omega\tau)\hat{e}_1^a + \sin(\omega\tau)\hat{e}_3^a$$
$$e_3^a = -\sin(\omega\tau)\hat{e}_1^a + \cos(\omega\tau)\hat{e}_3^a$$
$$e_2^a = \hat{e}_2^a$$

- Find these are related by:

$$\mathcal{L}_k e_i^a \equiv 0 \quad , \quad \frac{D\hat{e}_i^a}{d\tau} \equiv 0$$

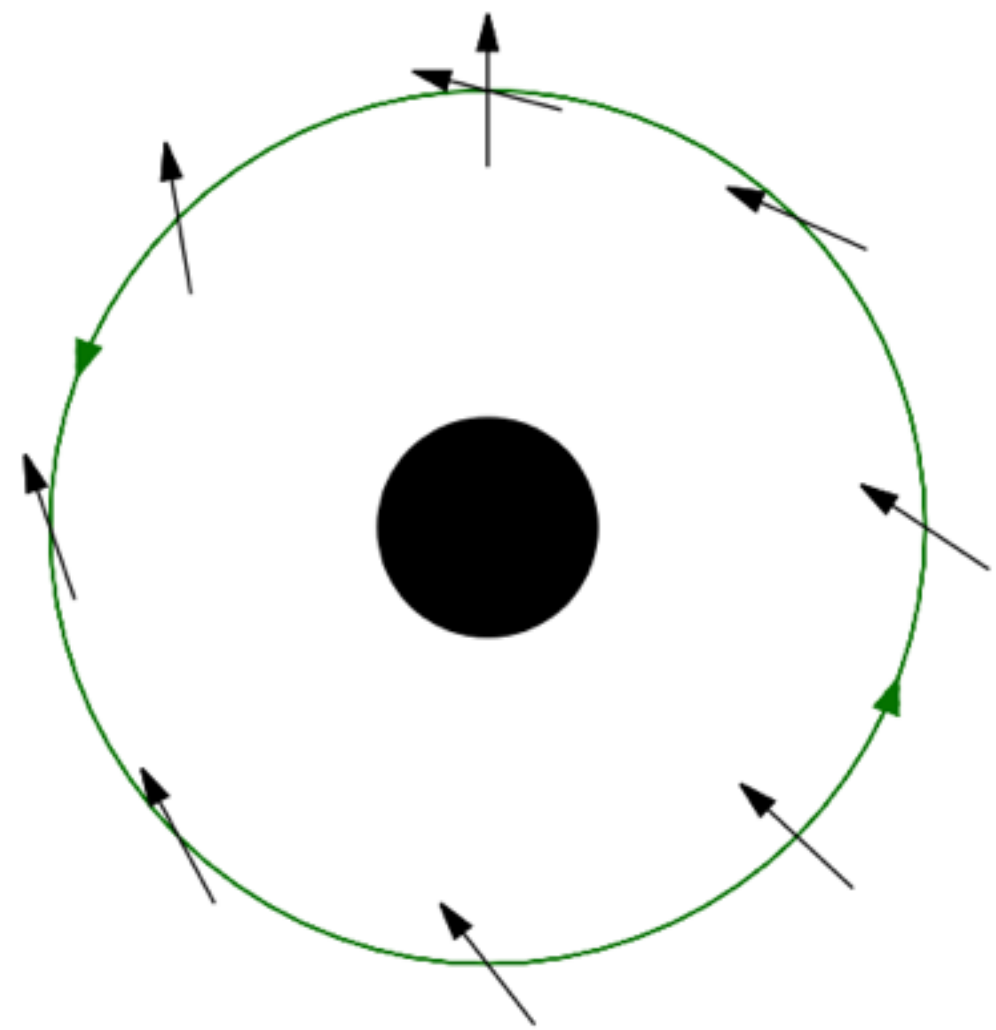
Spin Precession II

- Looking for physical quantities proportional to 1st derivative of the metric ($\Gamma^{\rho}_{\mu\nu}$).
- Spin precession $\Delta\Psi$ is exactly the quantity we're looking for:

$$\Psi \sim \omega \ = \Gamma_{abc} e_3^a u^b e_1^c$$

- Find variation $\Delta\Psi$ by calculating the terms like

$$\Delta\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu}(g+h)_{\mu\nu} - \Gamma^{\rho}_{\mu\nu}(g)_{\mu\nu}$$



Variation

- Total variation given by two terms

$$\Delta\Psi = \delta(\Gamma_{abc}e_3^a u^b e_1^c) - \frac{1}{3M\mu} r_0^3 v^2 F_r \frac{d\bar{\Psi}}{dr_0}$$

$\delta\Psi$

$\delta\Omega$,gauge invariant orbital frequency

- Reduces to just an expression as function of metric perturbation

Tidal Tensor I

- Newtonian Example: Have potential Φ, and particles x and z located at $x^a, z^a = x^a + \eta^a$, so

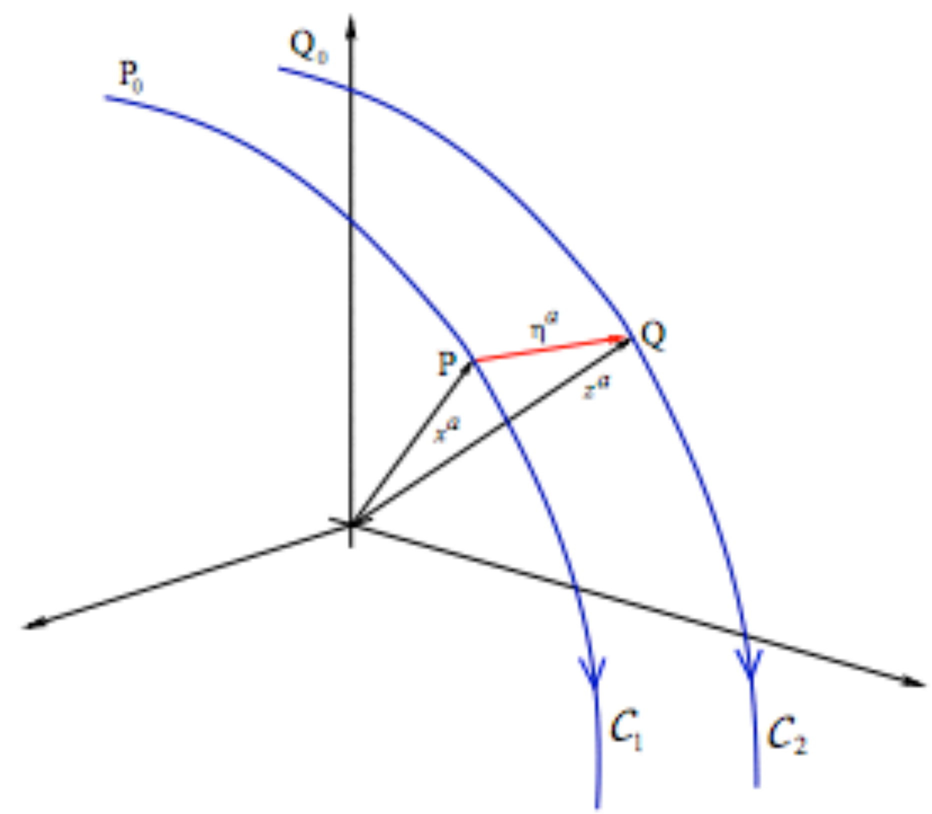
$$\ddot{x}^a = -(\partial^a \Phi)_P,$$
$$\ddot{z}^a = -(\partial^a \Phi)_Q = \ddot{x} + \ddot{\eta}$$

- Now Taylor Expand Potentials:

$$\Phi_{\rm Q} = \Phi_P + \eta^b \partial_b \Phi_P + \mathcal{O}(\eta^2)$$
$$-\partial^a \Phi_Q = \ddot{x} + \ddot{\eta} = -\partial^a(\Phi_P + \eta^b \partial_b \Phi_b)$$

Gives us a "geodesic deviation":

$$\ddot{\eta}^a = -(\partial^a \partial_b \Phi)\eta^b \equiv -K^a_{\ b}\eta^b$$



Tidal Tensor II

- Have similar equation in GR, where the $K^a{}_b$ previously (describing curvature) is constructed from R_{abcd} :

$$\frac{D^2\zeta^a}{d\tau^2} = -\mathcal{E}^a{}_b\zeta^b \quad , \quad \mathcal{E}_{ac} = R_{abcd}u^b u^d$$

- Also have similar equation for R^*_{abcd}, the hodge dual corresponding to Papapetrou eqns:

$$\frac{Dp^a}{d\tau} = -\mathcal{B}^a{}_b s^b \quad , \quad \mathcal{B}_{ac} = R^*_{abcd}u^b u^d$$

- Can make a suitable choice of basis tetrad such that matrices $\mathcal{E}$ and $\mathcal{B}$ seperate. Find 5 L.I. invariants:

$$\{\Delta\lambda_1^E, \Delta\lambda_2^E, \Delta\lambda_3^E, \Delta\lambda^B, \Delta\chi\}$$

Octupolar Terms

- Begin by defining a tetrad as for tidal case,so that $\mathcal{E}_{ij}$ is diagonal in perturbed spacetime.
- Consider 3rd derivs of metric $\Rightarrow R_{adbe;c}$
- Define contractions similar to quadrupole case for octupole moments:

$$\mathcal{E}_{abc} = R_{adbe;c}u^d u^e$$

$$\mathcal{B}_{abc} = R^*_{adbe;c}u^d u^e$$

Octupoles II

- Have 64 terms, but lots of symmetry.
  - Circular orbit+reflection symmetry $\Rightarrow$ many terms zero.
  - Time independence $\Rightarrow$ many terms given by quadrupoles.
  - Bianchi Identity, Trace conditions $\Rightarrow$ relations between octupole terms.
- Left with just 4+3 new independent terms:

$\{\delta\mathcal{E}_{111}, \delta\mathcal{E}_{122}, \delta\mathcal{E}_{133}\}$	Electric Conservative, 1 trace eqn
$\{\delta\mathcal{E}_{113}, \delta\mathcal{E}_{223}, \delta\mathcal{E}_{333}\}$	Electric Dissipative, 1 trace eqn
$\{\delta\mathcal{B}_{123}, \delta\mathcal{B}_{211}, \delta\mathcal{B}_{222}, \delta\mathcal{B}_{233}\}$	Magnetic Dissipative, 1 trace eqn

Non zero unperturbed terms given by

$$\mathcal{E}_{111} = \mathcal{A}(6r_0^2 - 9Mr_0) \qquad \mathcal{B}_{211} = \mathcal{C}(4r_0^2 - 8Mr_0)$$
$$\mathcal{E}_{122} = -\mathcal{A}(3r_0^2 - 2Mr_0) \qquad \mathcal{B}_{222} = \mathcal{C}(3r_0^2 - 6Mr_0)$$
$$\mathcal{E}_{333} = \mathcal{A}(3r_0^2 - 7Mr_0) \qquad \mathcal{B}_{233} = \mathcal{C}(r_0^2 - 2Mr_0)$$

Progress

Collected numerous gauge invariants up to octupolar moment in the perturbed field:

$$\{\Delta\psi\}$$

$$\{\Delta\lambda_1^E, \Delta\lambda_2^E, \Delta\lambda_3^E, \Delta\lambda^B, \Delta\chi\}$$

$$\{\Delta\mathcal{E}_{111}, \Delta\mathcal{E}_{122}, \Delta\mathcal{E}_{133}, \Delta\mathcal{E}_{333}, \Delta\mathcal{B}_{211}, \Delta\mathcal{B}_{222}, \Delta\mathcal{B}_{233}\}$$

All just dependent on $h_{\alpha\beta}, \partial_r h_{\alpha\beta}$; Calculating these gives us gauge invariant quantities.

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Circular Orbits: Lorenz

- Compute frequency domain circular orbit metric perturbation for two gauges, Lorenz & Regge-Wheeler.

- Lorenz takes the physically motivated gauge choice $\nabla^{\alpha} h_{\alpha\beta} = 0$ to simplify a standard wave equation:

$$\Box h_{\alpha\beta} - R^{\mu\nu}{}_{\alpha\beta} h_{\mu\nu} = 8\pi T_{\alpha\beta}$$

- Spherical decomposition leaves {6 even +4 odd} fields, - (3 even +1odd) gauge equations to solve.

- 4 +2 coupled equations. Awkward to solve but easier to regularise.

Circular Orbits: RW

- Regge-Wheeler gauge is defined by setting a selection of 3 even +1 odd field to zero.
- Reduces Einstein's equations to just one even plus one odd field equation:

$$\partial_{r_*}^2 + (\omega^2 - V_{lm}^{\mathrm{e/o}})\phi^{\mathrm{e/o}} = S_{\mu\nu}$$

- Usually can't be regularised, due to singularities in perturbation expressions; For GIs however, can just use Lorenz gauge RPs.
- Single field means quicker and easier to solve.

Solving Perturbations

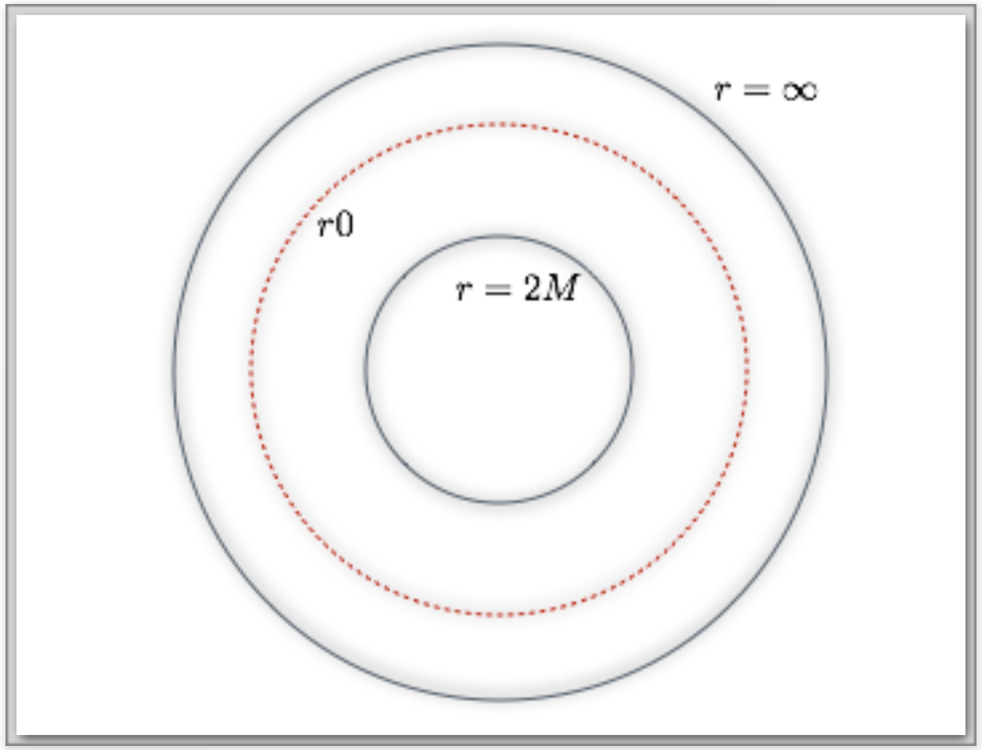
- Whichever gauge we choose, the wave equations must be solved numerically.

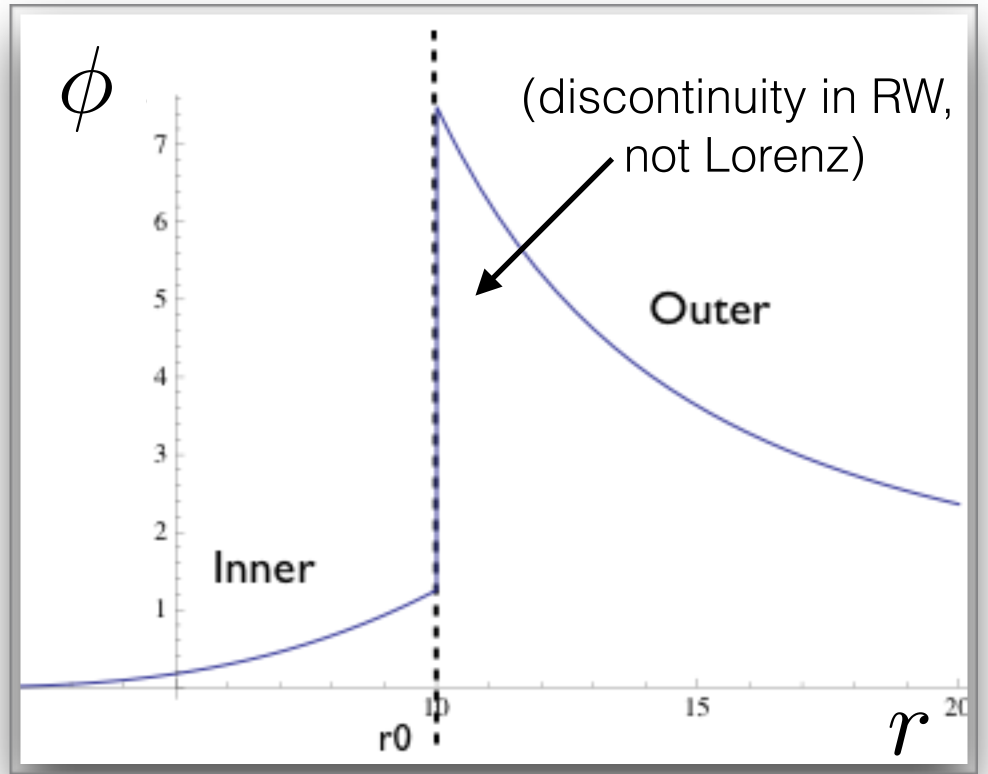
- First find boundary conditions via power law expansions at boundaries:

$$\phi_\infty = e^{i\omega r_\star} \sum_{n=0}^{n_\infty} \frac{a_n}{r^n}$$

$$\phi_H = e^{-i\omega r_\star} \sum_{n=0}^{n_H} (r - 2M)^n$$

- Numerically solve inwards from boundaries to get solutions at particle.





MST

- Using MST (Chris's talk tomorrow), construct analytical expressions for the PN series in R-W gauge.
- Can compare these against our numerical results to provide a 3rd check of our calculations.

Regularisation

- Taking higher derivatives comes at a cost: extra derivatives of harmonics increase l-mode divergence

$$\partial_\theta^n Y_{lm}(\theta,\phi) \sim l^n Y_{lm}(\theta,\phi)$$

- Must subtract off regularisation parameters l by l :

$$\delta X_{reg} = \sum_{\underbrace{l=0}_{\text{tensor } l}}^{\infty} \delta X^l - \underbrace{\delta X_{-3}}_{\text{(Octupole)}} - \underbrace{\delta X_{-2}}_{\text{(Tidal)}} - \underbrace{\delta X_{-1}}_{\text{(Spin)}} - \underbrace{\delta X_0}_{\text{(Redshift)}} - ..$$

So codes need higher accuracy with each derivative. (lose 2-3 orders of magnitude per factor of l)

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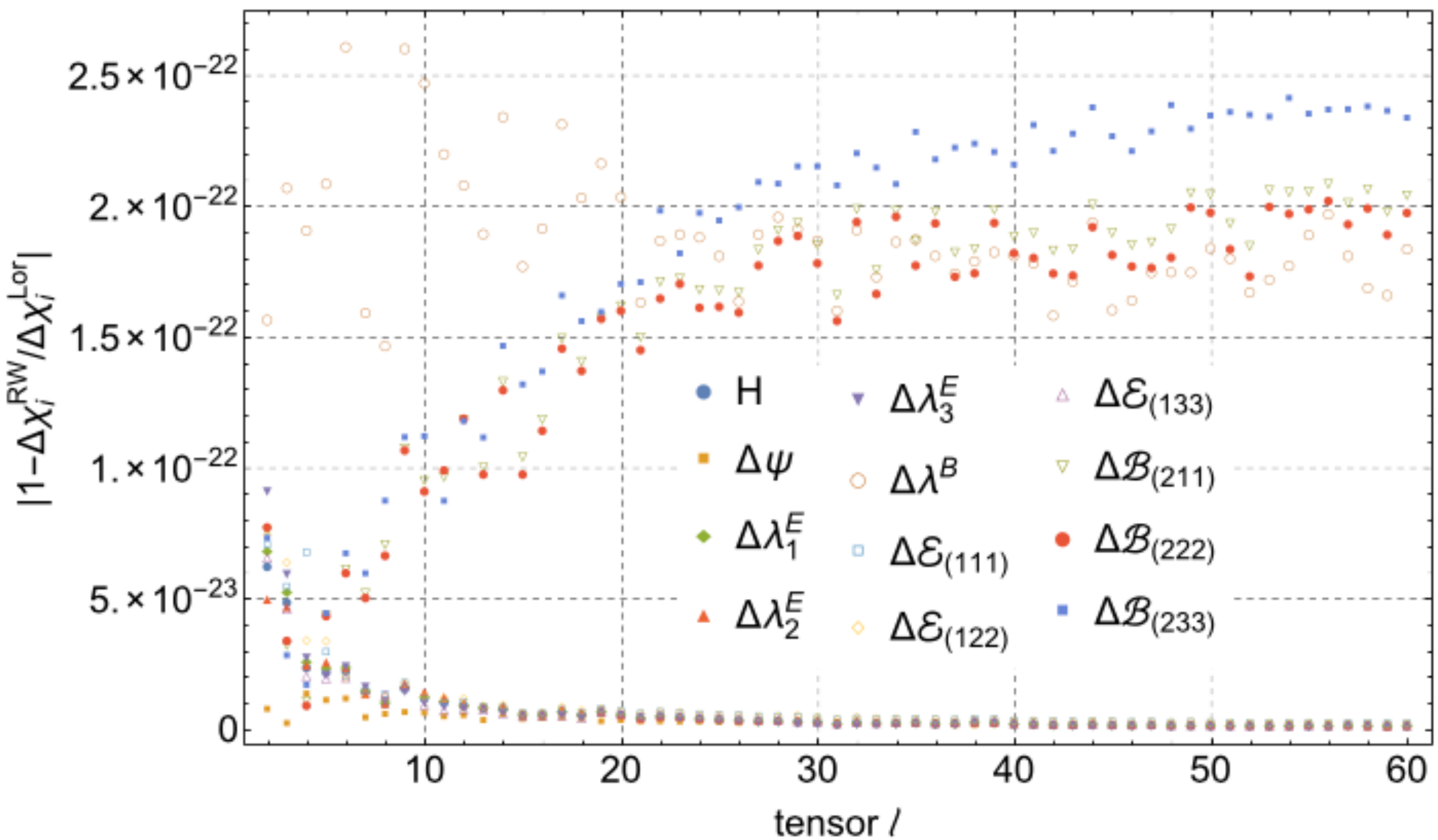
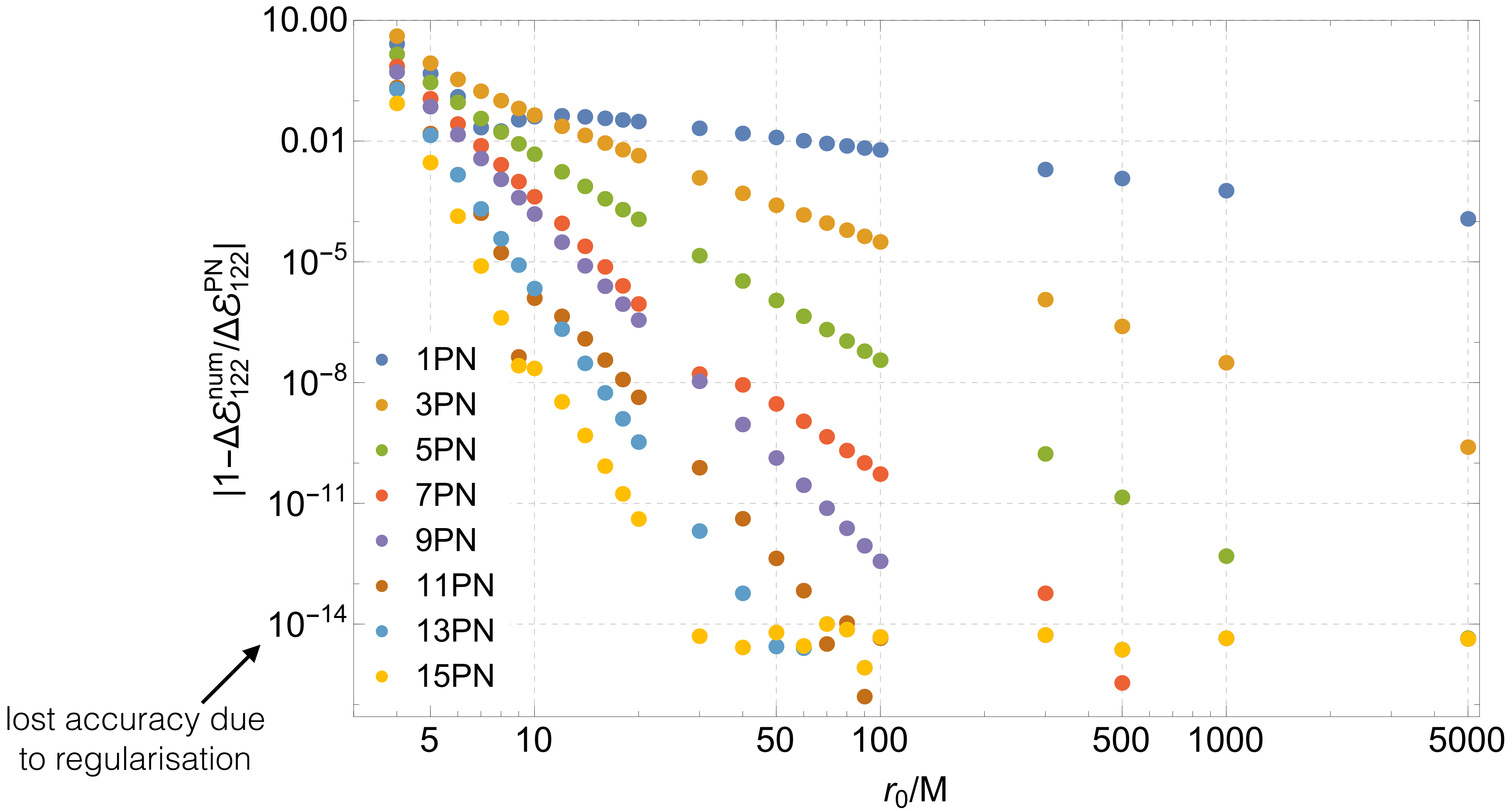
R-W v Lorenz



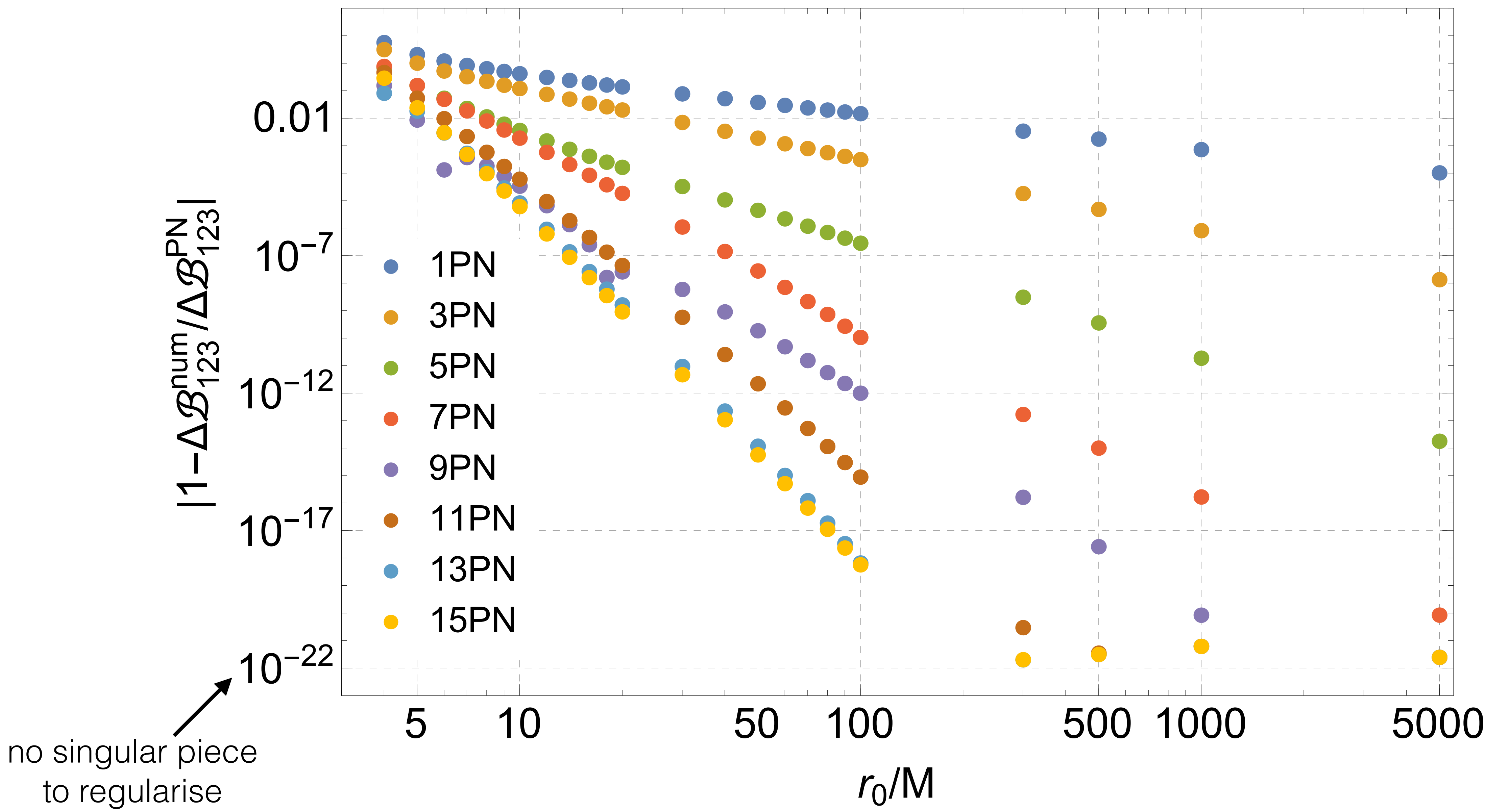
Comparing RW to Lorenz codes. See excellent agreement to ~22 sf for all gauge invariants.

E-type Octupoles



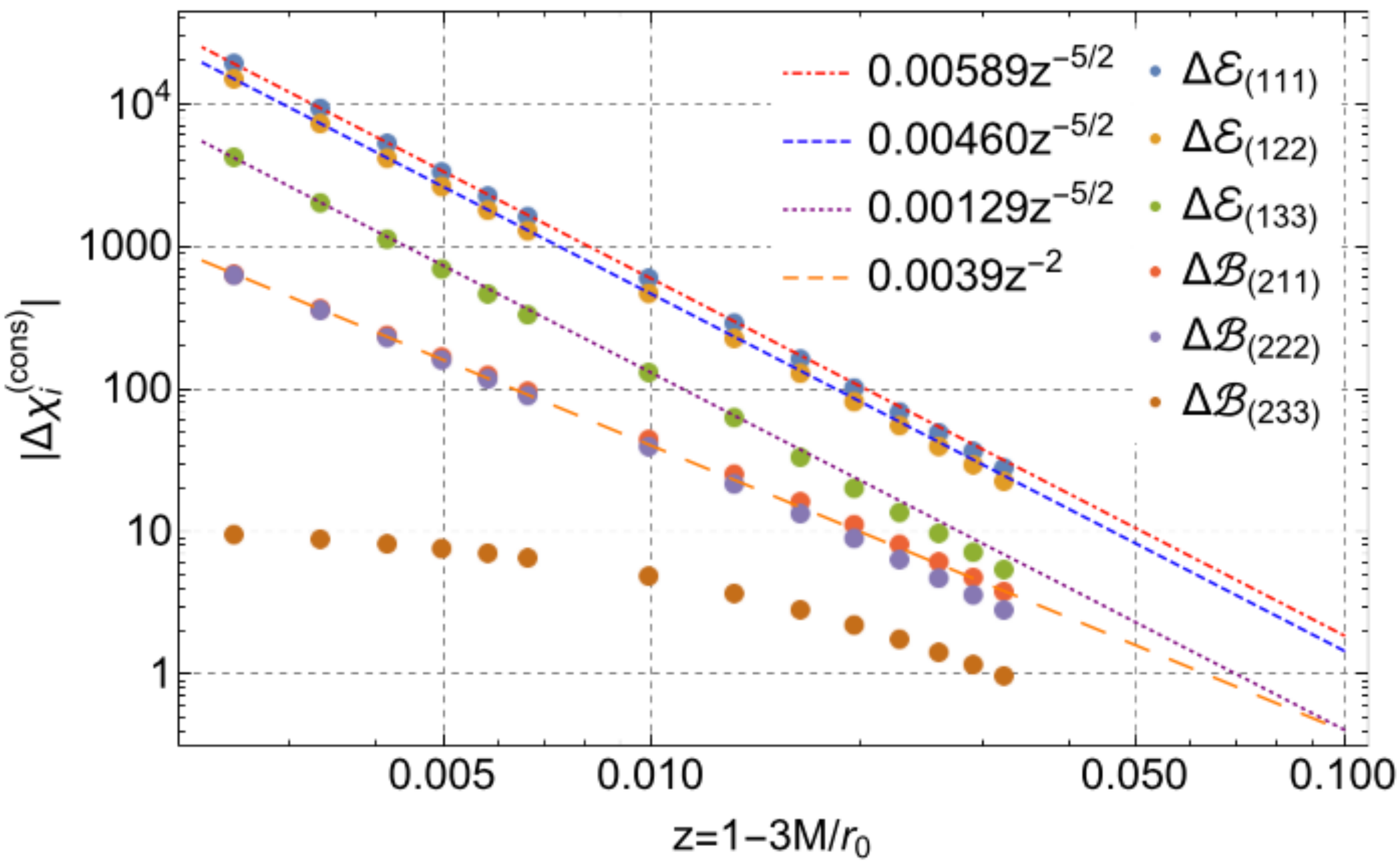
Comparing Codes with PN parameters extracted from MST method, for $\Delta\mathcal{E}_{122}$.

B-type Octupoles



Similar comparison for magnetic-type fields $\Delta\mathcal{B}_{123}$. again see accuracy floor at ~22 s.f.

Light Ring Divergence



Near the light ring, can extract divergent behaviour for 5 of 6 conservative fields. $\Delta\mathcal{B}_{233}$ can't quite be resolved in this range.

Conclusions

- Results have extracted very high order PN expressions, and combined with Light Ring data can be used for EOB calibrations.

- Found as with tidal case that most fields were conservative, but again have some dissipative

$\Rightarrow$ modify particle orbits at higher order.

- Next steps: extend gauge invariant calculations to Kerr (Chris) and generic orbits (Sarp).