Using an Analytical Regularization Scheme to Find the Self-Force on an Accelerated Scalar Charge

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Background

- Mano, Susuki and Takasugi (MST) – 1996
  - Analytic solutions to Teukolsky Equation
  - Can get expressions for general ℓ that are accurate as long as ℓ is “large”

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  - Can get expressions for general ℓ that are accurate as long as ℓ is "large"
- Hikida et al. – 2004-2005
  - Shows a method to regularize the self-force analytically to a given PN Order
  - Demonstartes the method by calculating self-force on a scalar charge moving in a circular geodesic on Schwarzschild background

This Work

- This work repeats Hikida's calculation for an accelerated scalar charge ($M/r \neq v^2$)

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Why do this for accelerated motion?

- Allows us to take limits as $M \to 0$ and $\Omega \to 0$ independently
- We can learn more about the physics involved

Outline

Notation

$$z = \frac{\omega r}{c}$$

$$\epsilon = \frac{2GM\omega}{c^3}$$

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$$\nu = \ell + O(\epsilon^2) = \text{“renormalized angular momentum''}$$

$$\phi_c^\nu = (2z)^\nu \left(1 - \frac{\epsilon}{z}\right)^\nu e^{-i(z-\epsilon)} \sum_{n=-\infty}^{\infty} (2i)^n a_n^\nu \frac{(\nu+1+i\epsilon)_n}{(2\nu+2)_{2n}} (z-\epsilon)^n$$
$$\times \ {}_1F_1(n+\nu+1+i\epsilon,\ 2n+2\nu+2,\ 2i(z-\epsilon))$$
$$= \frac{\Gamma(2\nu+2)}{\Gamma(\nu+1+i\epsilon)} R_c^\nu \ \text{(MST)}$$

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$$= \frac{\Gamma(2\nu + 2)}{\Gamma(\nu + 1 + i\epsilon)} R_c^\nu \text{ (MST)}$$

Green Functions

$$\psi^{ret} = -[4\pi]q \int d\tau \, G^{ret}(x, z(\tau))$$

$$G^{ret}(x, x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \sum_{\ell,m} g^{ret}_{\ell m\omega}(r, r') Y_{\ell m}(\theta, \phi) Y^*_{\ell m}(\theta', \phi')$$

$$g^{ret}_{\ell m\omega}(r, r') = \frac{-1}{W_{\ell m\omega}(\phi^{\nu}_{in}, \phi^{\nu}_{up})} \phi^{\nu}_{in}(r_<) \phi^{\nu}_{up}(r_>)$$

Green functions

Good News

We can get ϕ^ν_{in} and ϕ^ν_{up} analytically from MST:

$$\phi^\nu_{in} = \phi^\nu_c + \beta^\nu_c \phi^{-\nu-1}_c$$
$$\phi^\nu_{up} = \gamma^\nu_c \phi^\nu_c + \phi^{-\nu-1}_c$$

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Bad News

- ϕ^ν_c has a factor of $(2z)^\nu$, whose expansion in ϵ contains $\ln(2z) \propto \ln(\omega)$
- Makes the inverse Fourier Transform difficult for eccentric orbits

Green Functions

Solution to the Bad News

With some algebra, we can show that

$$g^{ret}_{\ell m\omega}(r,r') = \frac{-1}{W_{\ell m\omega}(\phi_c^{\nu},\phi_c^{-\nu-1})}\phi_c^{\nu}(r_<)\phi_c^{-\nu-1}(r_>) + \frac{1}{(1-\beta_c^{\nu}\gamma_c^{\nu})W_{\ell m\omega}(\phi_c^{\nu},\phi_c^{-\nu-1})}\Big[\gamma_c^{\nu}\phi_c^{\nu}(r)\phi_c^{\nu}(r') + \beta_c^{\nu}\phi_c^{-\nu-1}(r)\phi_c^{-\nu-1}(r') + \beta_c^{\nu}\gamma_c^{\nu}(\phi_c^{\nu}(r)\phi_c^{-\nu-1}(r') + \phi_c^{-\nu-1}(r)\phi_c^{\nu}(r'))\Big]$$

$$= g^{\tilde{S}}_{\ell m\omega}(r,r') + g^{\tilde{R}}_{\ell m\omega}(r,r')$$

The $\tilde{S}$ and $\tilde{R}$ Parts

The $\tilde{S}$ Part of the Field

- is singular (though not the Detweiler-Whiting singular field)
- satisfies sourced field equation
- is an even polynomial in ω

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The $\tilde{R}$ Part of the Field

- is regular
- satisfies the homogeneous field equation
- contains the non-conservative part of the force
- Each ℓ-mode except zero contributes at $(\ell - 1)^{\mathrm{th}}$ PN order

Why this decomposition?

$$\phi_c^\nu = (2\omega r)^\nu \Phi^\nu,$$

where Φ^ν is an even polynomial in ω.

$$\begin{aligned} g^{\tilde{S}}_{\ell m\omega}(r, r') &= \frac{-1}{W_{\ell m\omega}(\phi_c^\nu, \phi_c^{-\nu-1})} \phi_c^\nu(r_<)\phi_c^{-\nu-1}(r_>) \\ &= \frac{-1}{W_{\ell m\omega}(\phi_c^\nu, \phi_c^{-\nu-1})} \frac{1}{2\omega r_>} \left(\frac{r_<}{r_>}\right)^\nu \Phi^\nu(r_<)\Phi^{-\nu-1}(r_>) \end{aligned}$$

- $g^{\tilde{S}}_{\ell m\omega}$ is a polynomial in ω.
- Fourier Transform becomes easy, should be possible to regularize analytically

Regularizing the $\tilde{S}$ Part

- At this point, specialize to scalar charge, circular orbit, Schwarzschild background

The ℓ-modes of the $\tilde{S}$ part of the force $F^{\tilde{S}}_{\alpha,\ell}$ are given by

$$F^{\tilde{S}}_{\alpha,\ell} = qP^{\beta}_{\alpha}\partial_{\beta}\left[\frac{q}{u^t}\sum_{m=-\ell}^{\ell} g^{\tilde{S}}_{\ell,m,m\Omega}(r,r_0)Y_{\ell m}(\theta,\phi-\Omega t)Y^{*}_{\ell m}\left(\frac{\pi}{2},0\right)\right]\Bigg|_{x=z(t)}$$

- $F^{\tilde{S}}_{t,\ell} = F^{\tilde{S}}_{\theta,\ell} = F^{\tilde{S}}_{\phi,\ell} = 0$
- $F^{\tilde{S}}_{r}$ needs to be regularized

Regularizing $F_r^{\tilde{S}}$

$$F_{r,\ell}^{\tilde{S}(\pm)} = \lim_{r\to r_0^\pm} \frac{q^2}{u^t} \sum_{m=-\ell}^{\ell} \partial_r \left(g_{\ell,m,m\Omega}^{\tilde{S}}(r,r_0) \right) \left| Y_{\ell m}\left(\frac{\pi}{2},0\right)\right|^2$$
$$= A_r^{(\pm)}(\ell+1/2) + B_r + \tilde{D}_{\alpha,\ell}$$

- The regularization parameters are determined from the high-ℓ behavior of $F_{r,\ell}^{\tilde{S}}$
- High-ℓ behavior of ϕ_c^{ν}, $\phi_c^{-\nu-1}$ determined from MST
- To sum over m, need to know

$$\sum_{m=-\ell}^{\ell} m^{2n} \left| Y_{\ell m}\left(\frac{\pi}{2},0\right)\right|^2 = (-1)^n \frac{2\ell+1}{4\pi} \frac{d^{2n}}{d\phi^{2n}} P_\ell(\cos\phi)\Big|_{\phi=0}$$

The Regularized $\tilde{S}$ Part

$$F_r^{\tilde{S}-S} = \sum_{\ell=0}^{\infty} F_{r,\ell}^{\tilde{S}} - A_r(\ell + 1/2) - B_r$$

- To go to N^{th} PN order, we must calculate $F_{r,\ell}^{\tilde{S}}$ explicitly from MST for $\ell = 0$ through $N + 1$
- For higher ℓ-values, we can use the "high"-ℓ expressions from MST

The $\tilde{R}$ Part

$$g^{\tilde{R}}_{\ell m\omega}(r,r') = \frac{1}{(1-\beta^\nu_c\gamma^\nu_c)W_{\ell m\omega}(\phi^\nu_c,\phi^{-\nu-1}_c)}\Big[\gamma^\nu_c\phi^\nu_c(r)\phi^\nu_c(r') + \beta^\nu_c\phi^{-\nu-1}_c(r)\phi^{-\nu-1}_c(r') + \beta^\nu_c\gamma^\nu_c(\phi^\nu_c(r)\phi^{-\nu-1}_c(r') + \phi^{-\nu-1}_c(r)\phi^\nu_c(r'))\Big]$$

$$F^{\tilde{R}}_{\alpha,\ell} = P^\beta_\alpha\partial_\beta\left[\frac{q^2}{u^t}\sum_{m=-\ell}^{\ell} g^{\tilde{R}}_{\ell m\omega}(r,r_0)Y_{\ell,m}(\theta,\phi-\Omega t)Y^*_{\ell,m}\left(\frac{\pi}{2},0\right)\right]\Bigg|_{x=z(t)}$$

$$F^{\tilde{R}}_{t,0} = F^{\tilde{R}}_{\phi,0} = 0$$

$$F^{\tilde{R}}_{t,\ell} = -\Omega F^{\tilde{R}}_{\phi,\ell} \sim \mathrm{PN}(\ell+1/2), \quad \ell > 0$$

$$F^{\tilde{R}}_{r,0} \sim \mathrm{PN}(0)$$

The $\tilde{R}$ Part

- Each ℓ-mode contributes at a higher PN order
- To get the self-force to finite PN order, we only need to sum the $\tilde{R}$ modes to finite ℓ
- Unfortunately, $g^{\tilde{R}}_{\ell m \omega}$'s ω-dependence is complicated

To 6^{th} PN order:

$$F_t^{\tilde{R}} = \sum_{\ell=0}^{5} F_{t,\ell}^{\tilde{R}}$$

$$F_r^{\tilde{R}} = \sum_{\ell=0}^{7} F_{r,\ell}^{\tilde{R}}$$

$F_r^{\tilde{R}}$

$$F_r^{\tilde{R}} = \frac{q^2}{[4\pi]r^2}\left\{\left[\frac{2}{7} + \frac{5}{19}\frac{r}{M}v^2\right] + \left[-\frac{4}{7}\frac{M}{r} - \frac{89}{133}v^2 + \frac{1417}{3002}\frac{r}{M}v^4\right] + \cdots\right\},$$

$$v = r\Omega$$

F_r^R

$$
\begin{aligned}
F_r^R = \frac{q^2}{[4\pi]r^2} \Bigg\{ & \left[\frac{7\pi^2}{64} \left(\frac{M}{r}\right)^2 v^2 + \left(-\frac{2}{9} - \frac{4}{3}(\gamma + \ln(2|v|)) \right) \frac{M}{r} v^4 \right] + \Bigg[\left(2 + \frac{7\pi^2}{64} \right) \left(\frac{M}{r}\right)^3 v^2 \\
& + \left(\frac{7}{9} - \frac{83\pi^2}{1024} + \frac{8}{3}(\gamma + \ln(2|v|)) \right) \left(\frac{M}{r}\right)^2 v^4 + \left(\frac{479}{45} - \frac{128}{15}\ln(2) - \frac{22}{3}(\gamma + \ln(2|v|)) \right) \frac{M}{r} v^6 \Bigg] \\
& + \left[-\frac{38\pi}{45} \left(\frac{M}{r}\right)^2 |v|^5 \right] + \Bigg[\left(4 + \frac{11\pi^2}{256} \right) \left(\frac{M}{r}\right)^4 v^2 + \left(-\frac{19}{9} - \frac{141\pi^2}{1024} - \frac{2}{3}(\gamma + \ln(2|v|)) \right) \left(\frac{M}{r}\right)^3 v^4 \\
& \left(-\frac{517}{15} + \frac{1529\pi^2}{2048} + \frac{512}{15}\ln(2) + 36(\gamma + \ln(2|v|)) \right) \left(\frac{M}{r}\right)^2 v^6 + \Bigg(\frac{54647}{1260} - \frac{1216}{105}\ln(2) \\
& - \frac{2187}{70}\ln(3) - \frac{119}{6}(\gamma + \ln(2|v|)) \Bigg) \frac{M}{r} v^8 \Bigg] + \left[\frac{76\pi}{45} \left(\frac{M}{r}\right)^3 |v|^5 - \frac{1783\pi}{315} \left(\frac{M}{r}\right)^2 |v|^7 \right] + \\
& + \Bigg[\left(\frac{341}{45} - \frac{23\pi^2}{256} + \frac{16}{3}\ln\left(\frac{2M}{r}\right) \right) \left(\frac{M}{r}\right)^5 v^2 + \left(-\frac{41}{18} + \frac{5221\pi^2}{2048} - \frac{76585\pi^4}{524288} \right) \left(\frac{M}{r}\right)^4 v^4 \\
& + \Bigg(\frac{6239}{500} - \frac{138469\pi^2}{30720} - \frac{1984}{45}\ln(2) - \frac{49207}{675}(\gamma + \ln(2|v|)) + \frac{152}{45}(\gamma + \ln(2|v|))^2 \\
& + \frac{8}{3}\psi^{(2)}(2) \Bigg) \left(\frac{M}{r}\right)^3 v^6 + \Bigg(-\frac{803219}{2520} + \frac{423951\pi^2}{131072} + \frac{1408}{315}\ln(2) + \frac{6561}{35}\ln(3) \\
& + \frac{5131}{27}(\gamma + \ln(2|v|)) \Bigg) \left(\frac{M}{r}\right)^2 v^8 + \Bigg(\frac{7319647}{68040} - \frac{35920}{189}\ln(2) + \frac{8019}{140}\ln(3) \\
& \frac{161}{\;}(\gamma + \ln(2|v|)) \Bigg) \frac{M}{r} v^{10} \Bigg] + \dots \Bigg\}
\end{aligned}
$$

F_r^R

$$F_r^R = \frac{q^2}{[4\pi]r^2}\Bigg\{\Big[\frac{7\pi^2}{64}\boxed{\left(\frac{M}{r}\right)^2 v^2} + \left(-\frac{2}{9} - \frac{4}{3}(\gamma + \ln(2|v|))\right)\boxed{\frac{M}{r}v^4}\Big] + \Bigg[\left(2 + \frac{7\pi^2}{64}\right)\boxed{\left(\frac{M}{r}\right)^3 v^2}$$

$$+ \left(\frac{7}{9} - \frac{83\pi^2}{1024} + \frac{8}{3}(\gamma + \ln(2|v|))\right)\boxed{\left(\frac{M}{r}\right)^2 v^4} + \left(\frac{479}{45} - \frac{128}{15}\ln(2) - \frac{22}{3}(\gamma + \ln(2|v|))\right)\boxed{\frac{M}{r}v^6}\Bigg]$$

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$$\left.-\frac{2187}{70}\ln(3) - \frac{119}{6}(\gamma + \ln(2|v|))\right)\boxed{\frac{M}{r}v^8}\Bigg] + \left[\frac{76\pi}{45}\boxed{\left(\frac{M}{r}\right)^3 |v|^5} - \frac{1783\pi}{315}\boxed{\left(\frac{M}{r}\right)^2 |v|^7}\right] + \cdots\Bigg\}$$

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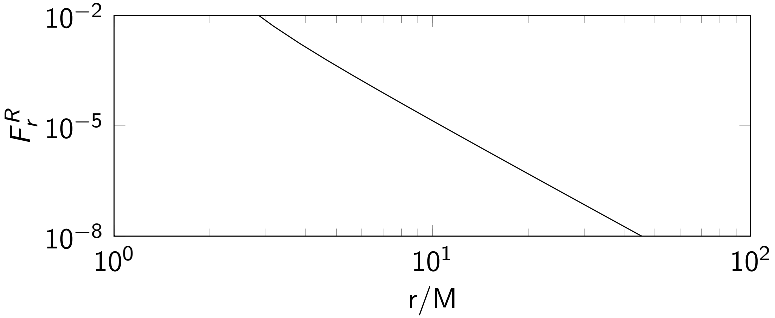
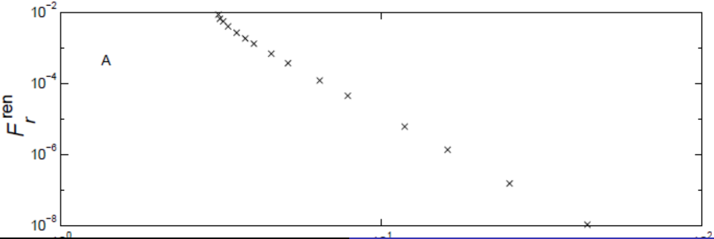
\end{aligned}
$$

Comparison with Burko





Comparison with Warburton and Barack (2010)

Converging to Warburton ($r = 50M$)

PN Order	F_r^R/q^2	Fractional Difference
3	$5.66868 \cdot 10^{-9}$	-0.10683
4	$6.37183 \cdot 10^{-9}$	$3.95878 \cdot 10^{-3}$
4.5	$6.34781 \cdot 10^{-9}$	$1.75506 \cdot 10^{-4}$
5	$6.35288 \cdot 10^{-9}$	$9.7452 \cdot 10^{-4}$
5.5	$6.35063 \cdot 10^{-9}$	$6.18665 \cdot 10^{-4}$
6	$6.34664 \cdot 10^{-9}$	$-8.9682 \cdot 10^{-6}$
Warburton	$6.3467 \cdot 10^{-9}$	–

Comparison with Detweiler, Messaritaki, and Whiting

Converging to DMW ($r = 10M$)

PN Order	F_r^R/q^2	Fractional Difference
3	$6.98505 \cdot 10^{-6}$	-0.4932
4	$1.42163 \cdot 10^{-5}$	0.0313
4.5	$1.33773 \cdot 10^{-5}$	-0.0295
5	$1.50205 \cdot 10^{-5}$	0.0897
5.5	$1.46263 \cdot 10^{-5}$	0.0611
6	$1.37594 \cdot 10^{-5}$	-0.0018
DMW	$1.378448171 \cdot 10^{-5}$	–

Summary

- Hikida et al. found new analytical regularization scheme for general orbits
- We used Hikida's scheme to calculate the self-force on a scalar charge following an accelerated circular orbit on a Schwarzchild background
- To get the force to 6^{th} PN order, explicit calculations for $\ell = 0$ through 7 were needed

Future Work

- Gravitational Case
- Eccentric Orbits
- Kerr