Second-order self-force: problems in the infrared

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(and special thanks to Takahiro Tanaka)

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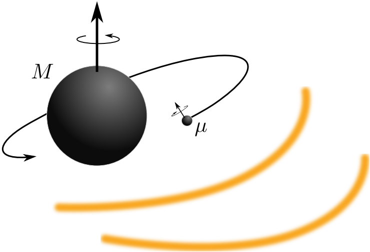
Recall our motivation

Several reasons for going to second order

- necessary for accurate EMRI evolution over inspiral time $\sim M^2/\mu$
- should yield highly accurate model for IMRIs
- will fix terms quadratic in mass in post-Newtonian and Effective One Body theory



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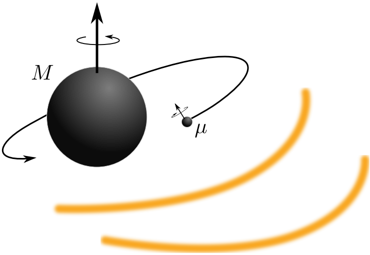

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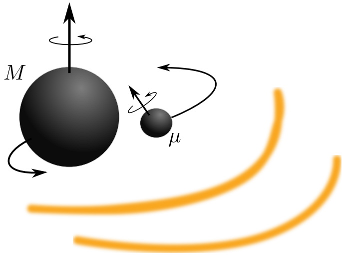

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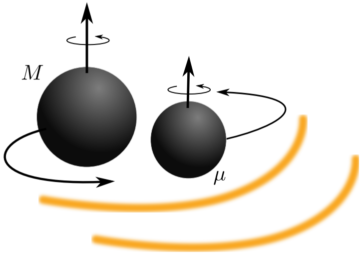
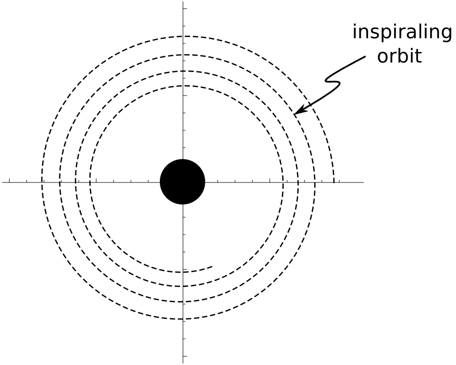


The simplest problem: quasicircular orbits in Schwarzschild



- assume slow evolution.
  i.e., $r_p = r_p(\epsilon t)$
  $(\epsilon = \mu/M)$

- at first order, we usually approximate the orbit as a geodesic
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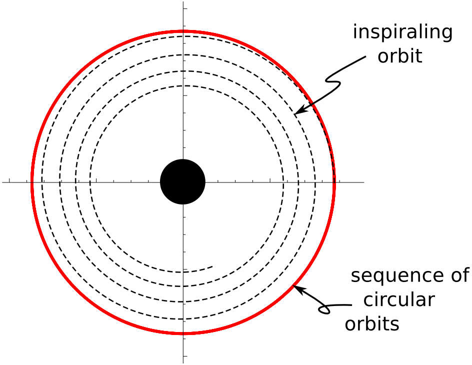

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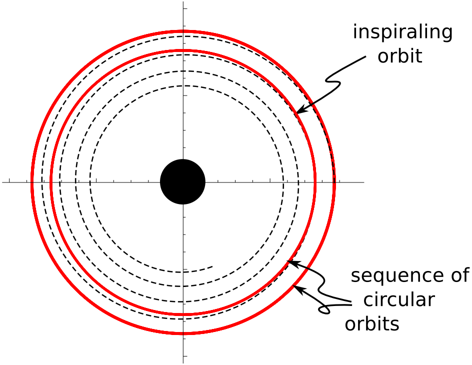

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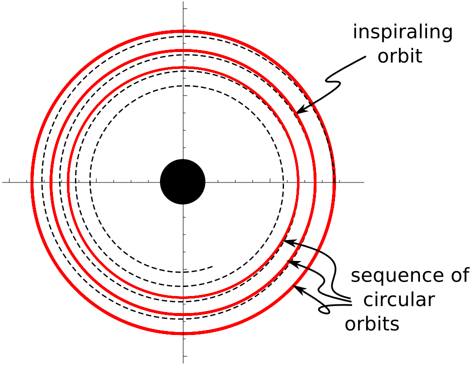

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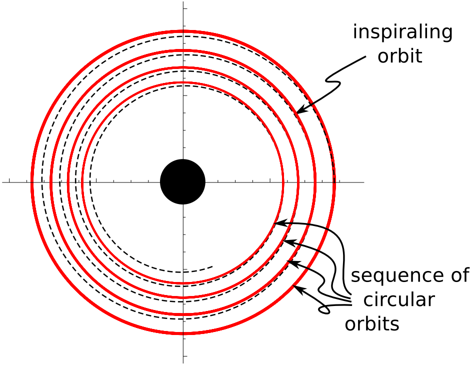

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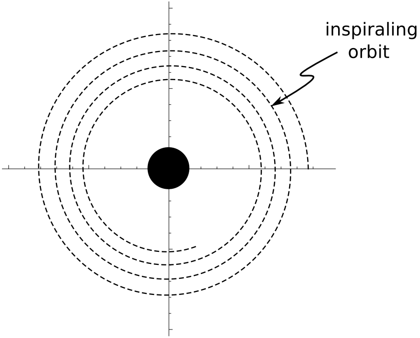
Things we can calculate in this scenario

- the actual slow evolution and resultant waveform
- Detweiler's redshift: $\tilde{u}^t$ in conservative part of effective metric

$$\tilde{u}^t = \frac{1}{\sqrt{1-\frac{3M}{r_0}}}\left\{1+\frac{1}{2}\frac{\mu}{M}h^{R1}_{u_0u_0}+\left(\frac{\mu}{M}\right)^2\left[\frac{1}{2}h^{R2}_{u_0u_0}+\frac{3}{8}\left(h^{R1}_{u_0u_0}\right)^2-\frac{r_0^2(r_0-3M)}{6M}(F_{1r})^2\right]\right\}$$



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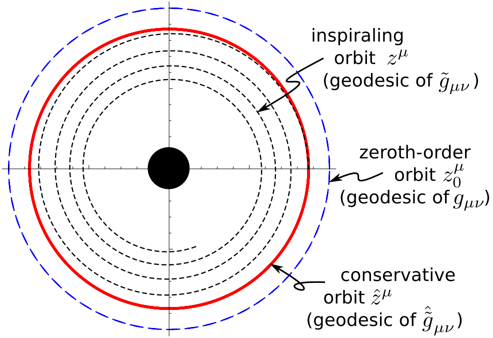

Outline

1. Treatments of quasicircular orbits

2. The infrared problem

3. Resolution of (half of) the infrared problem

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Governing equations at second order

- Lorenz-gauge field equations

$$
\begin{aligned}
E_{\mu\nu}[h^1] &= -16\pi\,\bar{T}^1_{\mu\nu} \\
E_{\mu\nu}[h^2] &= 2\delta^2 R_{\mu\nu}[h^1,h^1] && \text{inside } \Gamma \\
E_{\mu\nu}[h^{\mathcal{R}2}] &= -16\pi\,\bar{T}^2_{\mu\nu} + 2\delta^2 R_{\mu\nu}[h^1,h^1] - E_{\mu\nu}[h^{\mathcal{P}2}] && \text{outside } \Gamma
\end{aligned}
$$

where $E_{\mu\nu}[h] = \Box h_{\mu\nu} + 2R_\mu{}^\alpha{}_\nu{}^\beta h_{\alpha\beta}$, and Γ is a tube around z^μ

- Coupled to equation of motion

$$
\frac{D^2 z^\mu}{d\tau^2} = -\frac{1}{2}(g^{\mu\nu} + u^\mu u^\nu)(g_\nu{}^\delta - h^{R\delta}_\nu)(2h^R_{\delta\beta;\gamma} - h^R_{\beta\gamma;\delta})u^\beta u^\gamma
$$

We want to solve this system in the frequency domain

- the $\ell = 0, 1$ modes are problematic in the time domain
- frequency domain (combined with a two-timescale expansion) is probably more suitable for long-term evolutions

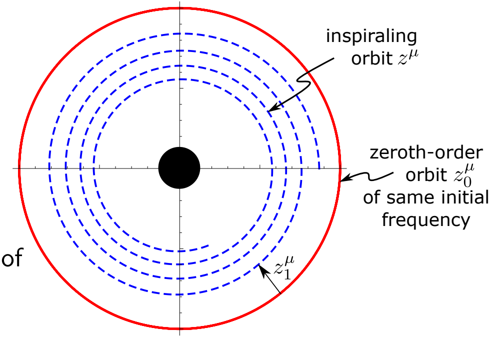
Treatment 1: expansion around a circular geodesic

- Expand around a zeroth-order orbit:

$$z^\mu(t,\epsilon) = z_0^\mu(t) + \epsilon z_1^\mu(t) + O\left(\epsilon^2\right)$$

where z_0^μ is a circular geodesic

- write $T^1_{\mu\nu}[z] = T^1_{\mu\nu}[z_0] + \epsilon\delta T^1_{\mu\nu}$
- add $\delta T^1_{\mu\nu}$ to $T^2_{\mu\nu}$
- $h^1_{\mu\nu}$ has simple time dependence $e^{-im\Omega_0 t}$
  $\Rightarrow$ *so does* $\delta^2 R_{\mu\nu}[h^1, h^1]$
- if we neglect dissipative part of z_1^μ, everything is periodic

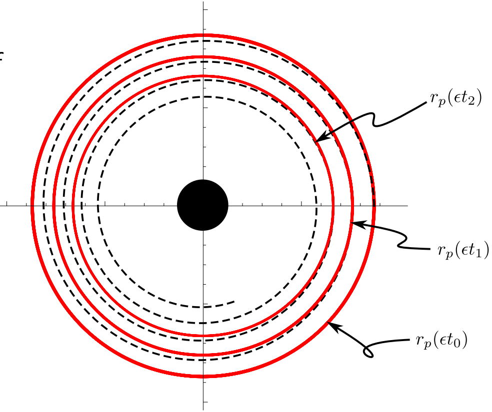


Treatment 2: Multiscale expansion

- Treat orbital radius as function of 'slow time' ϵt:

$$r_p(t,\epsilon) = r_p^\mu(\epsilon t,\epsilon)$$

- in $E_{\mu\nu}[h^1]$, treat derivatives of $r_p(\epsilon t)$ as small
- move those terms into second-order source
- at fixed slow time, all terms in field equations are periodic
- evolve from slow time to slow time using equation of motion for $r_p(\epsilon t)$



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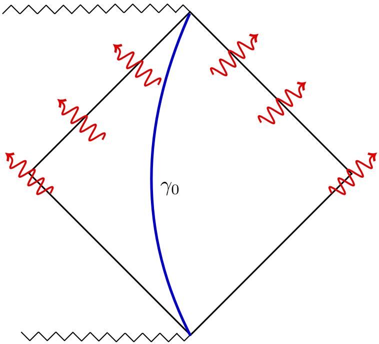
Typical calculation at first order



- we approximate the source orbit as a bound geodesic
- it radiates forever
- the system has infinite ADM mass
- at any given advanced time, the BH has already absorbed an infinite amount of energy and angular momentum
- the approximation breaks down after *dephasing time* $\sim 1/\sqrt{\epsilon}$, when $z^\mu - z_0^\mu \sim M$

  $\Rightarrow$ also breaks down at *distances* $\sim 1/\sqrt{\epsilon}$

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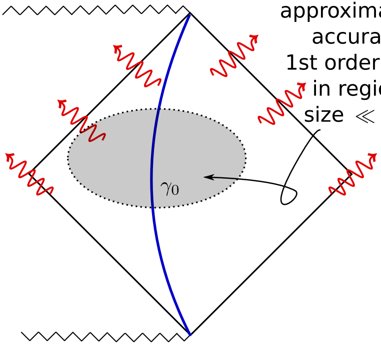

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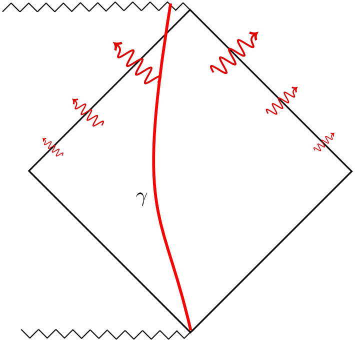
The infrared problem at second order



- we no longer neglect evolution of the orbit, and in the full retarded solution, the radiation goes to zero in the infinite past
- but in some "near zone" we still want to use expansion of worldline
- or a two-timescale expansion
- in this near zone, we solve periodic equations with periodic solutions (e.g., at fixed slow time)
  $\Rightarrow$ what boundary conditions do we impose?

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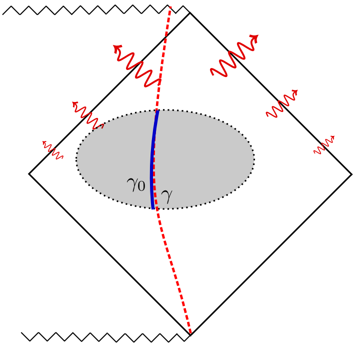

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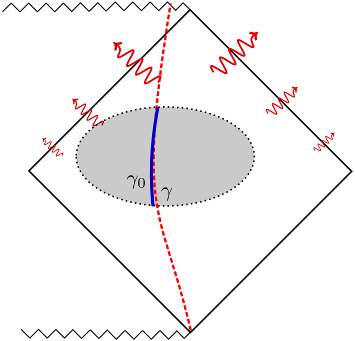

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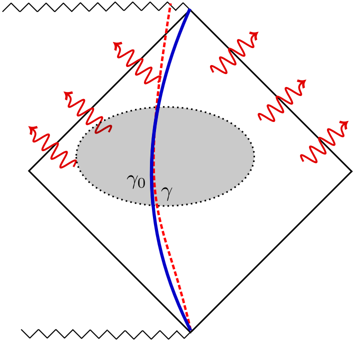

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Its mathematical manifestation in our calculations

- we decompose into harmonics: $h^1_{\mu\nu} = \sum_{ilm} h_{ilm}(r) e^{-im\Omega_0 t} Y^{ilm}_{\mu\nu}$
  $\Rightarrow$ the source also gets decomposed:
  $\delta^2 R_{\mu\nu} = \sum_{ilm} \delta^2 R_{ilm}(r) e^{-im\Omega_0 t} Y^{ilm}_{\mu\nu}$, where

$$\delta^2 R_{i\ell m} = \sum_{\substack{i'\ell' m' \\ i''\ell'' m''}} \mathcal{D}^{i\ell m}_{i'\ell' m' i''\ell'' m''} h^1_{i'\ell' m'} h^1_{i''\ell'' m''}$$

- at large r, $h^1_{i'\ell' m'} \sim \frac{e^{im'\Omega r}}{r}$
  $\Rightarrow$ from terms like $\frac{e^{im'\Omega r}}{r}\frac{e^{-im'\Omega r}}{r}$, we get $\delta^2 R_{i\ell 0} \sim \frac{1}{r^2}$ for all $m = 0$ modes of $\delta^2 R_{\mu\nu}$
- focus on $\ell = 0$. The retarded solution behaves like $h^2_{i00} \sim \int_r^\infty \frac{dr'}{r'}$
  $\Rightarrow$ *the integral diverges logarithmically*

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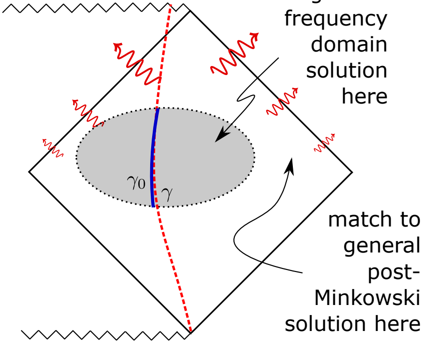
Matching to a post-Minkowski solution



Matched asymptotic expansions

- use the two-timescale expansion or expansion of worldline in the near zone
- use the Blanchet-Damour (or Will-Wiseman) general post-Minkowski solution in the far zone
- match the two to get the correct large-r behavior of the near-zone solution

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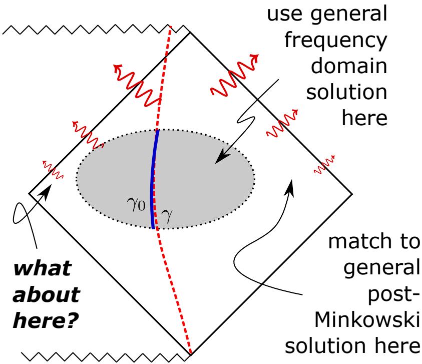

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Toy problem

- second-order gravity is hard
  $\Rightarrow$ use a simple scalar toy model as a testbed
- scalar point charge on quasicircular orbit in flat space

$$\Box\varphi_1 = -4\pi q \int \frac{\delta^4(x-z)}{r^2 \sin\theta}\, d\tau$$
$$\Box\varphi_2 = t^{\mu\nu}\nabla_\mu\varphi_1\nabla_\nu\varphi_1 =: S_2$$

  with $t^{\mu\nu} = \mathrm{diag}(1,1,1,1)$
- in near zone, expand worldline z^μ around circular orbit
  $\Rightarrow$ exactly the same infrared problems as in gravity

“Post-Minkowski” solution

From PM theory, the retarded field outside the particle's orbit always looks like

- First order

$$\phi_1 = \sum_{\ell\geq 0} \frac{(-1)^\ell}{\ell!}\partial_L \frac{F_L(u)}{r}$$

  where $L = i_1 \cdots i_\ell$ and $u = t - r$

- Second-order

$$\phi_2 = \sum_{\ell\geq 0} \frac{(-1)^\ell}{\ell!}\partial_L \frac{G_L(u)}{r} + \phi_2^{\rm part}$$

  where $\phi_2^{\rm part}$ is a certain integral over quadratic combinations of $F'_L s$

All the problems lie in hereditary terms

- Source looks like $S_2 = \sum_{\ell m} \frac{S^{(2)}_{\ell m}(u)\, Y_{\ell m}}{r^2} + O(1/r^3)$
- $O(1/r^3)$ source generates terms in φ_2 that depend only on current value of u, easily matched to terms in near-zone solution
- $\frac{S^{(2)}_{\ell m} Y_{\ell m}}{r^2}$ produces "hereditary terms" in φ_2. In particular,

$$\varphi^{\text{hered}}_{00} = \frac{1}{2r}\int_r^\infty ds S^{(2)}_{00}(t-s)[\ln(s-r) - \ln(s+r)]$$

$\Rightarrow$ diverges if we use a source from a precisely circular orbit ($S^{(2)}_{00} = constant$), but it converges for an inspiraling source

Matching I

- to match the near-zone solution to the well-behaved far-zone solution, we reexpand the far-zone solution in the near zone.
- rewrite the hereditary monopole term as

$$\varphi_{00}^{\rm hered} = (-1+\ln 2r)S_{00}^{(2)} + \frac{1}{2r}\int_0^\infty ds \ln\left(\frac{s}{2}\right)\left[S_{00}^{(2)}(t-s-r) - S_{00}^{(2)}(t-s+r)\right]$$

  expand for small r (corresponding to $r \ll 1/\epsilon$)
  $\Rightarrow$ in the near zone, the integral is higher order in ϵ
  $\Rightarrow \varphi_{00}^{\rm hered} = (-1+\ln 2r)S_{00}^{(2)}$ is the behavior of the actual retarded solution at large r in the near zone
- here, outside the integral, $S_{00}^{(2)}$ is constructed from the φ_1 of a circular orbit

Matching II

We can ensure our near-zone solution has the correct asymptotic behavior by

- introducing a puncture $\varphi^{\mathcal{P}}_{200} = (-1 + \ln 2r) S^{(2)}_{00}$
- solving $[\Box \varphi^{\mathcal{R}}_2]_{00} = S_{200} - [\Box \varphi^{\mathcal{P}}]_{00}$ in the near zone
- writing $\varphi_{200} = \varphi^{\mathcal{R}}_{200} + \varphi^{\mathcal{P}}_{200}$

Remaining work

Two things remain to be done

- apply the matching to PM expansion in gravity case
- figure out correct behavior at the horizon (work with Takahiro Tanaka)