九州大学
基幹教育院
Faculty of Arts and Science
Kyushu University

Calculation of radiation reaction effect on orbital parameters in Kerr spacetime

Kyushu University
Norichika Sago

with Ryuichi Fujita (Instituto Superior Técnico, Portugal)

based on arXiv:1505.01600 [gr-qc]

18th Capra meeting on radiation reaction
July 1, 2015 at YITP, Kyoto

九州大学
KYUSHU UNIVERSITY

Introduction

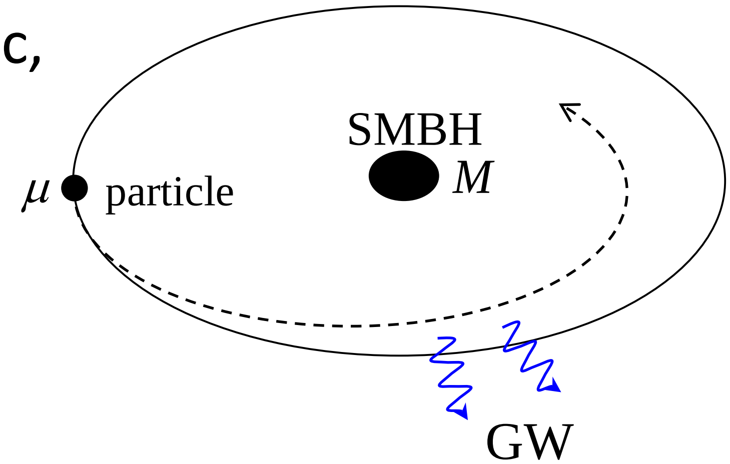
Consider the motion of a particle orbiting the Kerr geometry.

Test particle case $[\approx O((\mu/M)^0)]$

The particle moves along a geodesic,

characterized by E, L, C.

E : energy
L : azimuthal angular momentum
C : Carter constant



At 1st order $[\approx O((\mu/M)^1)]$

The particle no longer moves along the geodesic because of the back-reaction.

Secular orbital evolution ⇨ Evolution of E, L, C

Adiabatic evolution

For bound orbits, we can express the evolution of the orbital parameters in form of Fourier series:

$$\begin{aligned}\frac{dI}{dt} &= 0 & \cdots \text{ test particle}\\ &+ \left[\overbrace{\left\langle\frac{dI}{dt}\right\rangle}^{\text{secular part}} + \overbrace{\sum_{kn}\left(\frac{dI}{dt}\right)_{kn} e^{i(k\Omega_\theta + n\Omega_r)\lambda}}^{\text{oscillatory part}}\right] & \cdots \text{ 1st order}\\ &+ O\left(\left(\frac{\mu}{M}\right)^2\right) & \cdots \text{ higher order}\end{aligned}$$

<u>Adiabatic approximation</u>

Orbital phase : $\Phi = \left(\frac{\mu}{M}\right)^{-1}\left[\underbrace{\Phi^{(0)}}_{\text{secular part}} + \left(\frac{\mu}{M}\right)\underbrace{\Phi^{(1)}}_{\text{oscillatory parts/averaged 2nd}} + \cdots\right]$

We focus on only the secular evolution of the orbital parameters.

Expressions with the asymptotic amplitudes of GW

We have the simple expressions for the secular parts with the amplitude of each partial wave. [NS et al. (2006)]

Infinity part　　Horizon part

$$\left\langle \frac{dE}{dt} \right\rangle_t = -\mu^2 \sum_{\tilde{\Lambda}} \frac{1}{4\pi\omega_{mn_rn_\theta}^2} \left(\boxed{\left| \tilde{Z}_{\tilde{\Lambda}}^{\infty} \right|^2} + \boxed{\alpha_{\ell m}(\omega_{mn_rn_\theta}) \left| \tilde{Z}_{\tilde{\Lambda}}^{\mathrm{H}} \right|^2} \right)$$

$$\left\langle \frac{dL}{dt} \right\rangle_t = -\mu^2 \sum_{\tilde{\Lambda}} \frac{m}{4\pi\omega_{mn_rn_\theta}^3} \left(\left| \tilde{Z}_{\tilde{\Lambda}}^{\infty} \right|^2 + \alpha_{\ell m}(\omega_{mn_rn_\theta}) \left| \tilde{Z}_{\tilde{\Lambda}}^{\mathrm{H}} \right|^2 \right)$$

$$\begin{aligned} \left\langle \frac{dC}{dt} \right\rangle_t = & -2 \left\langle a^2 E \cos^2\theta \right\rangle_\lambda \left\langle \frac{dE}{dt} \right\rangle_t + 2 \left\langle L \cot^2\theta \right\rangle_\lambda \left\langle \frac{dL}{dt} \right\rangle_t \\ & -\mu^3 \sum_{\tilde{\Lambda}} \frac{n_\theta \Omega_\theta}{2\pi\omega_{mn_rn_\theta}^3} \left(\left| \tilde{Z}_{\tilde{\Lambda}}^{\infty} \right|^2 + \alpha_{\ell m}(\omega_{mn_rn_\theta}) \left| \tilde{Z}_{\tilde{\Lambda}}^{\mathrm{H}} \right|^2 \right) \end{aligned}$$

$\tilde{Z}_{\tilde{\Lambda}}^{\infty/\mathrm{H}}$: amplitude of the partial wave　　$\tilde{\Lambda} = \{\ell, m, n_r, n_\theta\}$

Analytic calculations in PN-eccentricity expansion

Analytic calculations with these expressions have already been done based on MST method. (e.g. Ganz et al. (2007)).

We update the results in Ganz et al. as follows:

	Sago et al. (2006)	Ganz et al. (2007)	This work
PN order	2.5	2.5	4
eccentricity	$O(e^2)$	$O(e^2)$	$O(e^6)$
inclination	$O(\theta_{\rm inc})$	No assumption	No assumption
BH absorption	neglected	neglected	included

Orbital parameters (for bound orbit)

We show the results in terms of the following parameters:

Semi-latus rectum p and eccentricity e

$$r_a = \frac{p}{1+e}, \quad r_p = \frac{p}{1-e}$$

r_a : apastron radius
r_p : periastron radius

Inclination angle ι

$$Y \equiv \cos\iota \equiv \frac{L}{\sqrt{L^2 + C}}$$

($C \to L_x^2 + L_y^2$ in Schwarzshild limit)

$$v = \sqrt{\frac{M}{p}}$$

: Post-Newtonian parameter

q : normalized spin parameter ($=a/M$)
γ : Euler's constant (=0.57721...)

Results (Infinity part of $\langle dE/dt \rangle$)

$$
\begin{aligned}
\left\langle \frac{dE}{dt} \right\rangle_t^{\infty} = & -\frac{32}{5}\left(\frac{\mu}{M}\right)^2 v^{10}(1-e^2)^{3/2} \\
& \times \Bigg[1 + \frac{73}{24}e^2 + \frac{37}{96}e^4 \\
& \quad + \left\{ -\frac{1247}{336} - \frac{9181}{672}e^2 + \frac{809}{128}e^4 + \frac{8609}{5376}e^6 \right\} v^2 \\
& \quad + \left\{ 4\pi - \frac{73}{12}Yq + \left(\frac{1375}{48}\pi - \frac{823}{24}Yq\right)e^2 \right. \\
& \qquad \left. + \left(\frac{3935}{192}\pi - \frac{949}{32}Yq\right)e^4 + \left(\frac{10007}{9216}\pi - \frac{491}{192}Yq\right)e^6 \right\} v^3 \\
& \quad + \left\{ -\frac{44711}{9072} + \frac{527}{96}Y^2q^2 - \frac{329}{96}q^2 + \left(-\frac{172157}{2592} - \frac{4379}{192}q^2 + \frac{6533}{192}Y^2q^2\right)e^2 \right. \\
& \qquad \left. + \left(-\frac{2764345}{24192} - \frac{3823}{256}q^2 + \frac{6753}{256}Y^2q^2\right)e^4 + \left(\frac{3743}{2304} - \frac{363}{512}q^2 + \frac{2855}{1536}Y^2q^2\right)e^6 \right\} v^4 \\
& \quad + \left\{ -\frac{8191}{672}\pi + \frac{3749}{336}Yq + \left(-\frac{44531}{336}\pi + \frac{1759}{56}Yq\right)e^2 \right. \\
& \qquad \left. + \left(-\frac{4311389}{43008}\pi - \frac{111203}{1344}Yq\right)e^4 + \left(\frac{15670391}{387072}\pi - \frac{49685}{448}Yq\right)e^6 \right\} v^5 + \cdots \Bigg].
\end{aligned}
$$

Results (Infinity part of $\langle dC/dt \rangle$)

$$
\begin{aligned}
\left\langle \frac{dC}{dt} \right\rangle_t^{\infty} = & -\frac{64}{5}\mu^3 v^6 (1-e^2)^{3/2}(1-Y^2) \\
& \times \Bigg[1 + \frac{7}{8} e^2 \\
& \quad + \left\{ -\frac{743}{336} + \frac{23}{42} e^2 + \frac{11927}{2688} e^4 \right\} v^2 \\
& \quad + \left\{ 4\pi - \frac{85}{8} Y q + \left(\frac{97}{8}\pi - \frac{211}{8} Y q \right) e^2 + \left(\frac{49}{32}\pi - \frac{517}{64} Y q \right) e^4 - \frac{49}{4608}\pi e^6 \right\} v^3 \\
& \quad + \left\{ -\frac{129193}{18144} - \frac{329}{96} q^2 + \frac{53}{8} Y^2 q^2 + \left(-\frac{84035}{1728} - \frac{929}{96} q^2 + \frac{163}{8} Y^2 q^2 \right) e^2 \right. \\
& \qquad \left. + \left(-\frac{1030273}{48384} - \frac{1051}{768} q^2 + \frac{387}{64} Y^2 q^2 \right) e^4 + \frac{100103}{8064} e^6 \right\} v^4 \\
& \quad + \left\{ -\frac{4159}{672}\pi + \frac{2553}{224} Y q + \left(-\frac{21229}{1344}\pi - \frac{553}{192} Y q \right) e^2 \right. \\
& \qquad \left. + \left(\frac{2017013}{43008}\pi - \frac{475541}{5376} Y q \right) e^4 + \left(\frac{6039325}{774144}\pi - \frac{153511}{3584} Y q \right) e^6 \right\} v^5 + \cdots \Bigg].
\end{aligned}
$$

Results (Horizon part of $\langle dE/dt \rangle$)

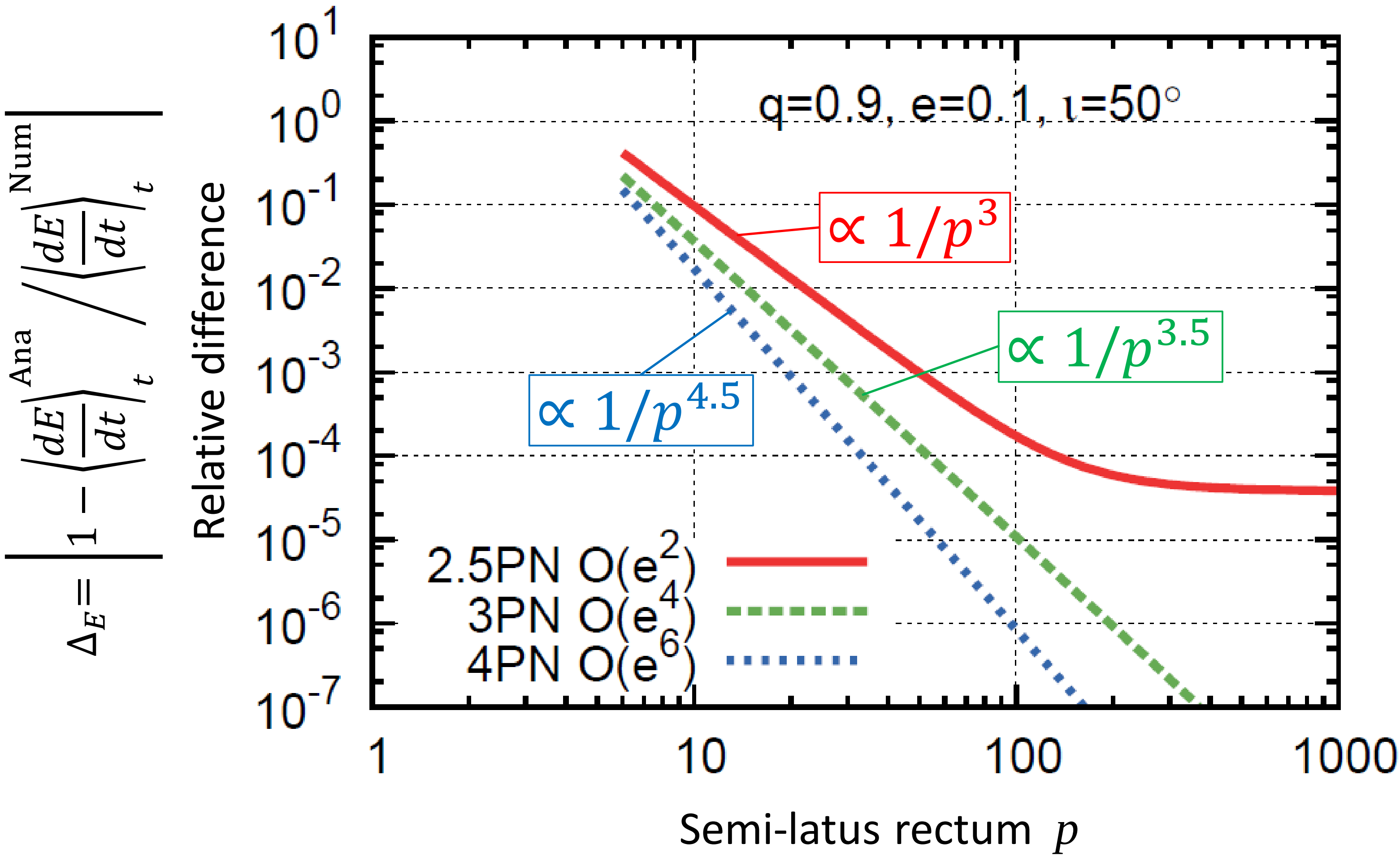
$$
\begin{aligned}
\left\langle \frac{dE}{dt} \right\rangle_t^H = & -\frac{32}{5}\left(\frac{\mu}{M}\right)^2 v^{10}(1-e^2)^{3/2} \\
& \times \Bigg[\Bigg\{-\frac{9}{32}q^3Y - \frac{15}{32}q^3Y^3 - \frac{1}{4}qY + \left(-\frac{135}{64}q^3Y - \frac{225}{64}q^3Y^3 - \frac{15}{8}qY\right)e^2 \\
& \quad + \left(-\frac{405}{256}q^3Y - \frac{675}{256}q^3Y^3 - \frac{45}{32}qY\right)e^4 + \left(-\frac{45}{512}q^3Y - \frac{75}{512}q^3Y^3 - \frac{5}{64}qY\right)e^6\Bigg\}v^5 \\
& + \Bigg\{-qY - \frac{81}{32}q^3Y + \frac{15}{32}q^3Y^3 + \left(\frac{195}{32}q^3Y^3 - \frac{1143}{32}q^3Y - \frac{57}{4}qY\right)e^2 \\
& \quad + \left(\frac{225}{32}q^3Y^3 - \frac{4455}{64}q^3Y - \frac{465}{16}qY\right)e^4 + \left(-\frac{75}{256}q^3Y^3 - \frac{6345}{256}q^3Y - \frac{355}{32}qY\right)e^6\Bigg\}v^7 + \cdots\Bigg]
\end{aligned}
$$

What we see from the PN formulae

- $\langle dC/dt \rangle = 0$ for Y=1 (equatorial) case.
  --> Equatorial orbits stay the equatorial plane
- $\langle d(L^2 + C)/dt \rangle$ is independent of Y for q=0 (Schwarzschild) case.
  --> $L^2 + C$ corresponds to the total angular momentum.
  (C corresponds to $L_x^2 + L_y^2$ in Schwarzschild case)
- The leading and next leading terms of $\langle dE/dt \rangle^{\mathrm{H}}$ can be positive for q>0 case.
  --> A superradiance can be possible.
- The superradiance terms are proportional to $(q \cdot \cos\iota)$ and disappear for Y=0 (polar) case.
  --> The terms come from the spin-orbit coupling.

Consistency check (Difference from numerical data)

We compare the analytic formulae with the numerical results from the Teukolsky code by Fujita-Tagoshi (2009).



Convergence check (with respect to PN expansion)

$$\left\langle \frac{dE}{dt} \right\rangle_t = \underbrace{-\frac{32}{5}\left(\frac{\mu}{M}\right)^2 v^{10}(1-e^2)^{3/2}}_{\text{Quadrupole part}} \underbrace{[\Delta_0 + \Delta_1 + \Delta_2 + \Delta_3 + \cdots]}_{\text{Each PN term}}$$

Leading : $\Delta_0 = 1 + \frac{73}{24}e^2 + \frac{37}{96}e^4$

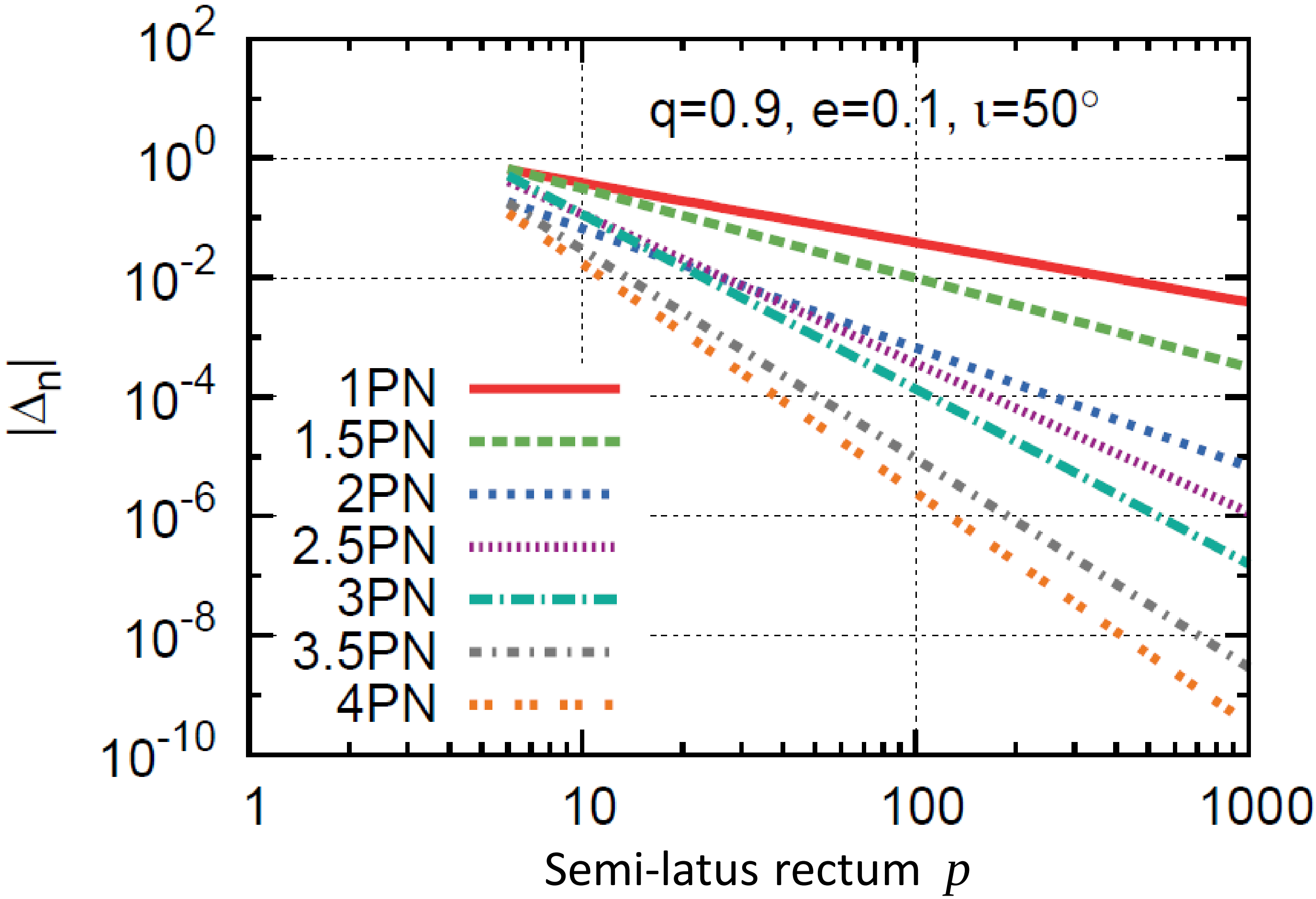
0.5PN : $\Delta_1 = 0$

1PN : $\Delta_2 = \left(-\frac{1247}{336} - \frac{9181}{672}e^2 + \frac{809}{128}e^4 + \frac{8609}{5376}e^6\right)v^2$

1.5PN : $\Delta_3 = \cdots$

Convergence check (with respect to PN expansion)

$$\left\langle \frac{dE}{dt} \right\rangle_t = -\frac{32}{5}\left(\frac{\mu}{M}\right)^2 v^{10}(1-e^2)^{3/2}[\Delta_0 + \Delta_1 + \Delta_2 + \Delta_3 + \cdots]$$

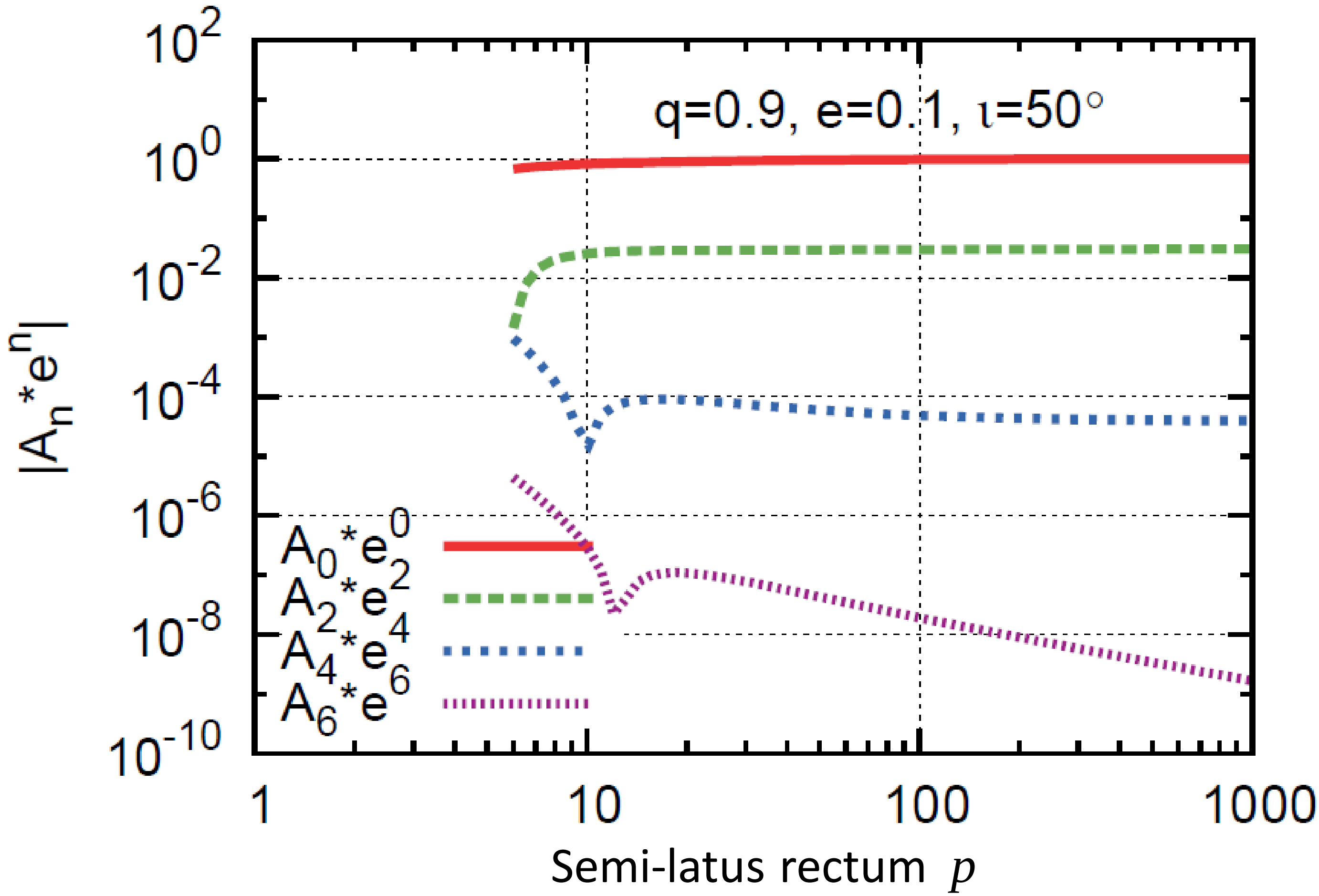


Convergence check (with respect to e-expansion)

$$\left\langle \frac{dE}{dt} \right\rangle_t = \underbrace{-\frac{32}{5}\left(\frac{\mu}{M}\right)^2 v^{10}(1-e^2)^{3/2}}_{\text{Quadrupole part}}\underbrace{[A_0 e^0 + A_2 e^2 + A_4 e^4 + A_6 e^6]}_{\text{Each order of eccentricity}}$$

Convergence check (with respect to e-expansion)

$$\left\langle\frac{dE}{dt}\right\rangle_t = -\frac{32}{5}\left(\frac{\mu}{M}\right)^2 v^{10}(1-e^2)^{3/2}[A_0e^0 + A_2e^2 + A_4e^4 + A_6e^6]$$



Exponential resummation

To improve the accuracy of the analytic formulae, we apply the following deformation:

$$\left\langle \frac{dE}{dt} \right\rangle_t = \left(\frac{dE}{dt}\right)_N [a_0 v^0 + \Delta_2 v^2 + \Delta_3 v^3 + \cdots]$$

original PN formula

$$= \left(\frac{dE}{dt}\right)_N \exp[\ln(a_0 v^0 + \Delta_2 v^2 + \Delta_3 v^3 + \cdots)]$$

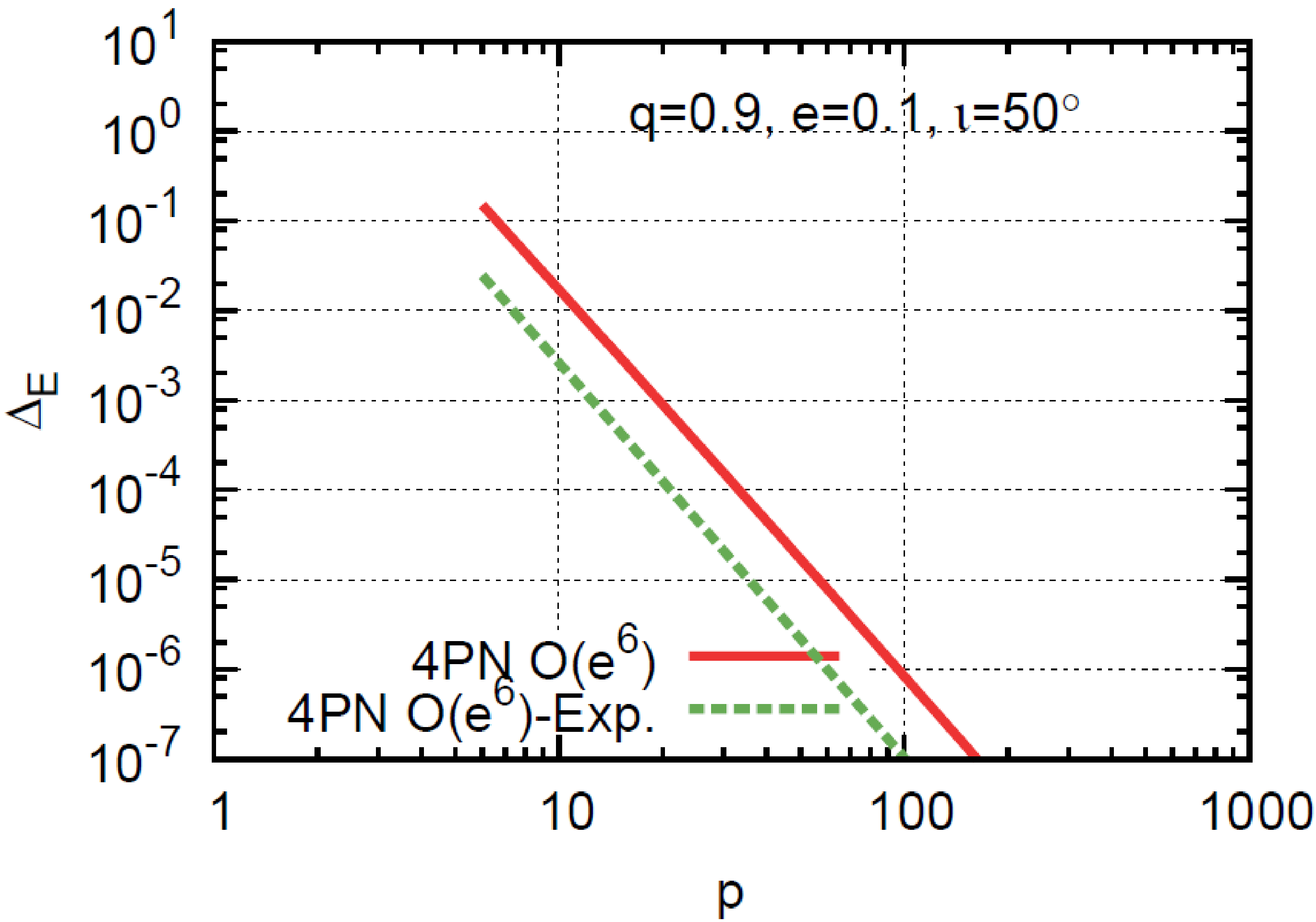
Expand the exponent in PN series

$$= \underline{\left(\frac{dE}{dt}\right)_N \exp\left[\ln a_0 + \frac{a_2}{a_0} v^2 + \frac{a_3}{a_0} v^3 + \cdots\right]}$$

Exponential resummation form

- Simple to implement
- Keep the sign (negative in this case)

Exponential resummation



Exponential resummation

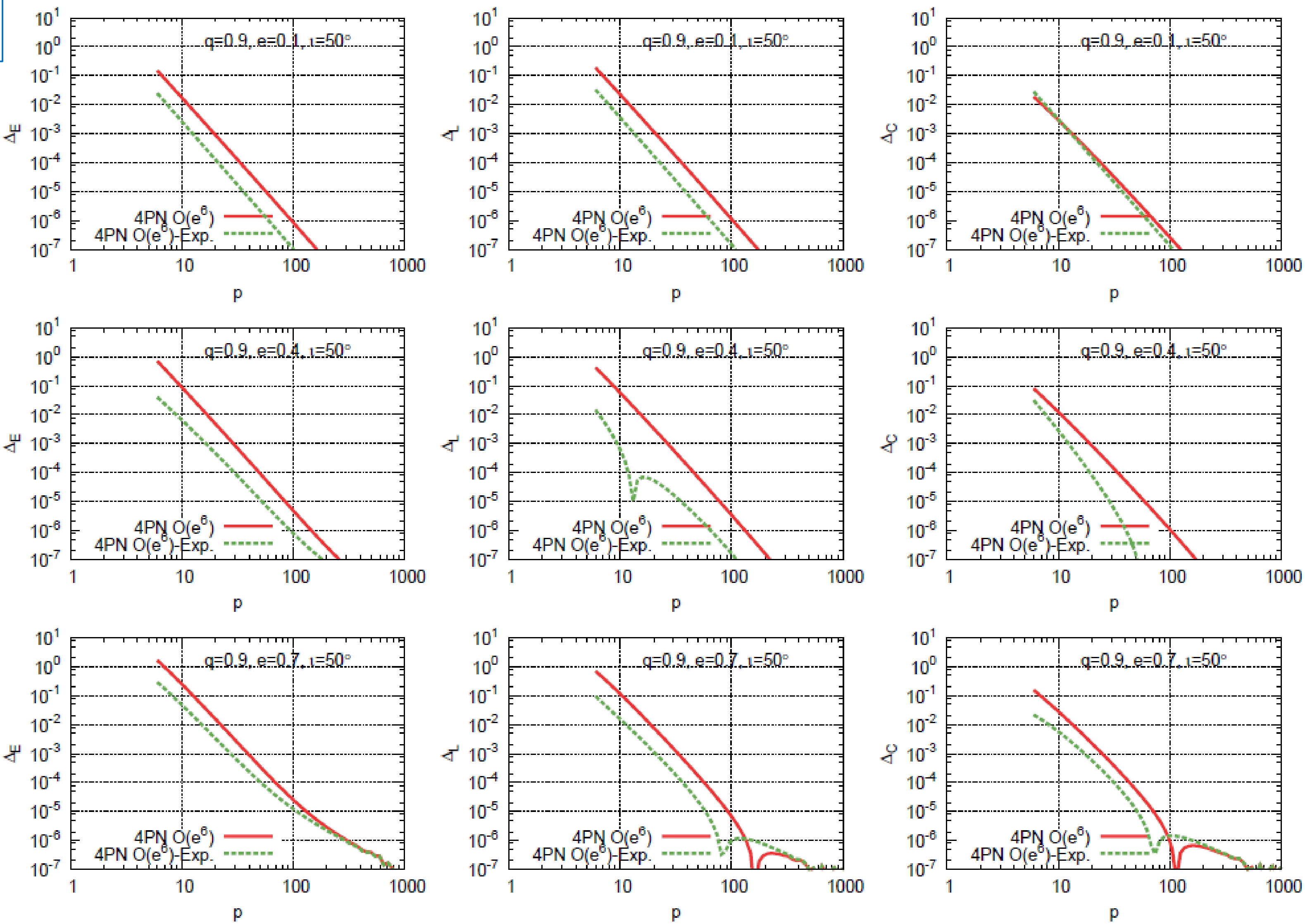
$q = 0.9$
$\iota = 50^\circ$

$\dot{E}$ | $\dot{L}$ | $\dot{C}$

$e = 0.1$

$e = 0.4$

$e = 0.7$



Summary and future issue

- The secular changes of E, L, C for bound orbits are calculated analytically at 4PN, $O(e^6)$ order, including the BH absorption.
- Comparison with the numerical data suggests the analytic formulae are correct.
- Exponential resummation may improve the accuracy of the analytic formulae.

- PN convergence becomes worse close to the central BH.
  The formulae are useless for highly eccentric orbits.
  --> Need higher order calculations.
- Extend the spinning particle case.
- Secular parts are not enough to know GW waveforms accurately.
  --> Need the conservative and 2nd order GSF anyway.