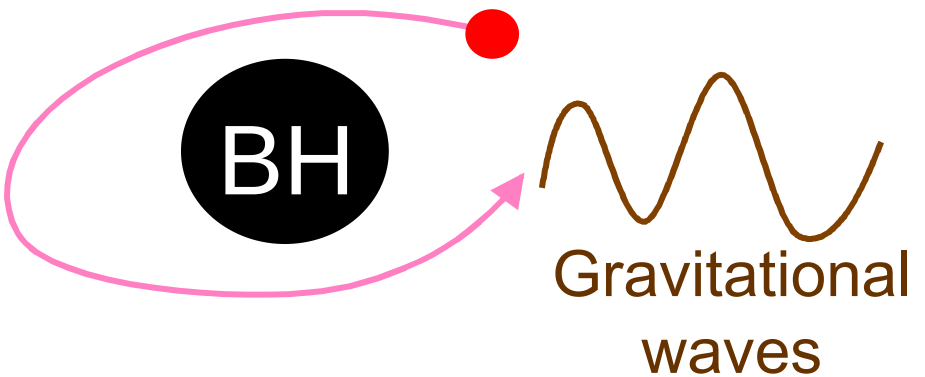
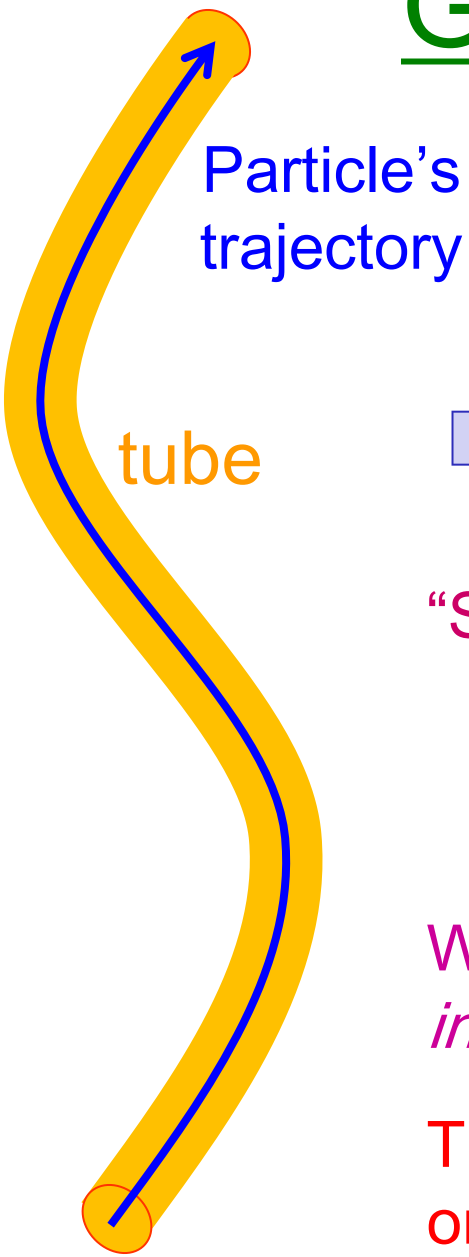
Some proposal on second order perturbation



Takahiro Tanaka
(Kyoto university)

Gauge invariance



Perturbation is everywhere small outside the world tube

"tube radius" >> μ (mass of satellite)

Unavoidable ambiguity in the perturbed trajectory of $O(\mu)$

"Self-force is *gauge dependent*"

$$F^{\mu}_{self}(\tau, \gamma)$$ has unnecessary information.

Source trajectory

While, "long term orbital evolution is *gauge invariant*", up to the above ambiguity of $O(\mu)$.

There must be a concise description keeping only the gauge invariant information

Use of canonical transformation

We just need to solve the geodesic equation on perturbed spacetime

$$S = \frac{1}{2}\int g^{\mu\nu} u_\mu u_\nu d\tau = \frac{1}{2}\int g_{(0)}^{\mu\nu} u_\mu u_\nu d\tau - \underbrace{\frac{1}{2}\int h_{full}^{\mu\nu} u_\mu u_\nu d\tau}$$

Interaction Hamiltonian $H_{\rm int}$

Change the variables to the "action variables" J_α
(~ constants of motion in the background $\left\{u^2/2, -E, L_z, Q\right\}$)
+ their conjugate "angle variables" w^α
using the standard canonical transformation.

Generating function:

$$W(x,J) = J_t t + J_\phi \phi + \int^r \underline{\tilde{u}_r(r',J)} dr' + \int^\theta \underline{\tilde{u}_\theta(\theta',J)} d\theta'$$

well-known fns for Kerr geodesic motion

$$J_r = \oint \tilde{u}_r(r',J) dr' \qquad J_\theta = \oint \tilde{u}_\theta(\theta',J) d\theta'$$

Radiation reaction to the constants of motion

“retarded” field= “radiative” + ”symmetric”

$$\frac{\text{"ret"}-\text{"adv"}}{2} \qquad \frac{\text{"ret"}+\text{"adv"}}{2}$$

regularization is unnecessary

$$\left\langle \frac{dJ_\alpha}{d\tau} \right\rangle = \left\langle \frac{\partial H_{\text{int}}}{\partial w^\alpha} \right\rangle \approx \int d\tau \int d\tau' \frac{\partial}{\partial w^\alpha} G^{(ret)}(\gamma, \gamma')\Big|_{\gamma'=\gamma}$$

1) $\partial / \partial w^a$ can be replaced with $\partial / \partial \underline{w^a_{(ini)}}$.

initial value

2) $\int d\tau \int d\tau' G^{(ret)}(\gamma, \gamma')\Big|_{\gamma'=\gamma}$ is independent of $w^a_{(ini)}$.

At the leading order in η, only the radiative part determines the change of “constants of motion”

except for resonance orbits, (Mino (2003))

Gauge invariance of the angle variables

Action angle variables w^{μ} are

gauge invariant in the context of long term evolution.

allowing $O(\mu)$ gauge ambiguity, which does not accumulate.

$\because$

$$W(x,J) = J_t t + J_\phi \phi + \int^r \tilde{u}_r(r',J)dr' + \int^\theta \tilde{u}_\theta(\theta',J)d\theta'$$

$$J_r = \oint \tilde{u}_r(r',J)dr' \qquad J_\theta = \oint \tilde{u}_\theta(\theta',J)d\theta'$$

$$W(x,J) = \tilde{W}(x,J) + n_r J_r + n_\theta J_\theta$$

where we introduce a single valued function with respect to x:

$$\tilde{W}(x,J) = J_t t + J_\phi \phi + \int_{r_0}^r \tilde{u}_r(r',J)dr' + \int_{\theta_0}^\theta \tilde{u}_\theta(\theta',J)d\theta'$$

$$w^I = \frac{\partial W(x,J)}{\partial J_I} = \frac{\partial \tilde{W}(x,J)}{\partial J_I} + n_I \ (I = r,\theta) \qquad w^i = \frac{\partial W(x,J)}{\partial J_i} = \frac{\partial \tilde{W}(x,J)}{\partial J_i} \ (i = t,\phi)$$

Small variations of x and J are not amplified in w.

Order counting in $\eta=\mu/M$ in wave form

Orbital frequencies:

$$\Omega^a \equiv \frac{dw^a}{d\tau}$$

Wave form is specified by $\dfrac{d\Omega^a}{d\tau}(\Omega)$

$$\frac{d\Omega^a}{dt} = \frac{\partial\Omega^a}{\partial J_b}\frac{dJ_b}{dt}$$

post adiabatic order ($O(1)$ phase)

linear perturbation

$$\frac{dJ_b}{dt} = 0 + O(\eta) + O(\eta^2)$$

$$\frac{\partial\Omega^a}{\partial J_b} = (\text{geodesic}) + O(\eta) + O(\eta^2)$$

adiabatic order ($O(\eta^{-1})$ phase); solved in 2005

Long term evolution

EOM)

$$\frac{dJ_a}{d\tau} = -\frac{\partial H_{\text{int}}}{\partial w^a}$$

We need to solve this equation to $O(\eta^2)$

$$\frac{dw^a}{d\tau} = \Omega^a_{(0)}(J) + \frac{\partial H_{\text{int}}}{\partial J^a}$$

and this equation to $O(\eta)$

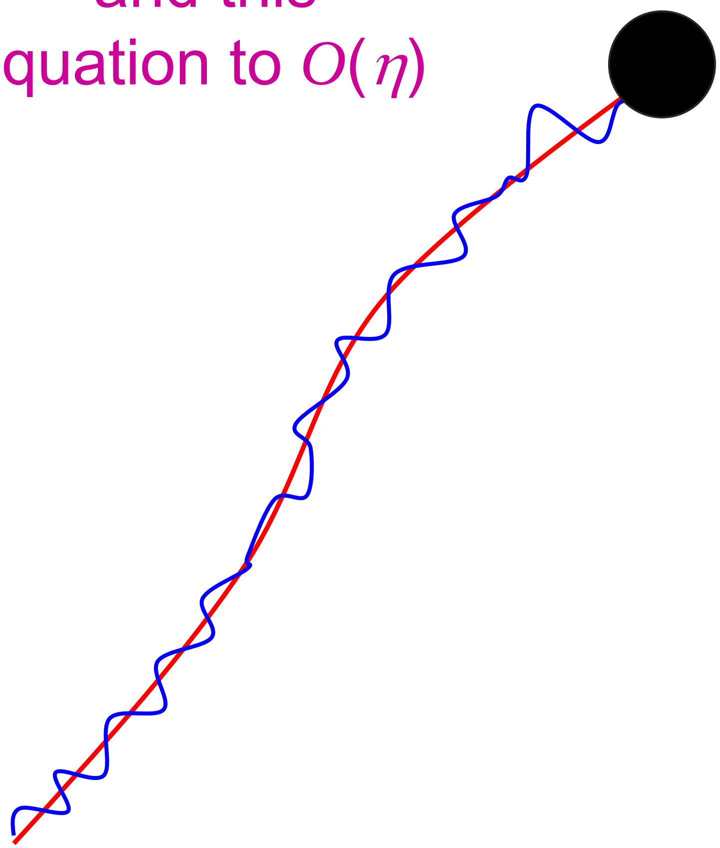
We separate the variables

$$J_a = \bar{J}_a + \delta J_a$$
$$w^a = \overline{w}^a + \delta w^a$$

so that

$(\bar{J}, \overline{w})$:slowly changing trend

$(\delta J, \delta w)$:rapidly oscillating but always small



Linear order metric perturbation)

$$H_{\rm int} = H_{\rm int}^{(1)} + H_{\rm int}^{(2)} + \cdots$$

Source orbit in computing $H_{\rm int}^{(1)} = -h_{(1)}^{\mu\nu} u_\mu u_\nu$
is approximated by the "osculating geodesic orbit":

$$\bar{J}_{osc}(\tau_0;\tau) = \bar{J}(\tau_0) \qquad \bar{w}^a_{osc}(\tau_0;\tau) = \bar{w}^a(\tau_0) + \Omega^a_{(0)}\left(\bar{J}(\tau_0)\right)(\tau - \tau_0)$$

We decompose EOM as

$$\frac{d\bar{J}_a}{d\tau} = -\left\langle \frac{\partial H_{\rm int}^{(1)}}{\partial w^a} \right\rangle_{osc} \qquad \frac{d\bar{w}^a}{d\tau} = \Omega^a_{(0)}\left(\bar{J}\right) + \left\langle \frac{\partial H_{\rm int}^{(1)}}{\partial J^a} \right\rangle_{osc} \quad \text{— ☆}$$

$$\frac{d\delta J_a^{(1)}}{d\tau} = -\frac{\partial H_{\rm int}^{(1)}}{\partial w^a} + \left\langle \frac{\partial H_{\rm int}^{(1)}}{\partial w^a} \right\rangle_{osc} \qquad \frac{d\delta w^a_{(1)}}{d\tau} = \frac{\partial \Omega^a_{(0)}\left(\bar{J}\right)}{\partial J_b} \delta J_b^{(1)} + \frac{\partial H_{\rm int}^{(1)}}{\partial J^a} - \left\langle \frac{\partial H_{\rm int}^{(1)}}{\partial J^a} \right\rangle_{osc}$$

Average $\langle \cdots \rangle_{osc}$ is also taken over the osculating geodesic orbit.

$\left(\delta J_a^{(1)}, \delta w^a_{(1)}, \left\langle \partial H_{\rm int}^{(1)} / \partial J^a \right\rangle_{osc}\right)$ can be erased by the choice of gauge with the rescaling of the proper time.

∵ Oscillating part of the force can be erased locally in GR. Both terms on the right hand side of Eq. ☆ are gauge dependent separately, while the left hand side is gauge invariant.

$\Rightarrow$ $$\frac{d\bar{J}_a}{d\tau} = -\left\langle \frac{\partial H_{\rm int}^{(1)}}{\partial w^a} \right\rangle_{osc} \qquad \frac{d\bar{w}^a}{d\tau} = \underline{\left(1 - \langle H_{\rm int} \rangle_{osc}\right)} \Omega^a_{(0)}\left(\bar{J}\right) \qquad \delta J_a^{(1)} = \delta w^a_{(1)} = 0$$

can be absorbed by proper time rescaling

Second order metric perturbation)

$$H_{\rm int} = H_{\rm int}^{(1)} + H_{\rm int}^{(2)} + \cdots$$

Source for $H_{\rm int}^{(2)} = -h_{(2)}^{\mu\nu} u_\mu u_\nu$ has two contributions:

1) Quadratic term of the first order metric perturbation "$\partial h\, \partial h$"

2) First order deviation of the source orbit from the osculating orbit

$$\Delta J_a(\tau_0;\tau) := \bar{J}_a(\tau) - \bar{J}_a^{osc}(\tau_0;\tau) = \left\langle \frac{dJ_a}{d\tau} \right\rangle_{osc} (\tau - \tau_0)$$

$$\Delta w^a(\tau_0;\tau) := \bar{w}^a(\tau) - \bar{w}^a_{osc}(\tau_0;\tau) = \frac{1}{2} \frac{\partial \Omega^a_{(0)}}{\partial J_b} \left\langle \frac{dJ_b}{d\tau} \right\rangle_{osc} (\tau - \tau_0)^2$$

$$\frac{dJ_a}{d\tau} = -\frac{\partial H_{\rm int}^{(1)}}{\partial w^a} - \underline{\frac{\partial H_{\rm int}^{(2,hh)}}{\partial w^a}} - \underbrace{\left(\Delta J_b \frac{\partial}{\partial J_b^{(s)}} + \Delta w^b \frac{\partial}{\partial w^b_{(s)}} \right) \frac{\partial H_{\rm int}^{(1)}}{\partial w^a}}_{\text{differentiation with respect to the source orbit}}$$

Recall that $\dfrac{\partial H_{\rm int}^{(1)}}{\partial w^a} = \left\langle \dfrac{\partial H_{\rm int}^{(1)}}{\partial w^a} \right\rangle_{osc}$ in our gauge.

$$\frac{d\bar{J}_a}{d\tau} = \underbrace{-\left(1 + 2\langle H_{\rm int}\rangle_{osc}\right)}_{\text{proper time rescaling}} \left\langle \frac{\partial H_{\rm int}^{(1)}}{\partial w^a} \right\rangle_{osc} - \left\langle \frac{\partial H_{\rm int}^{(2,hh)}}{\partial w^a} \right\rangle_{osc} - \left\langle \left(\Delta J_b \frac{\partial}{\partial J_b^{(s)}} + \Delta w^b \frac{\partial}{\partial w^b_{(s)}} \right) \frac{\partial H_{\rm int}^{(1)}}{\partial w^a} \right\rangle_{osc}$$

Using $H_{\text{int}}^{(1)} = \int d\tau' G_{(ret)}(J, w; \gamma(\tau')) \approx u_\mu u_\nu \int d\tau' G_{(ret)}^{\mu\nu\rho\sigma}(x, z(\tau')) u_\rho(\tau') u_\sigma(\tau')$,

the third term can be rewritten as

$$\left\langle \left(\Delta J_b \frac{\partial}{\partial J_b^{(s)}} + \Delta w^b \frac{\partial}{\partial w_{(s)}^b} \right) \frac{\partial H_{\text{int}}^{(1)}}{\partial w^a} \right\rangle_{osc}$$

$$= \left\langle \frac{\partial}{\partial w^a} \int d\tau' \left(G_{(ret)}(J, w; \gamma + \Delta\gamma(\tau; \tau')) - G_{(ret)}(J, w; \gamma(\tau')) \right) \right\rangle_{osc}$$

Retarded Green function contains singular part, but using the symmetry we used before, we can replace (ret) with (rad). So, regularization is not needed.

Anyway, necessary terms are all written as the average over the osculating orbit.

$$\frac{d\bar{J}_a}{d\tau} = -\left(1 + 2\langle H_{\text{int}} \rangle_{osc}\right) \left\langle \frac{\partial H_{\text{int}}^{(1)}}{\partial w^a} \right\rangle_{osc} - \left\langle \frac{\partial H_{\text{int}}^{(2,hh)}}{\partial w^a} \right\rangle_{osc}$$

$$- \left\langle \frac{\partial}{\partial w^a} \int d\tau' \left(G_{(rad)}(J, w; \gamma + \Delta\gamma(\tau; \tau')) - G_{(rad)}(J, w; \gamma(\tau')) \right) \right\rangle_{osc}$$

$$\frac{d\bar{w}^a}{d\tau} = \Omega_{(0)}^a(\bar{J})$$

$$\left\langle \frac{\partial}{\partial w^a} \int d\tau' \left(G_{(rad)}\left(J, w; \gamma + \Delta\gamma(\tau;\tau')\right) - G_{(rad)}\left(J, w; \gamma(\tau')\right)\right) \right\rangle_{osc}$$ can be rewritten as

$$= \frac{1}{2T}\int_{-T}^{T} d\tau \frac{\partial}{\partial w^a} \int d\tau' \delta\gamma^a(\tau;\tau') \frac{\partial}{\partial \gamma^a} G_{(rad)}(J, w; \gamma)$$

To deal with the factor $\tau - \tau'$ in $\delta\gamma^a$, we replace it with the differentiation with respect to ϖ, introducing the factor $e^{i\varpi(\tau-\tau')}$.

By using the mode sum expression for the radiative Green function, we can rewrite the above expression as

$$\begin{aligned}
&\int d\omega \sum_{l,m,n_r,n_\theta,n'_r,n'_\theta} \frac{1}{2T}\int_{-T}^{T} d\tau \int d\tau' \partial_{\varpi}^{n} e^{i\varpi(\tau-\tau')} \mathcal{I}_{l,m,n_r,n_\theta}(\omega) e^{-i(\omega-\omega_{l,m,n_r,n_\theta})\tau} \mathcal{I}'_{l,m,n'_r,n'_\theta}(\omega) e^{i(\omega-\omega_{l,m,n'_r,n'_\theta})\tau'} \\
&\approx \int d\omega \sum_{l,m,n_r,n_\theta,n'_r,n'_\theta} \frac{(2\pi)^2}{2T} \partial_{\varpi}^{n} \left[\delta(\varpi - (\omega - \omega_{l,m,n_r,n_\theta}))\delta(\varpi - (\omega - \omega_{l,m,n'_r,n'_\theta}))\right] \mathcal{I}_{l,m,n_r,n_\theta}(\omega) \mathcal{I}'_{l,m,n'_r,n'_\theta}(\omega) \\
&\approx \int d\omega \sum_{l,m,n_r,n_\theta} (2\pi)^2 \delta(\omega - \omega_{l,m,n'_r,n'_\theta}) \partial_{\omega}^{n} \left[\mathcal{I}_{l,m,n_r,n_\theta}(\omega) \mathcal{I}'_{l,m,n'_r,n'_\theta}(\omega)\right] \\
&= \sum_{l,m,n_r,n_\theta} (2\pi)^2 \partial_{\omega}^{n} \left[\mathcal{I}_{l,m,n_r,n_\theta}(\omega) \mathcal{I}'_{l,m,n'_r,n'_\theta}(\omega)\right]_{\omega=\omega_{l,m,n'_r,n'_\theta}} .
\end{aligned}$$

Computation is similar to the one necessary to compute the adiabatic order radiation reaction to the constants of motion.

Furthermore, $\left\langle \frac{\partial H_{\text{int}}^{(2,hh)}}{\partial w^a} \right\rangle_{osc}$ can be rewritten as

$$\int d\tau \int d\tau' \int d\tau'' \int d^4x' \hat{V}_{x'} \frac{\partial}{\partial w^a} G_{(ret)}(x, x'_1)|_{x=\gamma(\tau)} G_{(ret)}(x'_2, \gamma(\tau')) G_{(ret)}(x'_3, \gamma(\tau'')),$$

Using the fact that $\partial/\partial w^a$ can be replaced with the differentiation with respect to the initial value $w_{(ini)}^a$, and that

$$\int d\tau \int d\tau' \int d\tau'' \int d^4x \hat{V}(x) G_*(\gamma(\tau), x) G_*(x, \gamma(\tau')) G_*(x, \gamma(\tau'')),$$

is independent of $w_{(ini)}^a$, $\left\langle \frac{\partial H_{\text{int}}^{(2,hh)}}{\partial w^a} \right\rangle_{osc}$ is reduced to

$$2 \int d\tau \int d\tau' \int d\tau'' \int d^4x' \hat{V}_{x'} \frac{\partial}{\partial w^a} G_{(rad)}(x, x'_1)|_{x=\gamma(\tau')} G_{(sym)}(x'_2, \gamma(\tau)) G_{(sym)}(x'_3, \gamma(\tau'')).$$

which is radiative field sourced by the expression given by the first order symmetric field.

Radiative field is independent of how we remove the singular part of the source as long as it is written in the form of "$\Box h$".

Conclusion

We discussed a method for long-term evolution that controls the error of the orbital phase to be O(η).

Choosing the gauge of the first order perturbation appropriately,

$$\frac{d\bar{J}_a}{d\tau} = -\left(1+2\langle H_{\rm int}\rangle_{osc}\right)\left\langle \frac{\partial H^{(1)}_{\rm int}}{\partial w^a}\right\rangle_{osc} - \left\langle \frac{\partial H^{(2,hh)}_{\rm int}}{\partial w^a}\right\rangle_{osc} - \underline{\left\langle \frac{\partial}{\partial w^a}\int d\tau'\left(G_{(rad)}\left(J,w;\gamma+\Delta\gamma(\tau;\tau')\right)-G_{(rad)}\left(J,w;\gamma(\tau')\right)\right)\right\rangle_{osc}}$$

similar computation to $<dQ/d\tau>$

$$\frac{d\bar{w}^a}{d\tau} = \Omega^a_{(0)}\left(\bar{J}\right)$$

$$\left\langle \frac{\partial H^{(1)}_{\rm int}}{\partial w^a}\right\rangle_{osc} = \left\langle \frac{\partial}{\partial w^a}\int d\tau'\left(G_{(rad)}\left(J,w;\gamma(\tau')\right)\right)\right\rangle_{osc}$$

can be also evaluated by using the radiative Green fn., which does not require any regularization.

First order self-force is needed to compute the quadratic source term "$\partial h\, \partial h$". However, since only the radiative part of the second order perturbation is needed, the second order regularization might be easier.