Compute second order gravitational self-force effects

Barry Wardell, Cornell University & University College Dublin
Collaborators: Adam Pound, Jeremy Miller & Leor Barack (University of Southampton), Niels Warburton (MIT)

18th Capra Meeting On Radiation Reaction In General Relativity, 1st July 2015

Supported by the Irish Research Council

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2015? Second order redshift invariant (Barack, Miller, Pound, Warburton, Wardell)

Second order conservative effects

Generalised redshift invariant for circular orbits
[Pound, Phys. Rev. D90, 084039]

$$U_0(\Omega) = \left(1 - 3M/r_\Omega\right)^{1/2} \qquad \Omega = \sqrt{\frac{M}{r_\Omega^3}}$$

$$\tilde{U} = U_0(\Omega)\Big[1 + \tfrac{1}{2}\epsilon h^{\mathrm{R1}}_{u_0u_0} + \epsilon^2\big(\tfrac{1}{2}h^{\mathrm{R2}}_{u_0u_0} + \tfrac{3}{8}(h^{\mathrm{R1}}_{u_0u_0})^2 - \tfrac{r_\Omega^3}{6M^2}(F_{1r})^2\left(1 - 3M/r_\Omega\right)\big)\Big]$$

Second order field equations

$$\boxed{\mathcal{D}_{\mu\nu}[h] \equiv \Box h_{\mu\nu} + 2R_\mu{}^\alpha{}_\nu{}^\beta h_{\alpha\beta}}$$

$$\mathcal{D}_{\mu\nu}[h^{\mathrm{R1}}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S1}}]$$

$$\mathcal{D}_{\mu\nu}[h^{\mathrm{R2}}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S2}}] + \delta^2 R_{\mu\nu}[h^1, h^1]$$

$$\begin{aligned}\delta^2 R_{\alpha\beta}[h,h] \equiv &- \tfrac{1}{2} h^{\mu\nu}\left(2h_{\mu(\alpha;\beta)\nu} - h_{\alpha\beta;\mu\nu} - h_{\mu\nu;\alpha\beta}\right) \\ &+ \tfrac{1}{4} h^{\mu\nu}{}_{;\alpha} h_{\mu\nu;\beta} + \tfrac{1}{2} h^{\mu}{}_{\beta}{}^{;\nu}\left(h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}\right) \\ &- \tfrac{1}{2} \bar{h}^{\mu\nu}{}_{;\nu}\left(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}\right)\end{aligned}$$

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Second order field equations

$$\boxed{\mathcal{D}_{\mu\nu}[h] \equiv \Box h_{\mu\nu} + 2R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} h_{\alpha\beta}}$$

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Second order field equations

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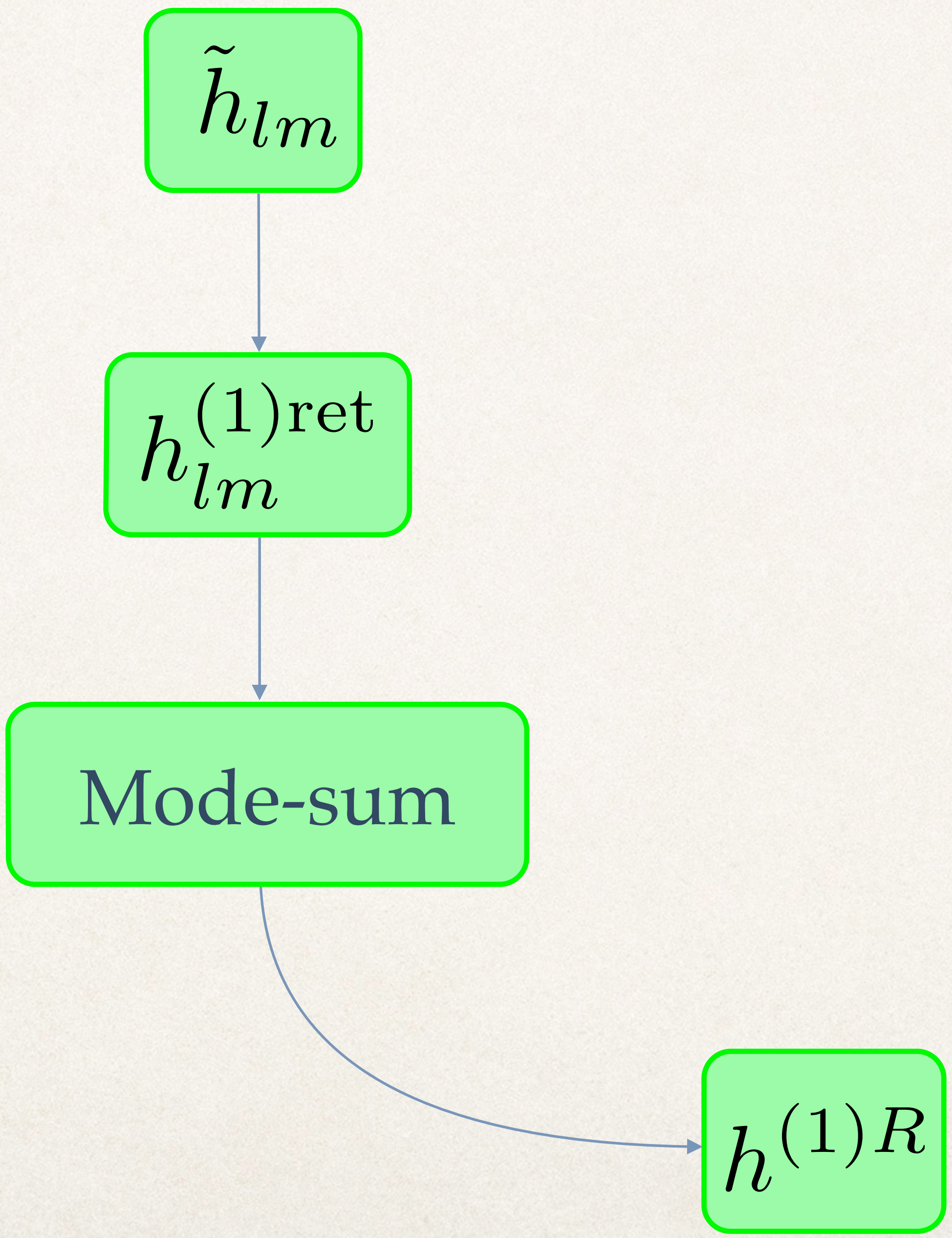
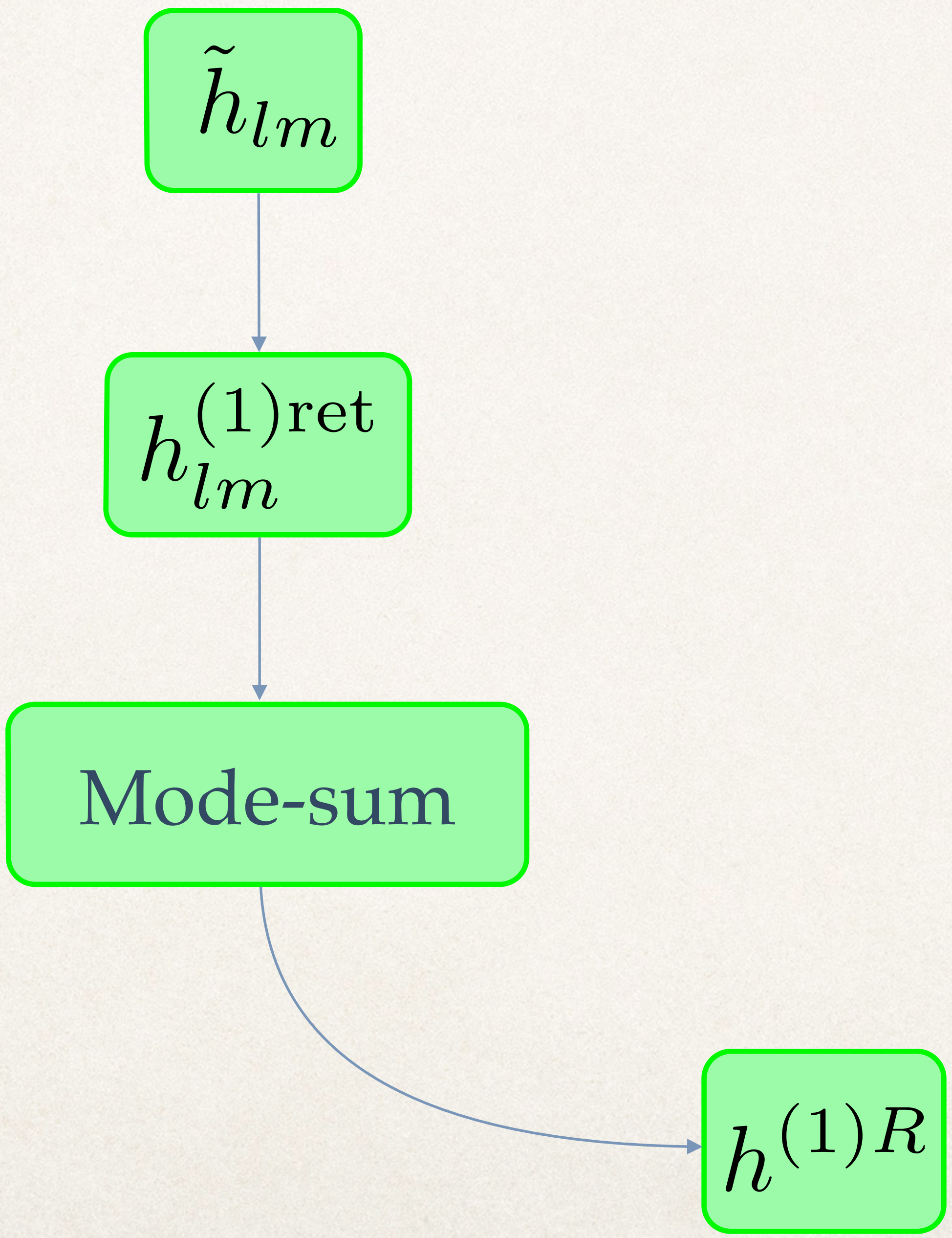
First order self-force



Towards second order self-force



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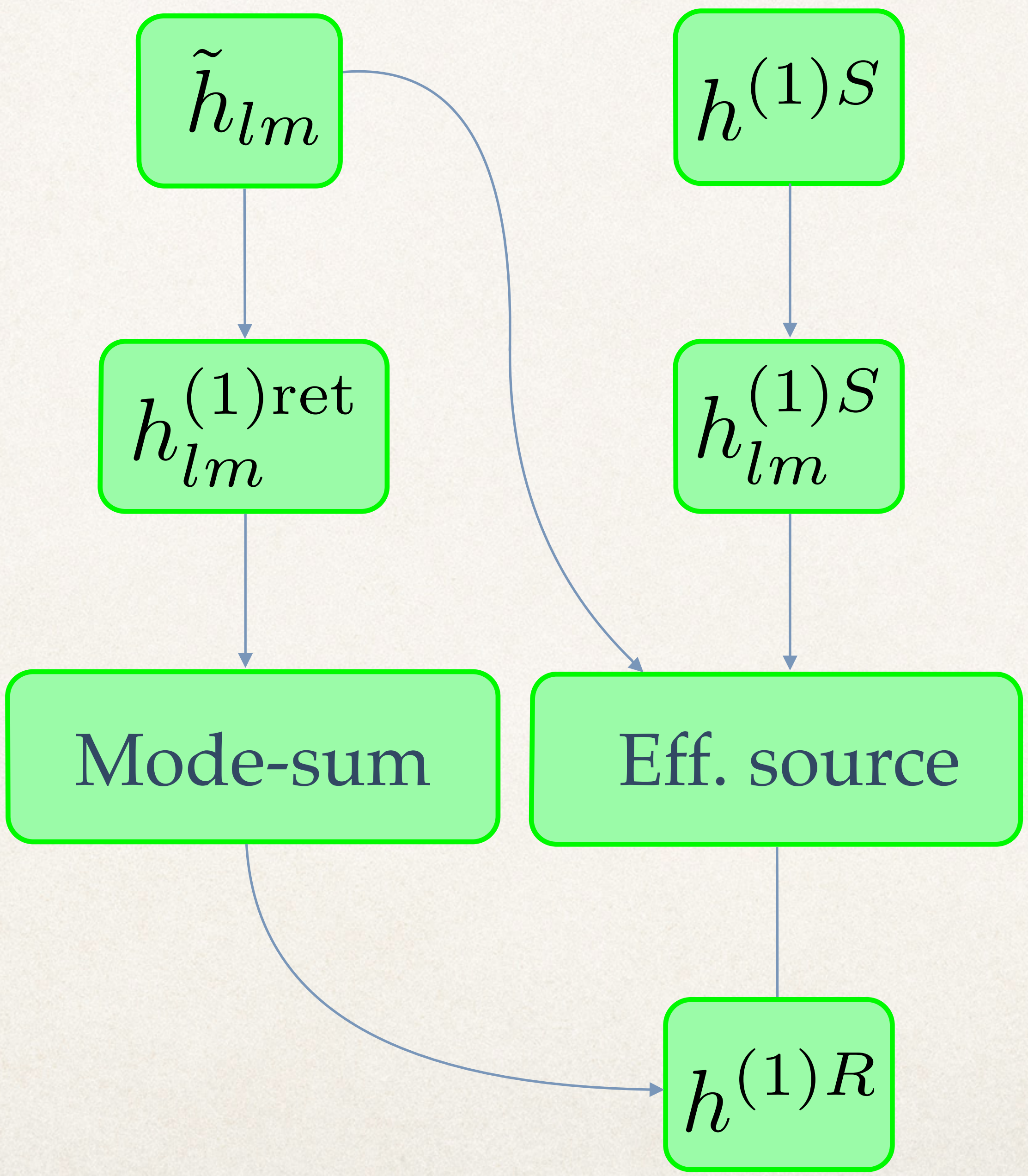
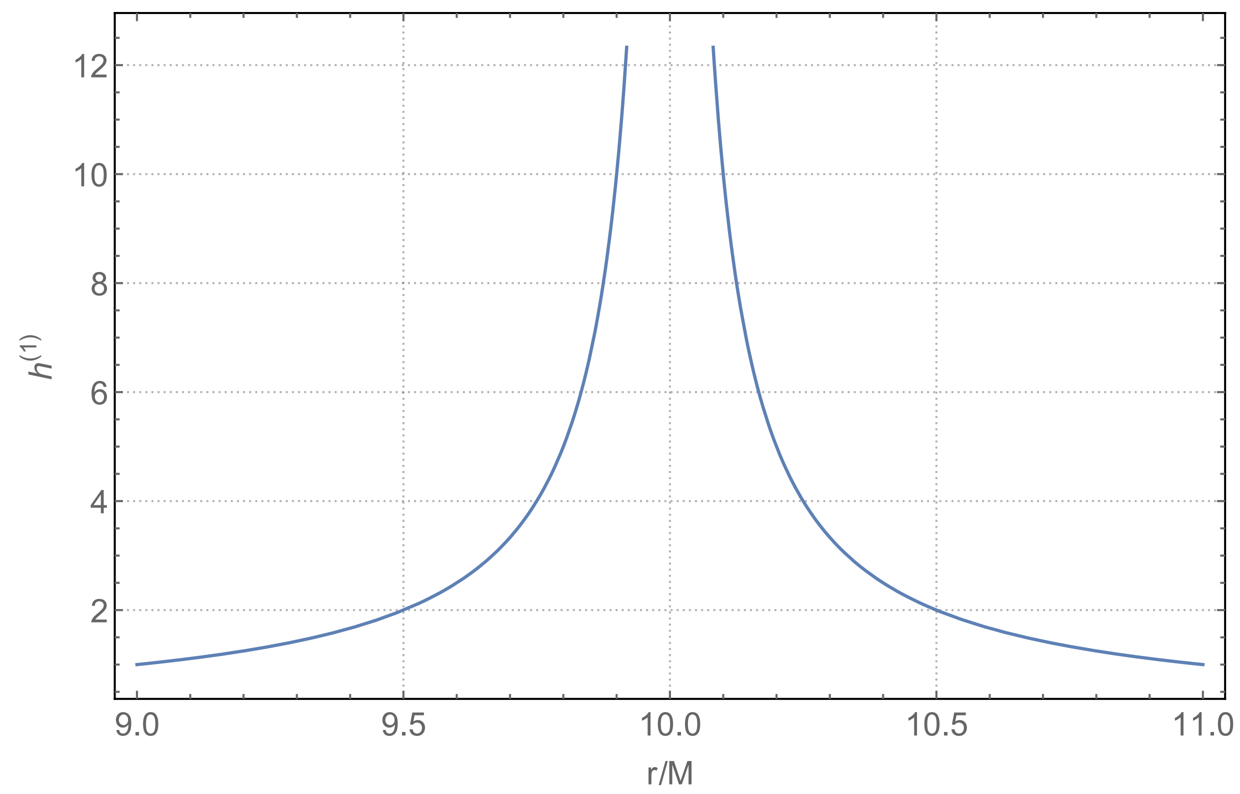


Motivation: Second order

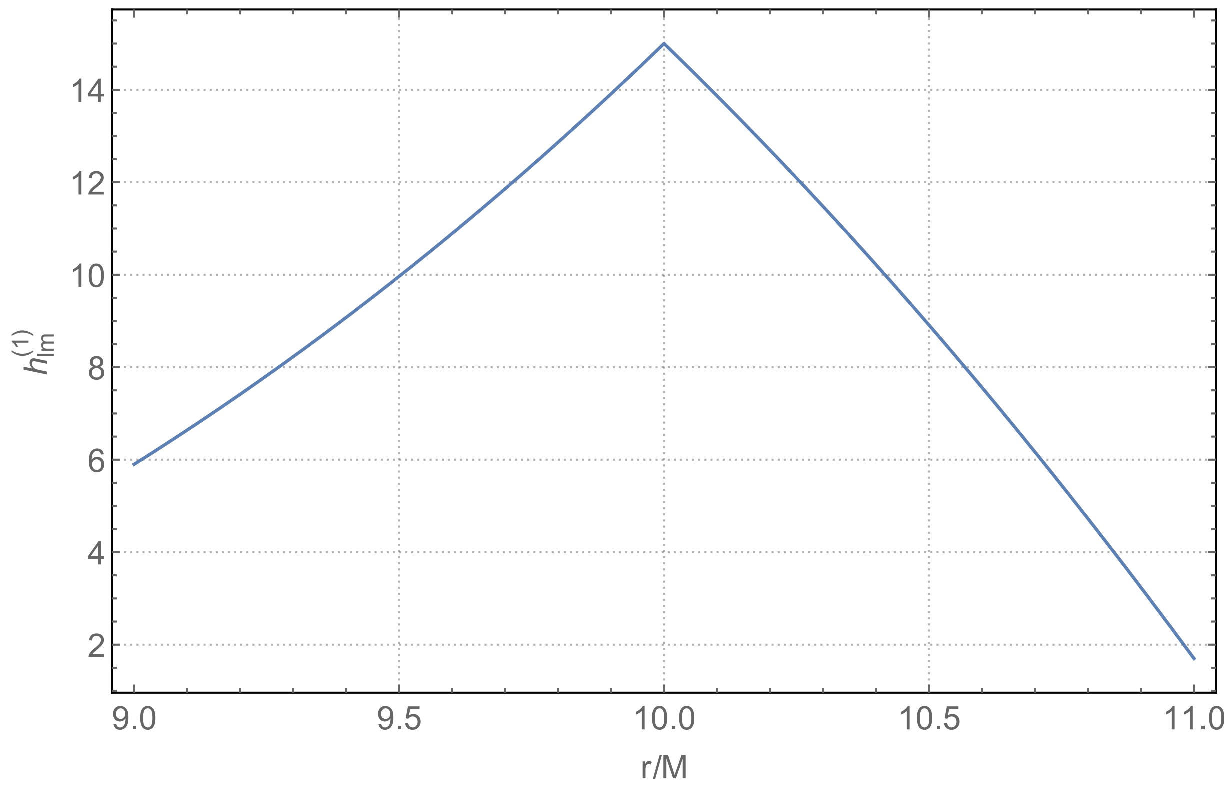
- Second order gravitational self-force will require high accuracy $\Rightarrow$ Frequency domain.



First order metric perturbation ~ $1/(r-r_0)$

Motivation: Second order

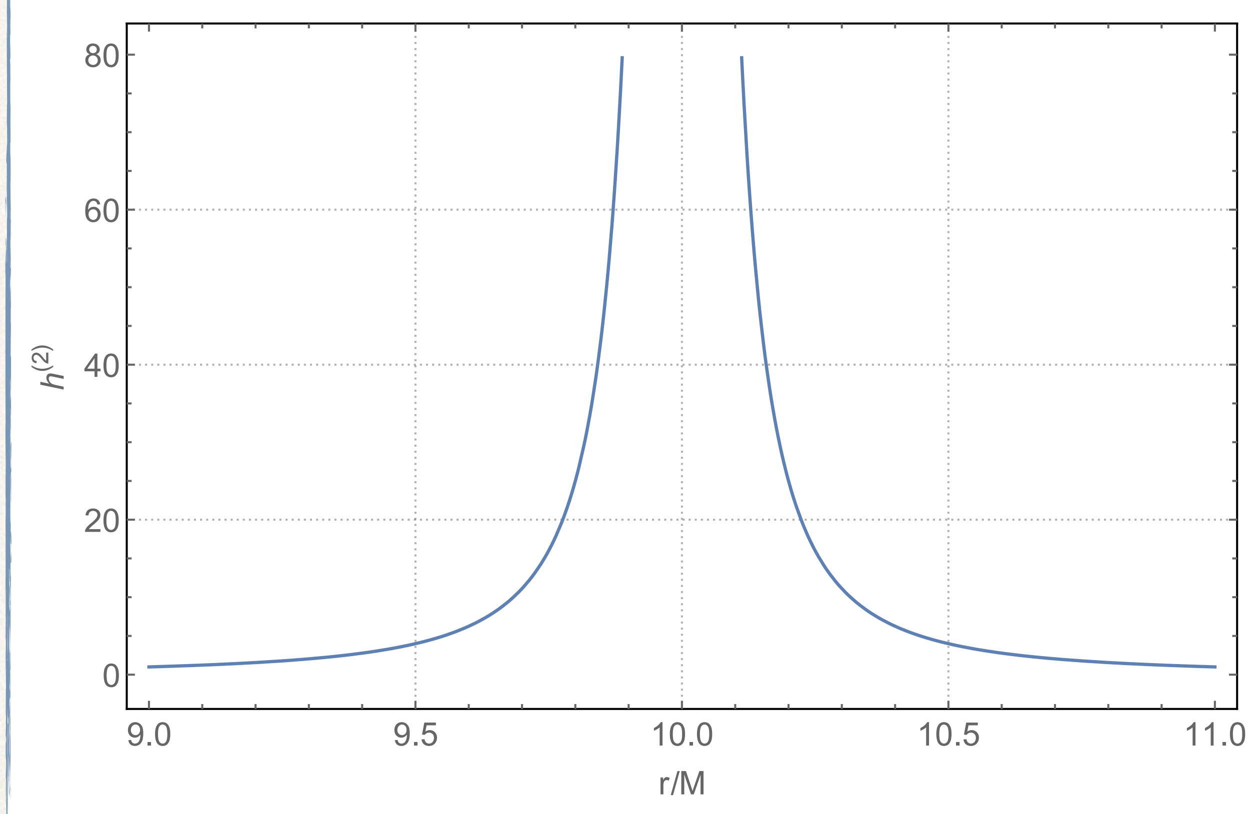
- Second order gravitational self-force will require high accuracy $\Rightarrow$ Frequency domain.
- Spherical harmonic modes at first order finite on world line $\Rightarrow$ mode-sum regularization.



First order modes ~ $|r-r_0|/(r-r_0)$

Motivation: Second order

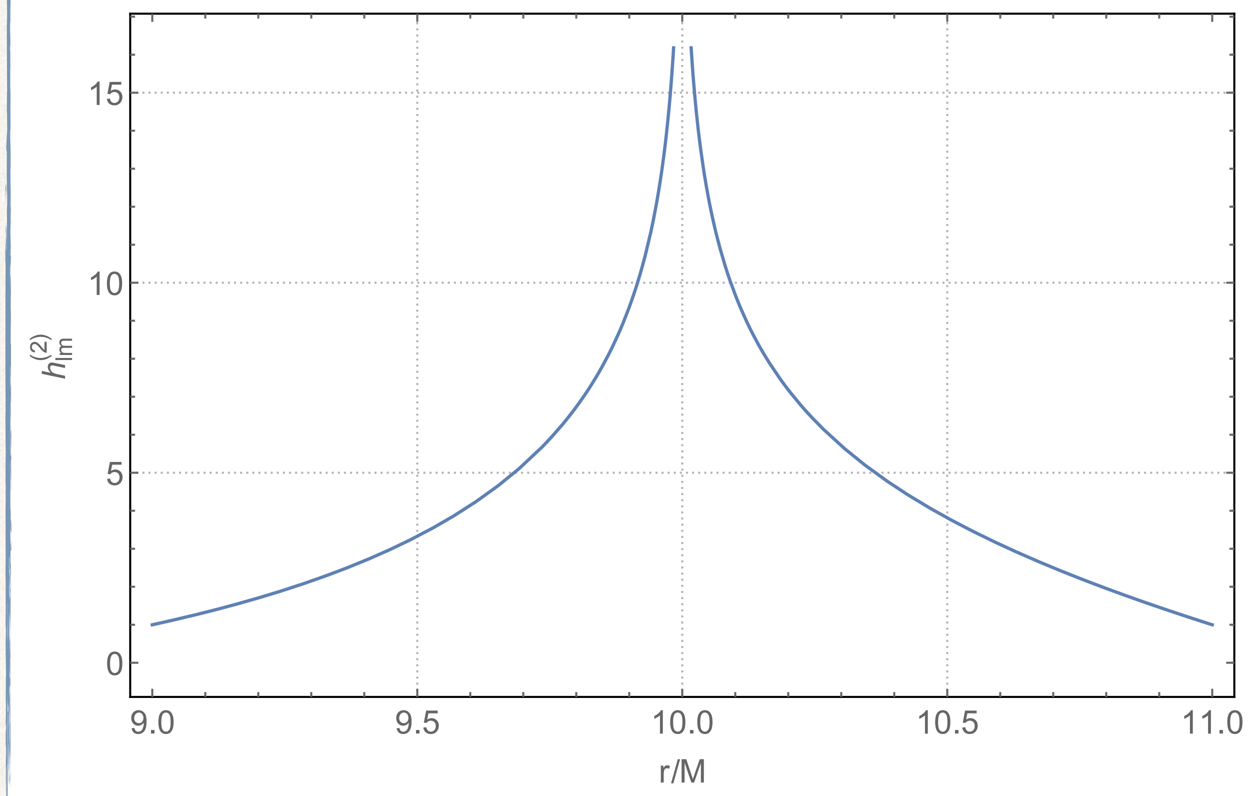
- Second order gravitational self-force will require high accuracy $\Rightarrow$ Frequency domain.
- Spherical harmonic modes at first order finite on world line $\Rightarrow$ mode-sum regularization.
- Second order metric more singular.



Second order perturbation ~ $1/(r-r_0)^2$

Motivation: Second order

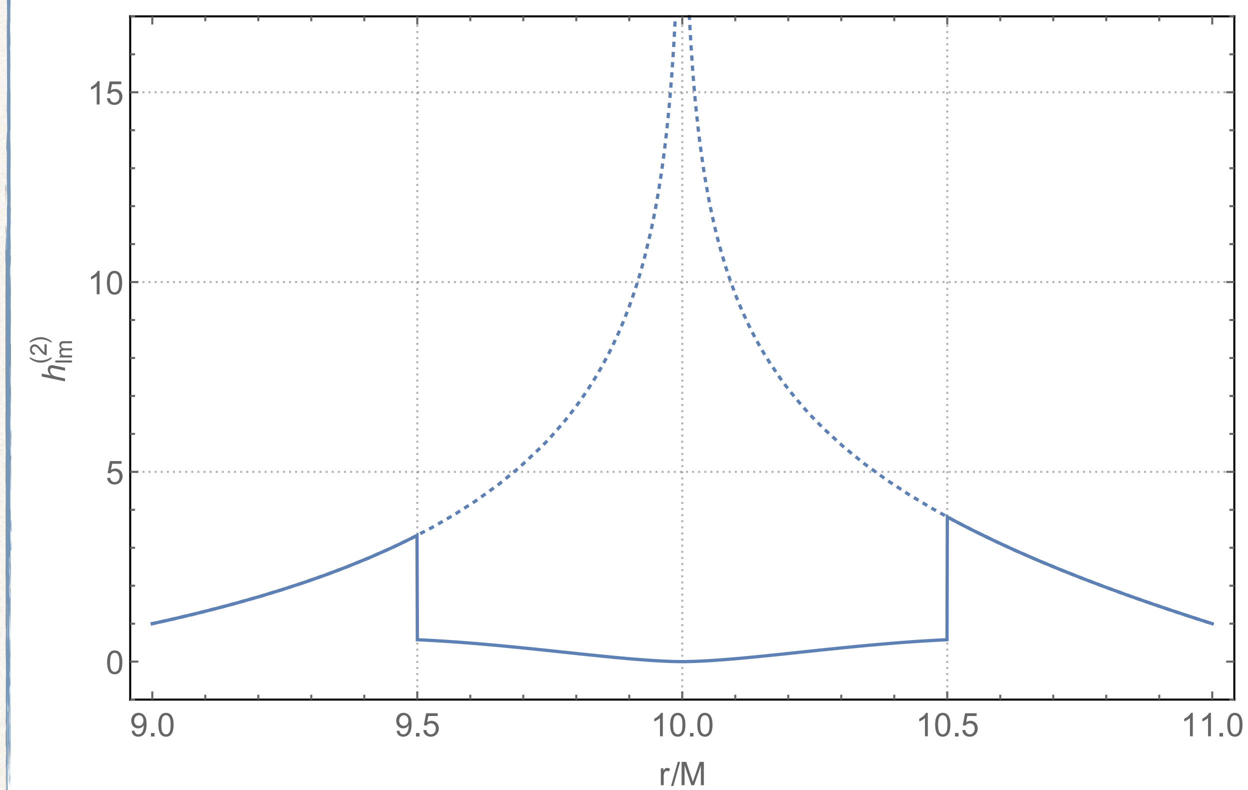
- Second order gravitational self-force will require high accuracy $\Rightarrow$ Frequency domain.
- Spherical harmonic modes at first order finite on world line $\Rightarrow$ mode-sum regularization.
- Second order metric more singular.
- Second order modes diverge logarithmically.



Second order modes ~ $\log | r-r_0 |$

Motivation: Second order

- Second order gravitational self-force will require high accuracy ⇒ Frequency domain.
- Spherical harmonic modes at first order finite on world line ⇒ mode-sum regularization.
- Second order metric more singular.
- Second order modes diverge logarithmically.
- Avoid computing retarded field on world line ⇒ effective source.



Second order modes ~ log | r-r_0 |

Frequency-domain scalar mode-sum self-force

$$\left[\frac{d^2}{dr^2} + \frac{2(r-M)}{fr^2}\frac{d}{dr} + \frac{1}{f}\left(\frac{\omega^2}{f} - \frac{\ell(\ell+1)}{r^2}\right)\right]\Phi^{\rm ret}_{\ell m} = \alpha_{\ell m}\delta(r-r_0)$$

Find solutions to homogeneous equation which satisfy outgoing boundary conditions on horizon and at infinity, respectively

$$\left[\frac{d^2}{dr^2} + \frac{2(r-M)}{fr^2}\frac{d}{dr} + \frac{1}{f}\left(\frac{\omega^2}{f} - \frac{\ell(\ell+1)}{r^2}\right)\right]\tilde{\Phi}^{\rm ret\pm}_{\ell m} = 0$$

Construct inhomogeneous solutions by matching on the world line

$$\Phi^{\rm ret\pm}_{\ell m} = c^{\pm}_{\ell m}\tilde{\Phi}^{\rm ret\pm}_{\ell m} \qquad c^{\pm}_{\ell m} = \alpha_{\ell m}\frac{\tilde{\Phi}^{\mp}_{\ell m}}{W}$$

where W is the Wronskian of homogeneous solutions

Frequency-domain scalar effective source

$$\left[\frac{d^2}{dr^2} + \frac{2(r-M)}{fr^2}\frac{d}{dr} + \frac{1}{f}\left(\frac{\omega^2}{f} - \frac{\ell(\ell+1)}{r^2}\right)\right]\Phi^{\mathrm{ret}}_{\ell m} = S^{\mathrm{eff}}_{\ell m}$$

Find solutions to homogeneous equation which satisfy outgoing boundary conditions on horizon and at infinity, respectively

$$\left[\frac{d^2}{dr^2} + \frac{2(r-M)}{fr^2}\frac{d}{dr} + \frac{1}{f}\left(\frac{\omega^2}{f} - \frac{\ell(\ell+1)}{r^2}\right)\right]\tilde{\Phi}^{\mathrm{ret}\pm}_{\ell m} = 0$$

Construct inhomogeneous solutions using variation of parameters

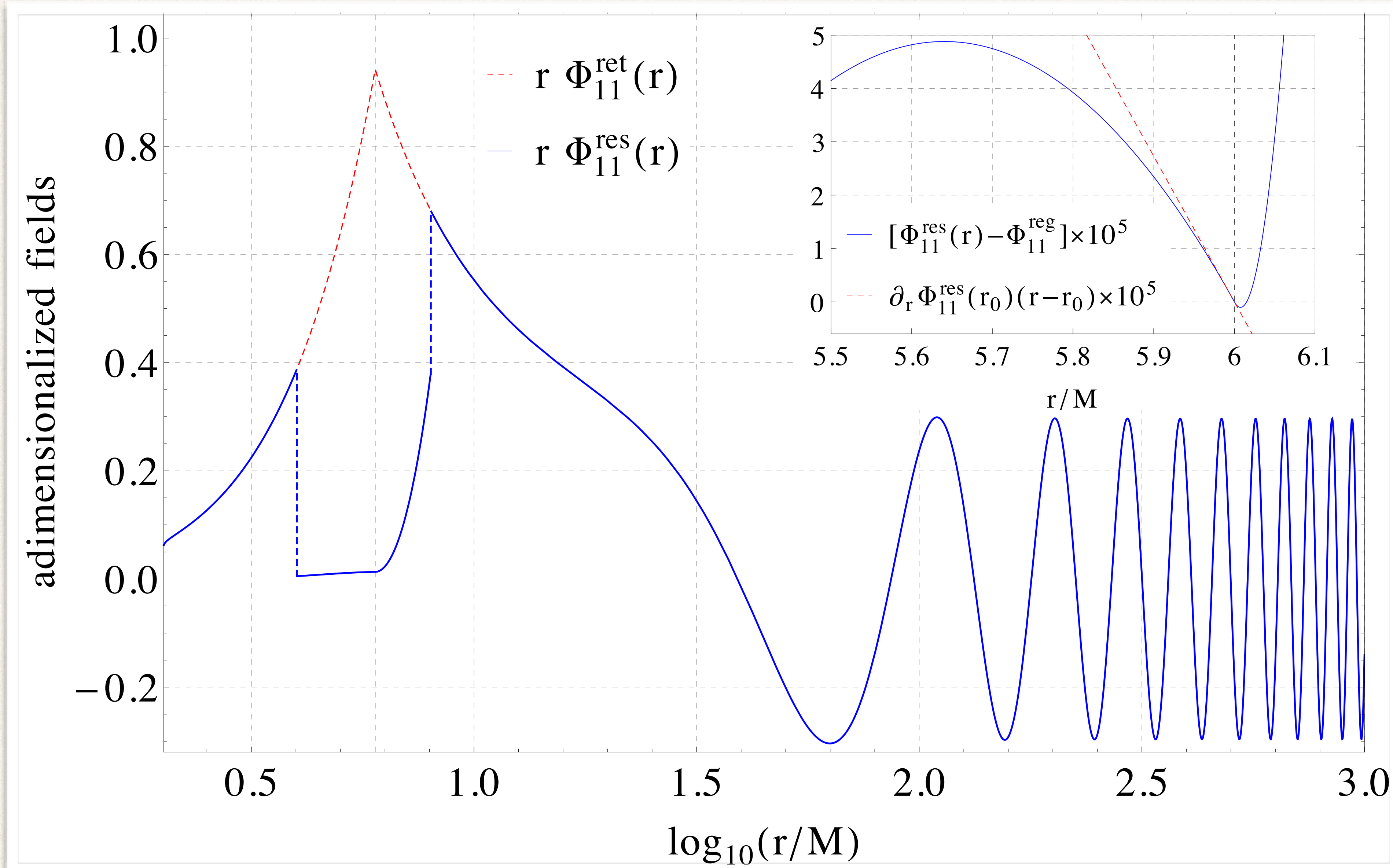
$$\Phi^{\mathrm{ret}}_{\ell m} = c^{+}_{\ell m}(r)\tilde{\Phi}^{\mathrm{ret}+}_{\ell m} + c^{-}_{\ell m}(r)\tilde{\Phi}^{\mathrm{ret}-}_{\ell m}$$

$$c^{+}_{\ell m}(r) = \int_{2M}^{r} \frac{\tilde{\phi}^{-}(r')}{W(r')} S^{\mathrm{eff}}_{\ell m}\, dr', \quad c^{-}_{\ell m}(r) = \int_{r}^{\infty} \frac{\tilde{\phi}^{+}(r')}{W(r')} S^{\mathrm{eff}}_{\ell m}\, dr'$$

where W is the Wronskian of homogeneous solutions

Results: scalar field

Phys. Rev. D. 89:044046
arXiv: 1311.3104



	r_0/M	eff. source$\times 10^3$	mode-sum$\times 10^3$	rel. diff.
$\Phi_0^{\rm res}$	6	5.454828078581	5.454828078597	3×10^{-12}
$\partial_r\Phi_0^{\rm res}$	6	0.16772830795	0.16772830804	5×10^{-10}
$\Phi_0^{\rm res}$	10	-1.049793165979	-1.049793165983	4×10^{-12}
$\partial_r\Phi_0^{\rm res}$	10	0.013784482250	0.013784482234	2×10^{-09}

Frequency-domain gravitational self-force (Lorenz gauge) [arXiv:1505.07841]

$$\Box \bar{h}_{\mu\nu} + 2R^{\alpha\ \beta}_{\ \mu\ \nu}\bar{h}_{\alpha\beta} = -16\pi T_{\mu\nu} \qquad \nabla_\mu \bar{h}^{\mu\nu} = 0$$

FD decomposition: $\bar{h}_{\mu\nu} = \frac{\mu}{r}\sum_{lm}\sum_{i=1}^{10} R^{(i)lm}(r) Y^{(i)lm}_{\mu\nu} e^{-im\Omega t}$

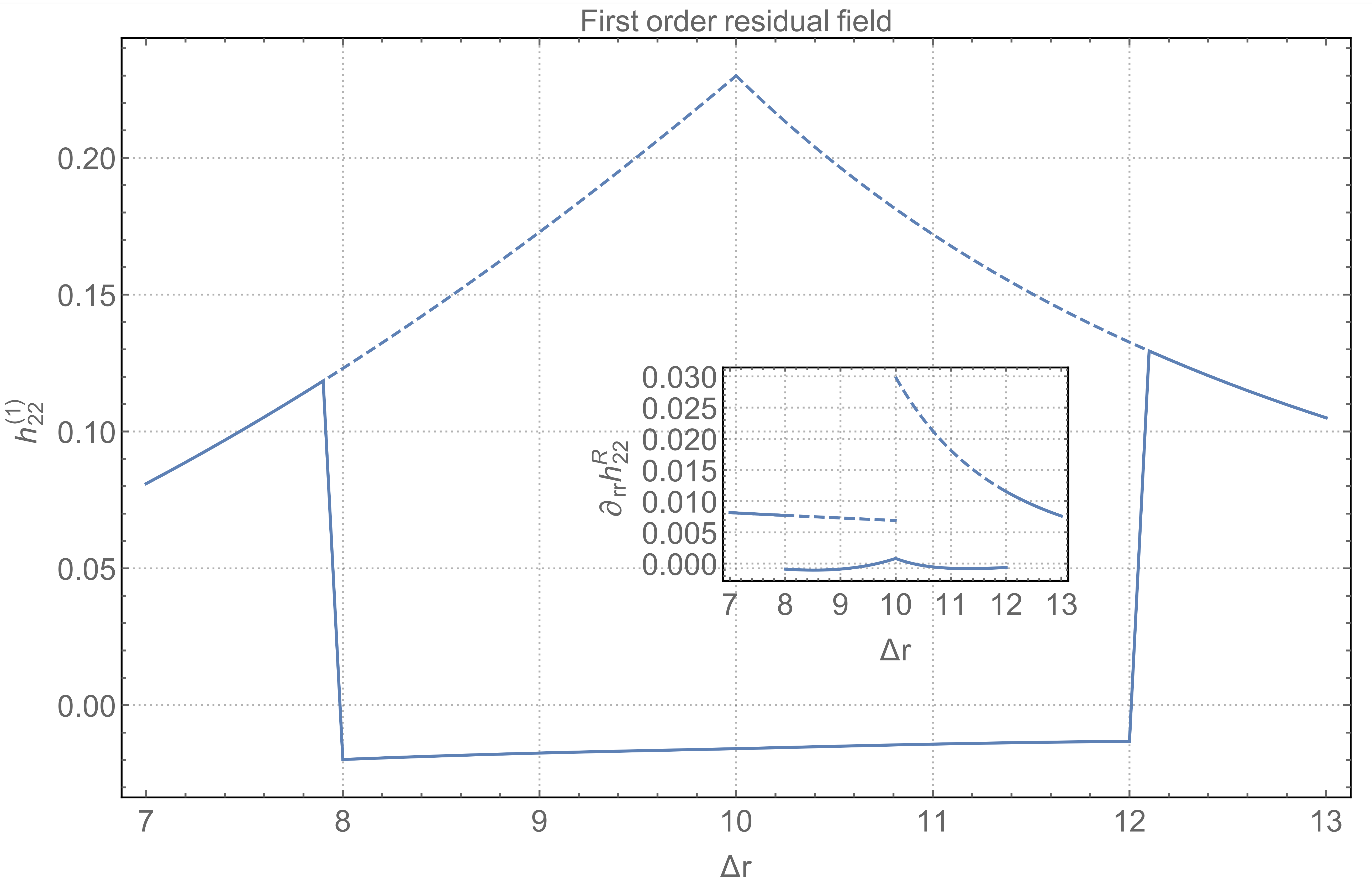
Radial field equations: $\Box_{lm} R^{(i)}_{lm} + 4\mathcal{M}^{(i)}_{\ (j)} R^{(j)}_{lm} = J^{(i)}_{lm}$

Decompose h^P into tensor harmonic modes $\bar{h}^{(i)P}_{lm}$

Construct effective source: $S^{(i)\text{eff}}_{lm} = \Box_{lm}\bar{h}^{(i)P}_{lm} + 4\mathcal{M}^{(i)}_{\ (j)}\bar{h}^{(i)P}_{lm}$

Variations of parameters to find the residual field

Results: Lorenz-gauge gravity [arXiv:1505.07841]



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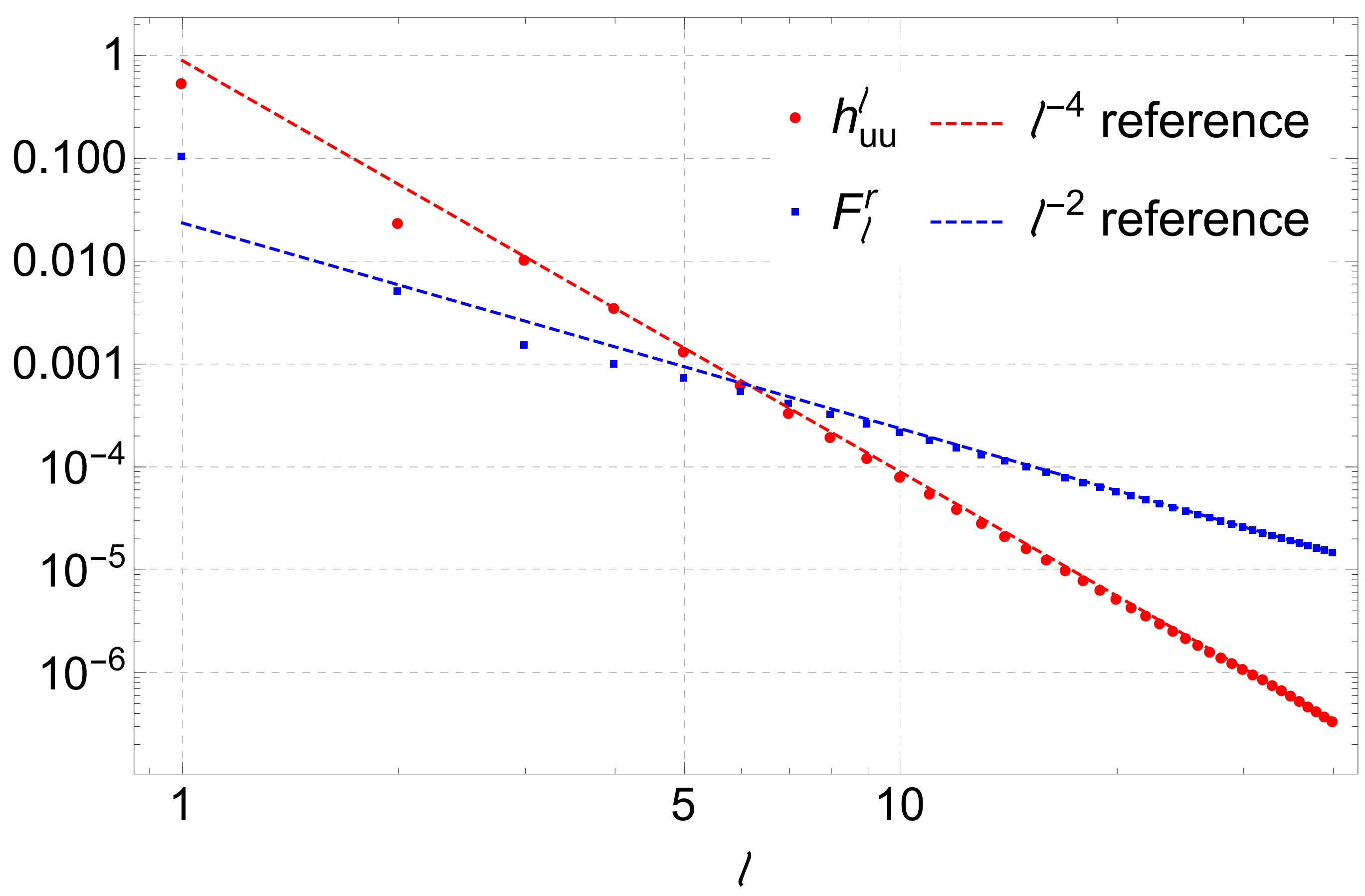


	r_0/M	this work	Akcay *et al.* [8, 30]	rel. diff.
h^R_{uu}	6	-1.047185497	$-1.0471854796(1)$	2×10^{-7}
F^r	6	2.446653×10^{-2}	$2.4466495(4) \times 10^{-2}$	2×10^{-6}
h^R_{uu}	10	-0.48925802	$-0.48925800172(4)$	4×10^{-8}
F^r	10	1.3389466×10^{-2}	$1.3389465(7) \times 10^{-2}$	3×10^{-8}

Tensor-mode regularisation

Punctures are directly relation to standard mode-sum regularisation

Subtract the the punctures from the individual lmi-modes of the retarded field

$$\bar{h}_{\mu\nu} = \frac{\mu}{r} \sum_{lm} \sum_{i=1}^{10} (R^{(i)}_{lm} - \bar{h}^{(i)P}_{lm}) Y^{(i)lm}_{\mu\nu} e^{-i\omega t}$$

Scalar:

$$\bar{h}^{(1)\mathrm{P}}_{\ell m} = 4(r_0 + \Delta r) D^{\ell}_{m,0} \frac{1}{\sqrt{\pi(2\ell+1)}} \left[\frac{(2\ell+1)(r_0 - 2M)\pi|\Delta r|}{r_0^{5/2}\sqrt{r_0 - 3M}} + \frac{2}{r_0^3}\sqrt{\frac{r_0 - 2M}{(r_0 - 3M)}} \left\{ 2r_0(r_0 - 2M)\mathcal{K} + \left[(r_0 - 2M)\mathcal{E} - 2(r_0 - 4M)\mathcal{K}\right]\Delta r \right\} \right]$$

Vector:

$$\bar{h}^{(4)\mathrm{P}}_{\ell m} = 4\left[D^{\ell}_{m,-1} - D^{\ell}_{m1}\right]\sqrt{\frac{\ell(\ell+1)}{\pi(2\ell+1)}} \left[\frac{(2\ell+1)\sqrt{M}\pi|\Delta r|}{r_0\sqrt{r_0 - 3M}} + 2\sqrt{\frac{r_0 - 2M}{M r_0^3 (r_0 - 3M)}} \left\{ 2r_0(r_0 - 2M)(\mathcal{E} - \mathcal{K}) + \frac{3M}{(2\ell - 1)(2\ell + 3)}\mathcal{K} + \left[2(r_0 - 2M)\big((r_0 - 5M)\mathcal{K} - (r_0 - 4M)\mathcal{E}\big) + \frac{3M}{(2\ell - 1)(2\ell + 3)}\big((r_0 - 2M)\mathcal{E} + 2M\mathcal{K}\big)\right]\Delta r \right\} \right]$$

Tensor-mode regularisation

Can re-write this as a mode-sum formula

$$\bar{h}^R_{\mu\nu} = \left[\sum_{l=0}^{\infty}\left(\sum_{mi}\frac{\mu}{r}R^{(i)}_{lm}Y^{(i)lm}_{\mu\nu}e^{-i\omega t}\right) - B\right] - D$$

Can regularize the tensor-harmonic modes directly - no mode coupling!

Compare with scalar-harmonic regularisation formula:

$$h^{lm}_{\alpha\beta}u^\alpha u^\beta = \left\{\mathcal{G}^{l+2,m}_{(+2)} + \mathcal{G}^{l+1,m}_{(+1)} + \mathcal{G}^{lm}_{(0)} + \mathcal{G}^{l-1,m}_{(-1)} + \mathcal{G}^{l-2,m}_{(-2)}\right\}Y^{lm}$$

$$\begin{aligned}
\mathcal{G}^{lm}_{(+2)} =& r^2(u^\varphi)^2\left[\alpha^{lm}_{(-2)}\bar{h}^{(3)} - \frac{(l-2)!}{(l+2)!}\left(\gamma^{lm}_{(-2)} - \beta^{lm}_{(-2)}\right)\bar{h}^{(7)}\right],\\
\mathcal{G}^{lm}_{(+1)} =& 2imr^2(u^\varphi)^2\frac{(l-2)!}{(l+2)!}\epsilon^{lm}_{(-1)}\bar{h}^{(10)} - \frac{2ru^tu^\varphi}{l(l+1)}\delta^{lm}_{(-1)}\bar{h}^{(8)} - \frac{2ru^ru^\varphi}{fl(l+1)}\delta^{lm}_{(-1)}\bar{h}^{(9)},\\
\mathcal{G}^{lm}_{(0)} =& \left(\bar{h}^{(1)} + f\bar{h}^{(6)}\right)(u^t)^2 + 2f^{-1}\bar{h}^{(2)}u^tu^r + f^{-2}\left(\bar{h}^{(1)} - f\bar{h}^{(6)}\right)(u^r)^2\\
&+ \frac{2imr\bar{h}^{(4)}}{l(l+1)}u^tu^\varphi + \frac{2imr\bar{h}^{(5)}}{fl(l+1)}u^ru^\varphi\\
&+ r^2(u^\varphi)^2\left[\alpha^{lm}_{(0)}\bar{h}^{(3)} - \frac{(l-2)!}{(l+2)!}\left(\gamma^{lm}_{(0)} - \beta^{lm}_{(0)} + m^2\right)\bar{h}^{(7)}\right],\\
\mathcal{G}^{lm}_{(-1)} =& 2imr^2\frac{(l-2)!}{(l+2)!}\epsilon^{lm}_{(+1)}\bar{h}^{(10)}(u^\varphi)^2 - \frac{2r\bar{h}^{(8)}}{l(l+1)}\delta^{lm}_{(+1)}u^tu^\varphi - \frac{2r\bar{h}^{(9)}}{fl(l+1)}\delta^{lm}_{(+1)}u^ru^\varphi,\\
\mathcal{G}^{lm}_{(-2)} =& r^2(u^\varphi)^2\left[\alpha^{lm}_{(+2)}\bar{h}^{(3)} - \frac{(l-2)!}{(l+2)!}\left(\gamma^{lm}_{(+2)} - \beta^{lm}_{(+2)}\right)\bar{h}^{(7)}\right].
\end{aligned}$$

$$\Lambda_1 = \frac{\ell(\ell+1)}{(2\ell-1)(2\ell+3)}$$

$$\Lambda_2 = \frac{(\ell-1)\ell(\ell+1)(\ell+1)}{(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)}$$

$$h^{[0]}_{tt} = \frac{4(r_0-M)\mathcal{K}}{\pi\, r_0^2}\sqrt{\frac{r_0-2M}{r_0-3M}}, \tag{6.11a}$$

$$h^{[0]}_{t\varphi} = -\frac{32M^{1/2}\mathcal{K}}{\pi\, r_0^{1/2}}\sqrt{\frac{r_0-2M}{r_0-3M}}\Lambda_1, \tag{6.11b}$$

$$h^{[0]}_{rr} = \frac{4\mathcal{K}}{\pi}\frac{(r_0-3M)^{1/2}}{(r_0-2M)^{3/2}}, \tag{6.11c}$$

$$h^{[0]}_{\theta\theta} = \frac{4r_0\mathcal{K}}{\pi}\sqrt{\frac{r_0-2M}{r_0-3M}} - \frac{64Mr_0\mathcal{K}}{\pi(r_0-2M)^{1/2}(r_0-3M)^{1/2}}\Lambda_2, \tag{6.11d}$$

$$h^{[0]}_{\varphi\varphi} = \frac{4r_0\mathcal{K}}{\pi}\sqrt{\frac{r_0-2M}{r_0-3M}} + \frac{64Mr_0\mathcal{K}}{\pi(r_0-2M)^{1/2}(r_0-3M)^{1/2}}\Lambda_2, \tag{6.11e}$$

$$h^{[-1]}_{tt,r} = \mp\frac{(r_0-M)}{r_0^{5/2}(r_0-3M)^{1/2}}, \tag{6.11f}$$

$$h^{[0]}_{tt,r} = \frac{2(r_0-M)[(r_0-2M)\mathcal{E}-2(r_0-4M)\mathcal{K}]}{\pi r_0^3(r_0-3M)^{1/2}(r_0-2M)^{1/2}}, \tag{6.11g}$$

$$h^{[-1]}_{rr,r} = \mp\frac{(r_0-3M)^{1/2}}{r_0^{1/2}(r_0-2M)^2}, \tag{6.11h}$$

$$h^{[0]}_{rr,r} = \frac{2(r_0-3M)^{1/2}[(r_0-2M)\mathcal{E}-2r_0\mathcal{K}]}{\pi r_0(r_0-2M)^{5/2}}, \tag{6.11i}$$

$$h^{[-1]}_{t\varphi,r} = \pm\left[\frac{2M^{1/2}}{r_0(r_0-3M)^{1/2}}\right]_{\ell\geq 1}, \tag{6.11j}$$

$$h^{[0]}_{t\varphi,r} = -\frac{16M^{1/2}[(r_0-2M)\mathcal{E}+2M\mathcal{K}]}{\pi r_0^{3/2}(r_0-3M)^{1/2}(r_0-2M)^{1/2}}\Lambda_1, \tag{6.11k}$$

$$h^{[-1]}_{\varphi\varphi,r} = \mp\sqrt{\frac{r_0}{r_0-3M}} \mp \left[\frac{Mr_0^{1/2}}{(r_0-2M)(r_0-3M)^{1/2}}\right]_{\ell\geq 2}, \tag{6.11l}$$

$$h^{[0]}_{\varphi\varphi,r} = \frac{2(\mathcal{E}+2\mathcal{K})}{\pi}\sqrt{\frac{r_0-2M}{r_0-3M}} + \frac{32M(\mathcal{E}+2\mathcal{K})}{\pi(r_0-3M)^{1/2}(r_0-2M)^{1/2}}\Lambda_2, \tag{6.11m}$$

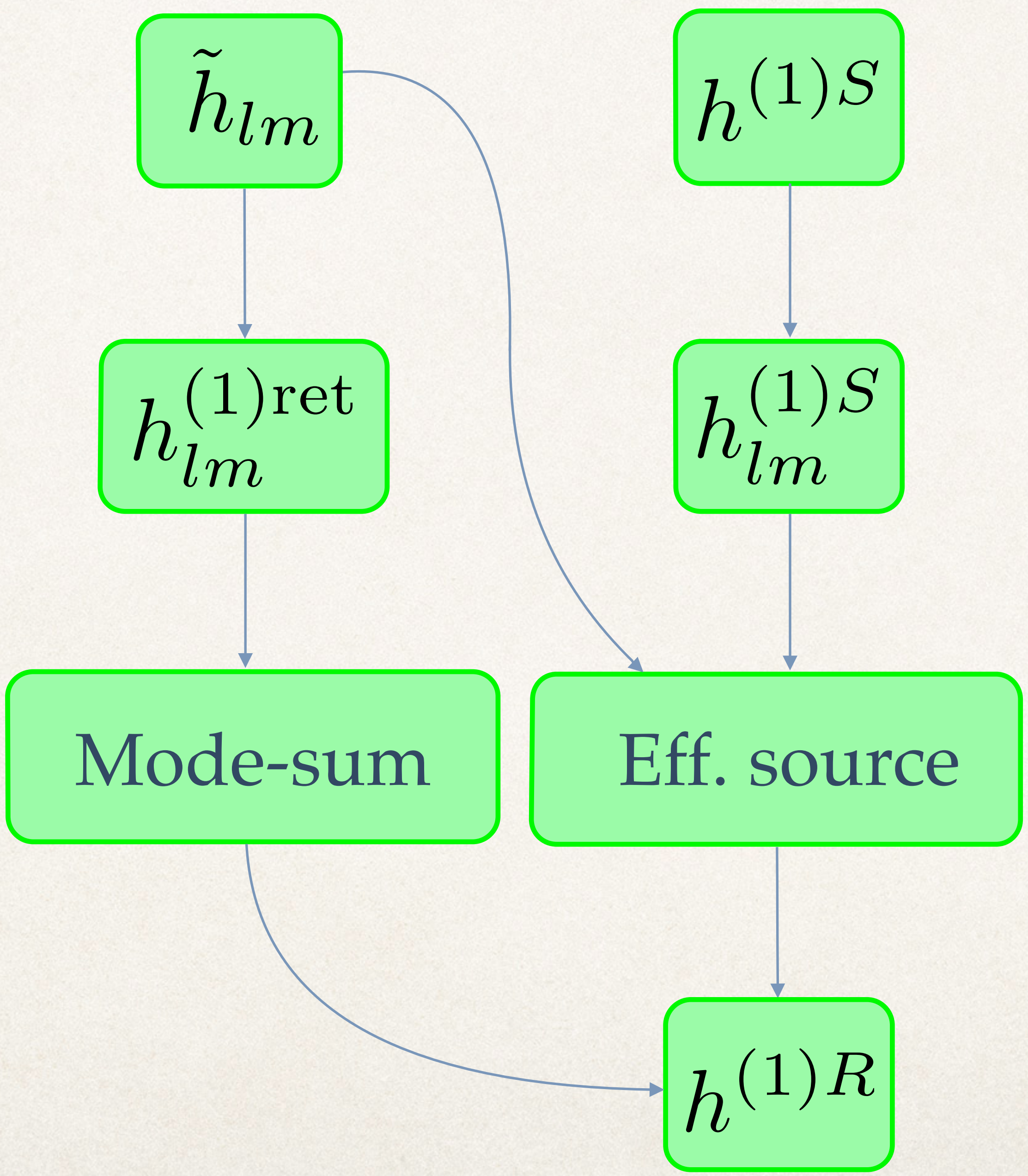
$$h^{[-1]}_{\theta\theta,r} = \mp\sqrt{\frac{r_0}{r_0-3M}} \pm \left[\frac{Mr_0^{1/2}}{(r_0-2M)(r_0-3M)^{1/2}}\right]_{\ell\geq 2}, \tag{6.11n}$$

$$h^{[0]}_{\theta\theta,r} = \frac{2(\mathcal{E}+2\mathcal{K})}{\pi}\sqrt{\frac{r_0-2M}{r_0-3M}} - \frac{32M(\mathcal{E}+2\mathcal{K})}{\pi(r_0-3M)^{1/2}(r_0-2M)^{1/2}}\Lambda_2, \tag{6.11o}$$

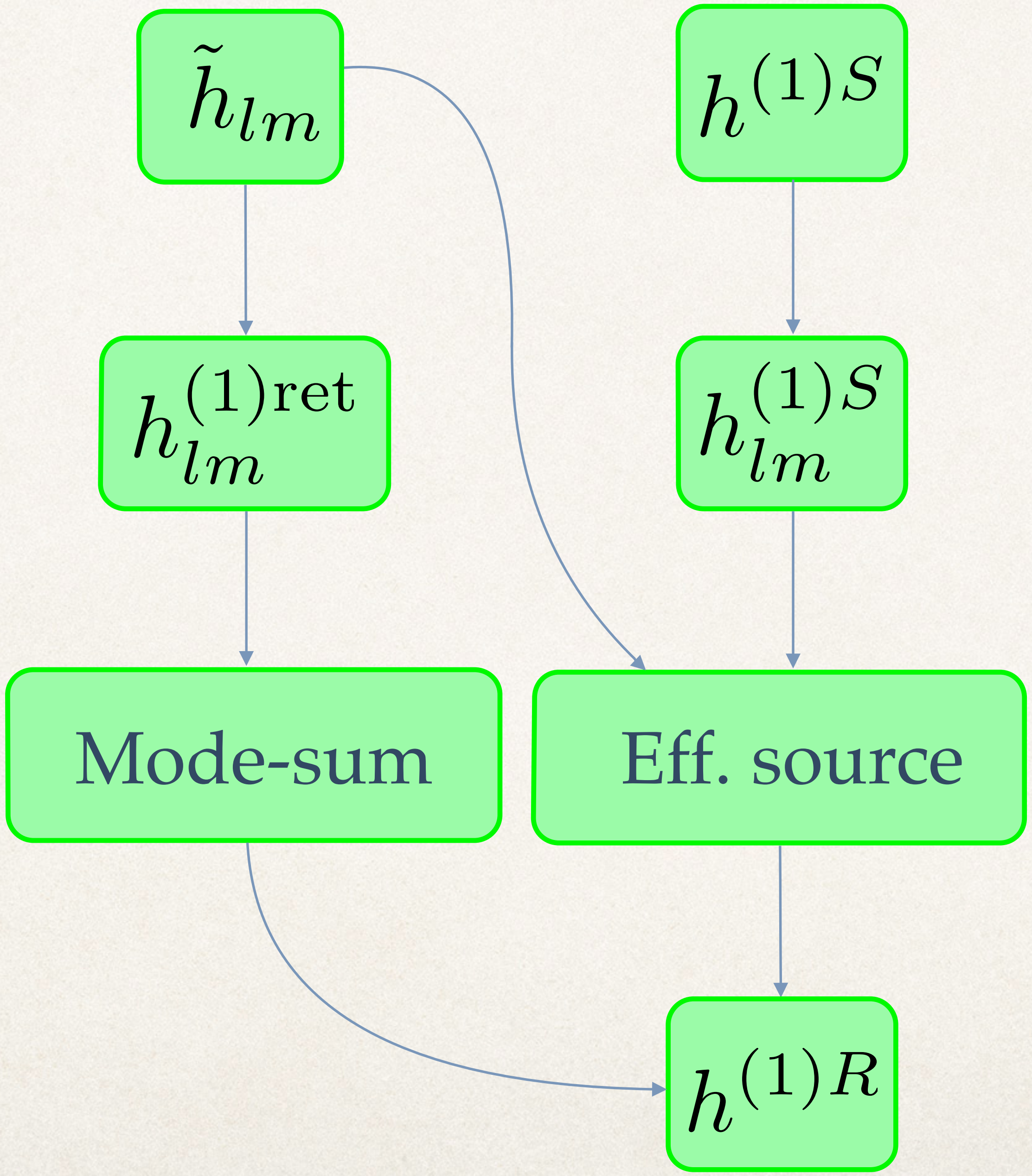
$$h^{[0]}_{tr,\varphi} = -\frac{32((r_0-2M)\mathcal{E}-(r_0-3M)\mathcal{K})}{\pi M^{1/2}r_0^{3/2}}\sqrt{\frac{r_0-2M}{r_0-3M}}\Lambda_1, \tag{6.11p}$$

$$h^{[0]}_{r\varphi,\varphi} = \frac{16[(r_0-2M)\mathcal{E}-(r_0-3M)\mathcal{K}]}{\pi(r_0-2M)^{1/2}(r_0-3M)^{1/2}}(\Lambda_1+4\Lambda_2), \tag{6.11q}$$

Towards second order self-force



Towards second order self-force



PHYSICAL REVIEW D **89,** 104020 (2014)

Practical, covariant puncture for second-order self-force calculations

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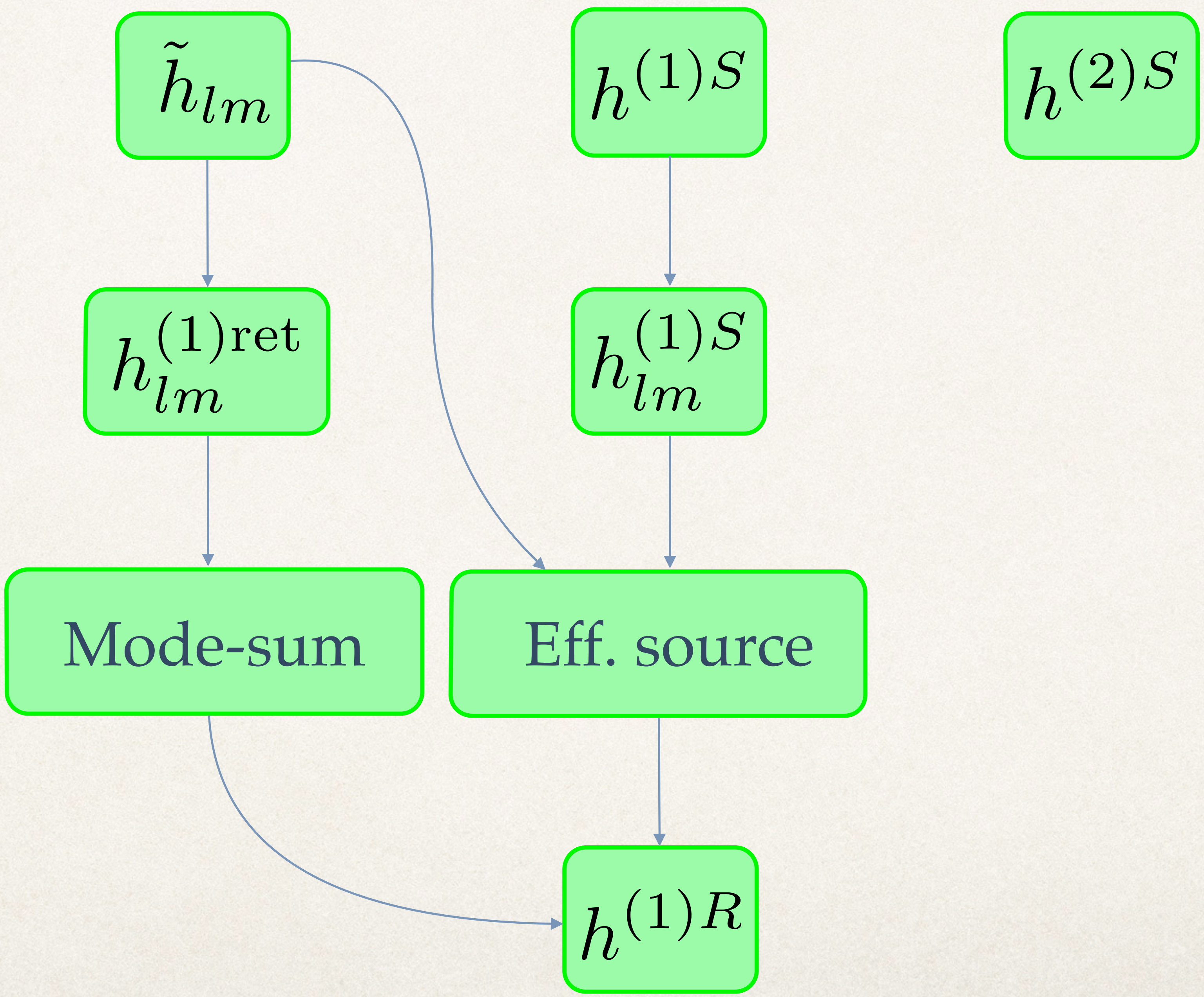
(Received 7 March 2014; published 13 May 2014)

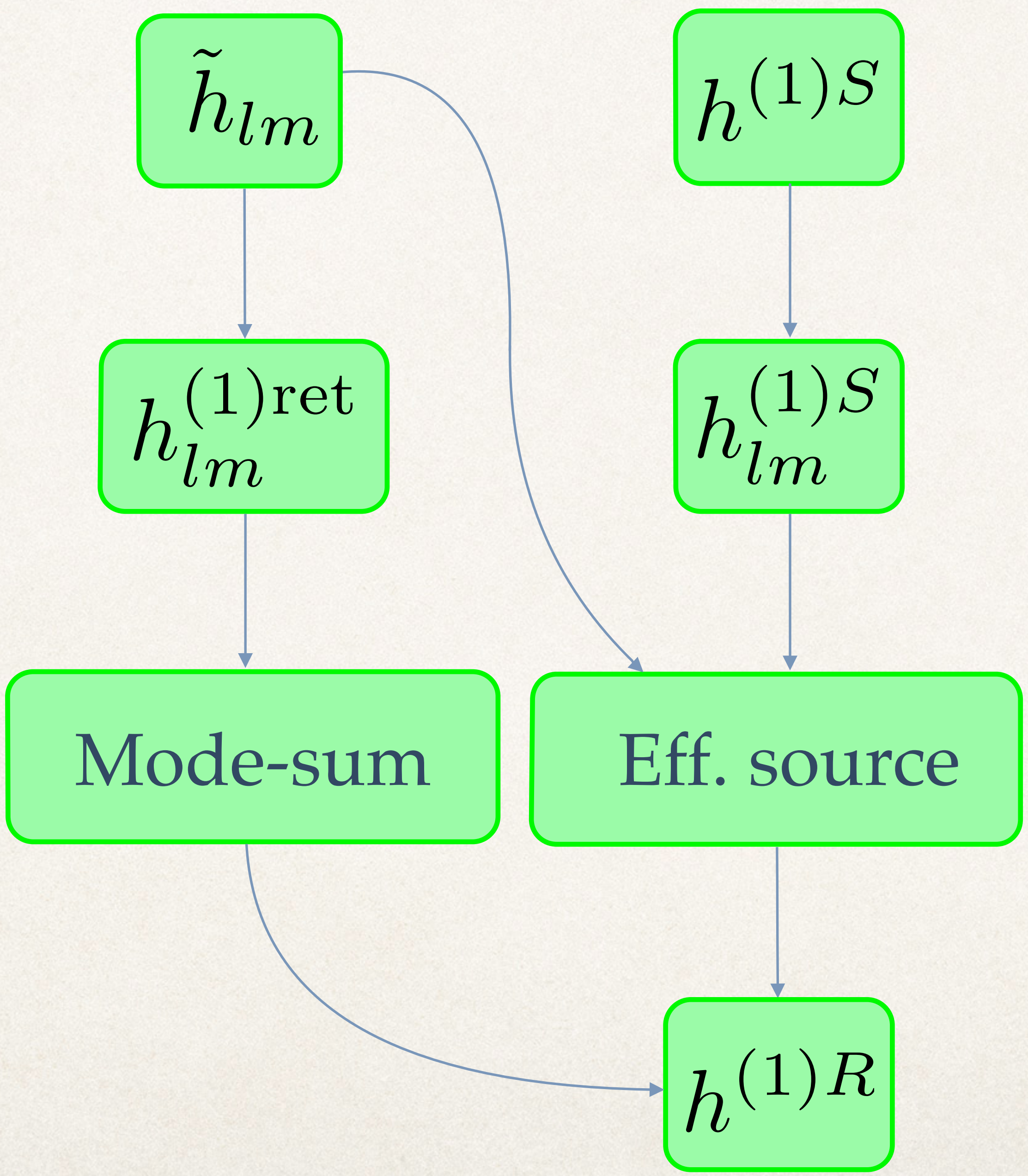
Accurately modeling an extreme-mass-ratio inspiral requires knowledge of the second-order gravitational self-force on the inspiraling small object. Recently, numerical puncture schemes have been formulated to calculate this force, and their essential analytical ingredients have been derived from first principles. However, the "puncture," a local representation of the small object's self-field, in each of these schemes has been presented only in a local coordinate system centered on the small object, while a numerical implementation will require the puncture in coordinates covering the entire numerical domain. In this paper we provide an explicit covariant self-field as a local expansion in terms of Synge's world function. The self-field is written in the Lorenz gauge, in an arbitrary vacuum background, and in forms suitable for both self-consistent and Gralla-Wald-type representations of the object's trajectory. We illustrate the local expansion's utility by sketching the procedure of constructing from it a numerically practical puncture in any chosen coordinate system.

DOI: 10.1103/PhysRevD.89.104020 PACS numbers: 04.20.-q, 04.25.-g, 04.25.Nx, 04.30.Db

Towards second order self-force



Towards second order self-force



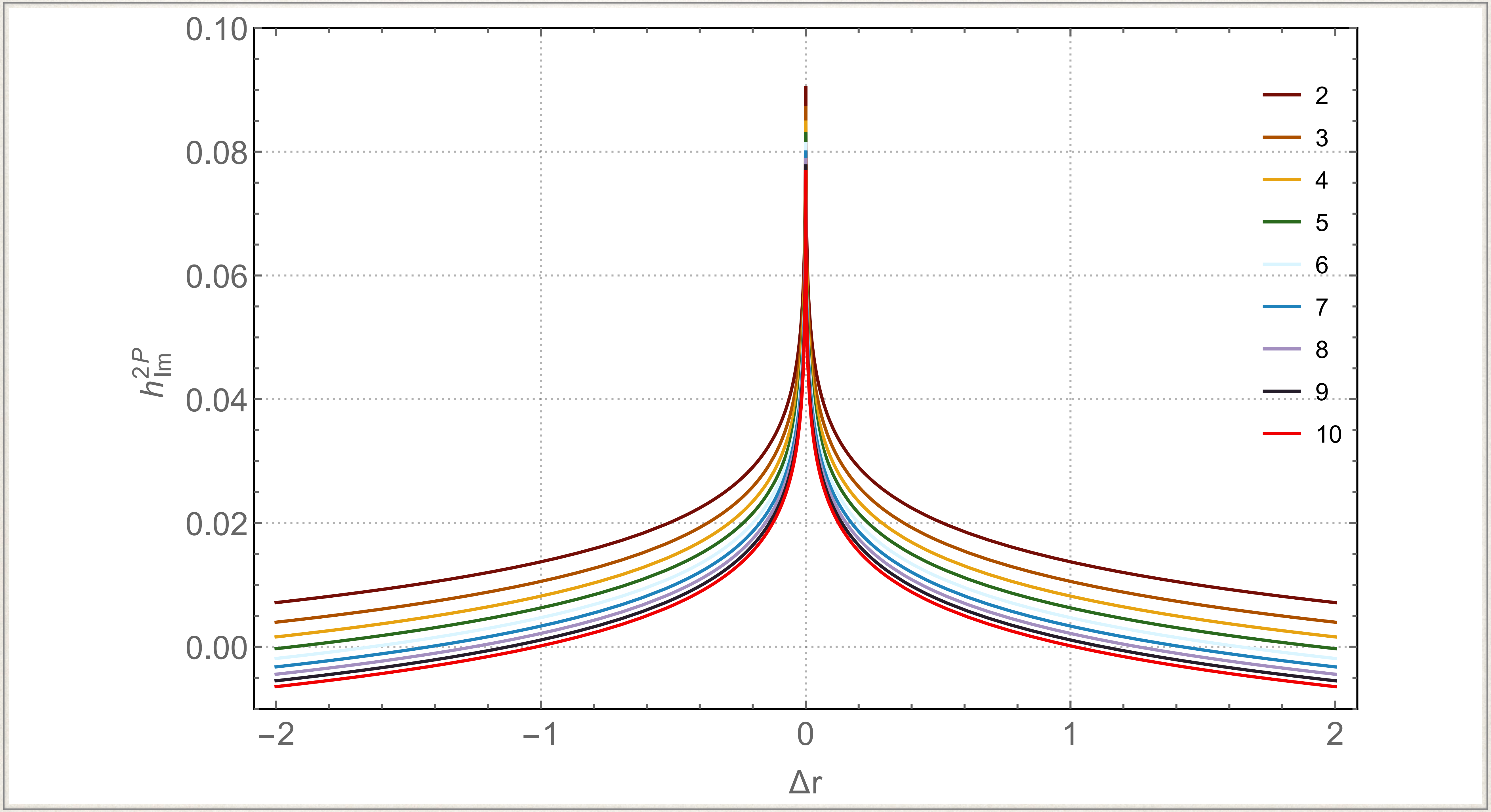
$$h^{S2}_{\ell m}$$

$$
\begin{aligned}
&\frac{14\,(r_0-3\,M)^2\,\text{Hypergeometric2F1}^{(0,1,0,0)}\!\left(\frac{1}{2},1,1,\frac{M}{r_0-2\,M}\right)}{(r_0-2\,M)^2}+\frac{2\,(3\,M\,(3\,M-14)-2\,(M-7)\,r_0)\,\text{Hypergeometric2F1}^{(0,1,0,0)}\!\left(\frac{1}{2},2,1,\frac{M}{r_0-2\,M}\right)}{2\,M-r_0}+\\
&\frac{4\,H_l\sqrt{\frac{r_0-3\,M}{r_0-2\,M}}\,(3\,M\,(3\,M-14)-2\,(M-7)\,r_0)}{3\,M-r_0}+\frac{14\,H_l\,(5\,M-2\,r_0)\sqrt{\frac{r_0-3\,M}{r_0-2\,M}}}{2\,M-r_0}+28\,H_l+\\
&\log(\Delta\mathrm{r})\left(\frac{14\sqrt{\frac{r_0-3\,M}{r_0-2\,M}}\,(5\,M-2\,r_0)}{2\,M-r_0}+\frac{4\sqrt{\frac{r_0-3\,M}{r_0-2\,M}}\,(3\,M\,(3\,M-14)-2\,(M-7)\,r_0)}{3\,M-r_0}+28\right)+\\
&\frac{7\,(5\,M-2\,r_0)\sqrt{\frac{r_0-3\,M}{r_0-2\,M}}}{2\,M-r_0}-28\sqrt{\frac{r_0-3\,M}{r_0-2\,M}}-14\log\!\left(\frac{4\,r_0\,(r_0-2\,M)^2}{r_0-3\,M}\right)+28\left(\log(2)-\log\!\left(\sqrt{\frac{r_0-3\,M}{r_0-2\,M}}+1\right)\right)-\\
&\frac{2\sqrt{\frac{r_0-3\,M}{r_0-2\,M}}\,(3\,M\,(3\,M-14)-2\,(M-7)\,r_0)\log\!\left(\frac{4\,r_0\,(r_0-2\,M)^2}{r_0-3\,M}\right)}{3\,M-r_0}-\frac{7\,(5\,M-2\,r_0)\sqrt{\frac{r_0-3\,M}{r_0-2\,M}}\log\!\left(\frac{4\,r_0\,(r_0-2\,M)^2}{r_0-3\,M}\right)}{2\,M-r_0}+14
\end{aligned}
$$

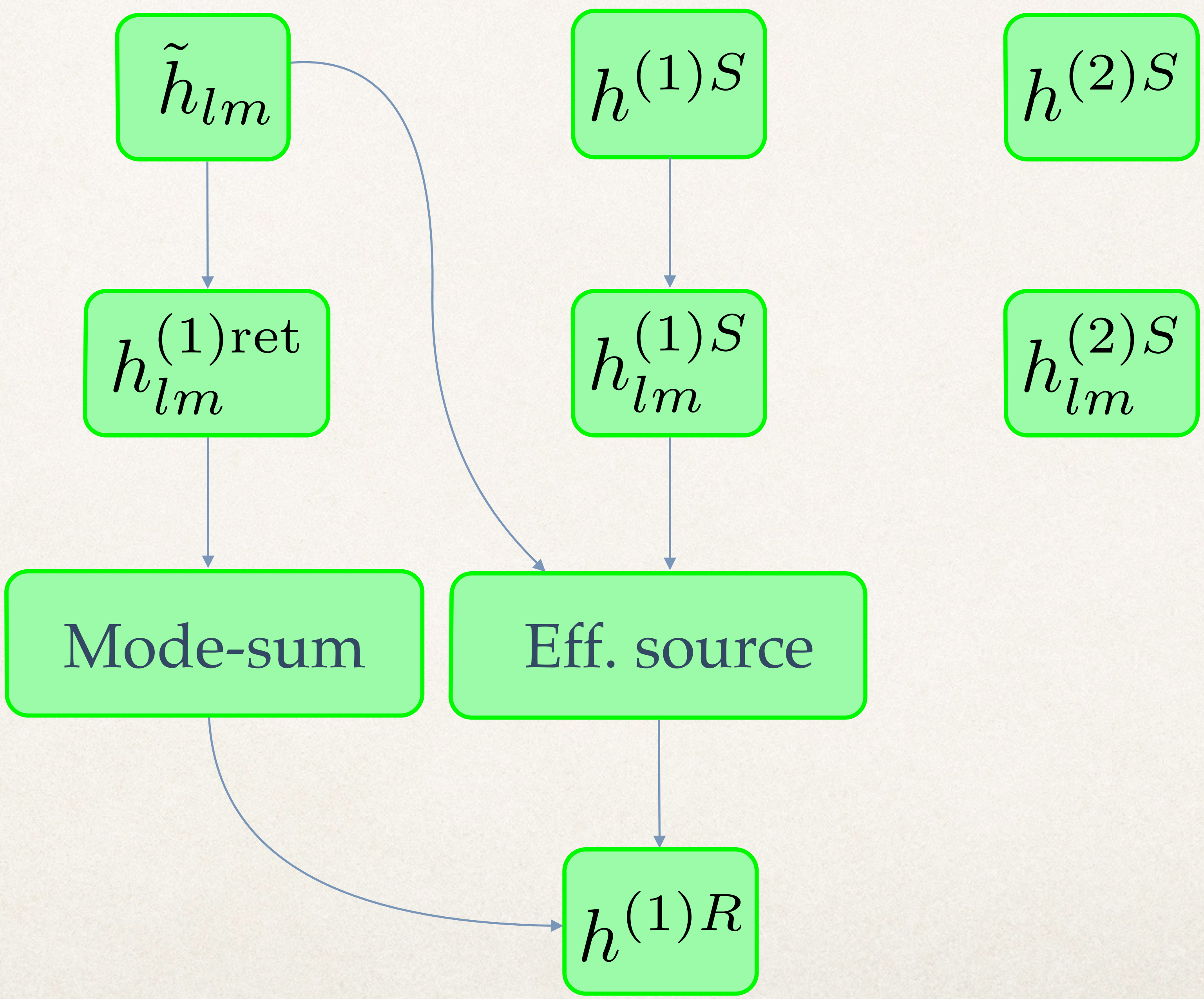
$h_{\ell m}^{S2}$

$$c_1 + c_2 \log \ell + c_3 \log \Delta r + c_4 \ell^{-1} + \cdots$$

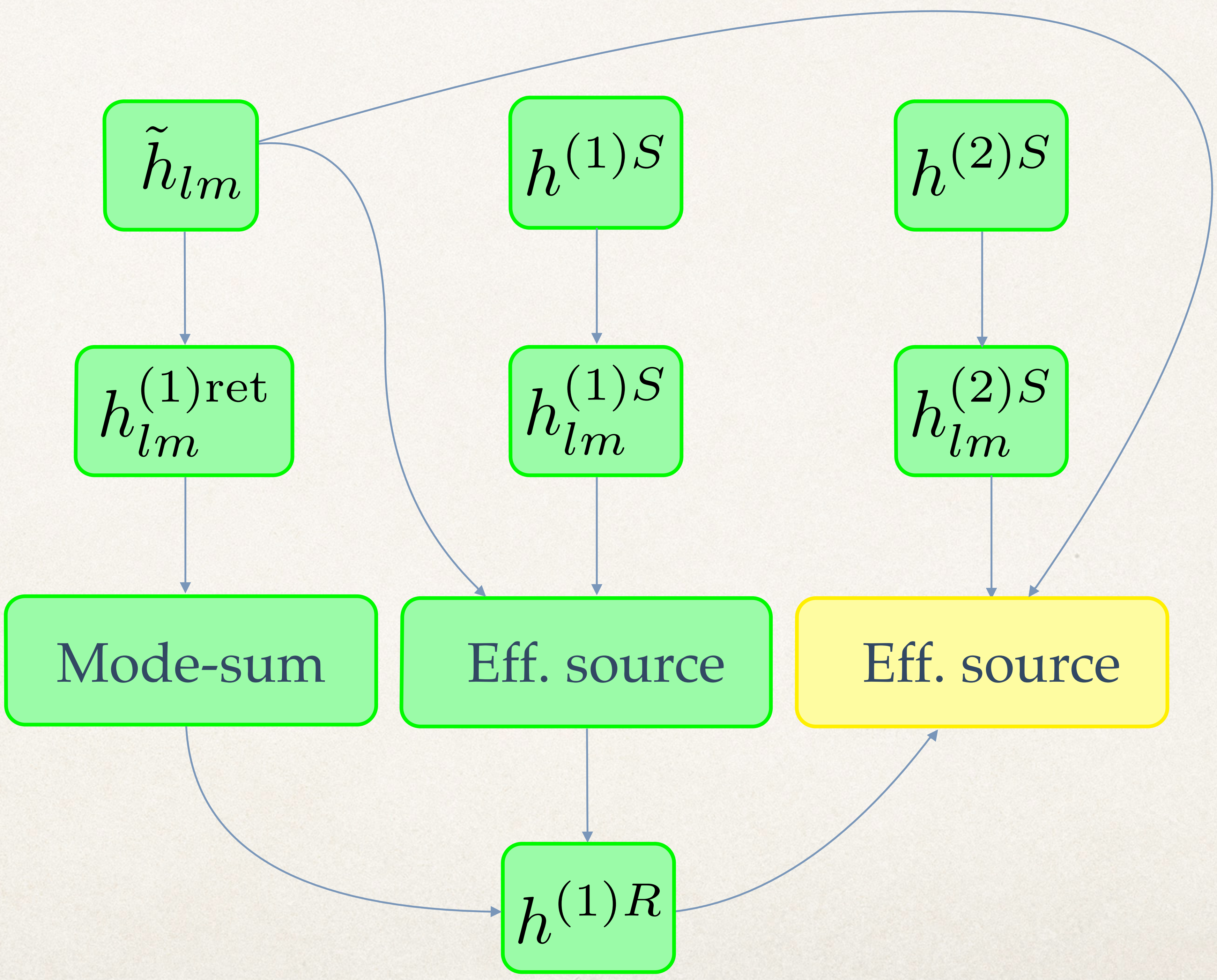
$$h_{\ell m}^{S2}$$



Towards second order self-force



Towards second order self-force



Second order effective source

$$\mathcal{D}_{\mu\nu}[h^{\mathrm{R1}}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S1}}]$$

$$\mathcal{D}_{\mu\nu}[h^{\mathrm{R2}}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S2}}] + \delta^2 R_{\mu\nu}[h^1, h^1]$$

$$\begin{aligned}\delta^2 R_{\alpha\beta}[h,h] \equiv &- \tfrac{1}{2}h^{\mu\nu}(2h_{\mu(\alpha;\beta)\nu} - h_{\alpha\beta;\mu\nu} - h_{\mu\nu;\alpha\beta}) \\ &+ \tfrac{1}{4}h^{\mu\nu}{}_{;\alpha}h_{\mu\nu;\beta} + \tfrac{1}{2}h^{\mu}{}_{\beta}{}^{;\nu}(h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}) \\ &- \tfrac{1}{2}\bar{h}^{\mu\nu}{}_{;\nu}(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu})\end{aligned}$$

$$\delta^2 R_{\mu\nu}[h^1, h^1] \;=\; \sum_{i\ell m} \delta^2 R_{i\ell m}(r;\hat{r}) e^{-im\Omega t} Y^{i\ell m}_{\mu\nu}(r, \theta^A)$$

Second order effective source

$$\boxed{\begin{aligned}\mathcal{D}_{\mu\nu}[h^{\mathrm{R1}}] &= -\mathcal{D}_{\mu\nu}[h^{\mathrm{S1}}] \\ \mathcal{D}_{\mu\nu}[h^{\mathrm{R2}}] &= -\mathcal{D}_{\mu\nu}[h^{\mathrm{S2}}] + \delta^2 R_{\mu\nu}[h^1, h^1]\end{aligned}}$$

$$\boxed{\begin{aligned}\delta^2 R_{\alpha\beta}[h,h] \equiv &- \tfrac{1}{2} h^{\mu\nu}\left(2h_{\mu(\alpha;\beta)\nu} - h_{\alpha\beta;\mu\nu} - h_{\mu\nu;\alpha\beta}\right) \\ &+ \tfrac{1}{4} h^{\mu\nu}{}_{;\alpha} h_{\mu\nu;\beta} + \tfrac{1}{2} h^{\mu}{}_{\beta}{}^{;\nu}\left(h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}\right) \\ &- \tfrac{1}{2} \bar{h}^{\mu\nu}{}_{;\nu}\left(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}\right)\end{aligned}}$$

$$\boxed{\delta^2 R_{\mu\nu}[h^1, h^1] \;=\; \sum_{i\ell m} \delta^2 R_{i\ell m}(r;\hat{r}) e^{-im\Omega t} Y^{i\ell m}_{\mu\nu}(r, \theta^A)}$$

Second order effective source

$$\mathcal{D}_{\mu\nu}[h^{\mathrm{R}1}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S}1}]$$

$$\mathcal{D}_{\mu\nu}[h^{\mathrm{R}2}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S}2}] + \delta^2 R_{\mu\nu}[h^1, h^1]$$

$$\begin{aligned}\delta^2 R_{\alpha\beta}[h,h] \equiv & -\tfrac{1}{2}h^{\mu\nu}\left(2h_{\mu(\alpha;\beta)\nu} - h_{\alpha\beta;\mu\nu} - h_{\mu\nu;\alpha\beta}\right) \\ & + \tfrac{1}{4}h^{\mu\nu}{}_{;\alpha}h_{\mu\nu;\beta} + \tfrac{1}{2}h^{\mu}{}_{\beta}{}^{;\nu}\left(h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}\right) \\ & - \tfrac{1}{2}\bar{h}^{\mu\nu}{}_{;\nu}\left(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}\right)\end{aligned}$$

$$\delta^2 R_{\mu\nu}[h^1, h^1] \;=\; \sum_{i\ell m} \delta^2 R_{i\ell m}(r;\hat{r})\, e^{-im\Omega t}\, Y^{i\ell m}_{\mu\nu}(r,\theta^A)$$

Second order effective source

$$\mathcal{D}_{\mu\nu}[h^{\mathrm{R1}}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S1}}]$$

$$\mathcal{D}_{\mu\nu}[h^{\mathrm{R2}}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S2}}] + \delta^2 R_{\mu\nu}[h^1, h^1]$$

$$\begin{aligned}\delta^2 R_{\alpha\beta}[h,h] \equiv & -\tfrac{1}{2} h^{\mu\nu}\left(2h_{\mu(\alpha;\beta)\nu} - h_{\alpha\beta;\mu\nu} - h_{\mu\nu;\alpha\beta}\right)\\ & + \tfrac{1}{4} h^{\mu\nu}{}_{;\alpha} h_{\mu\nu;\beta} + \tfrac{1}{2} h^{\mu}{}_{\beta}{}^{;\nu}\left(h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}\right)\\ & - \tfrac{1}{2}\bar{h}^{\mu\nu}{}_{;\nu}\left(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}\right)\end{aligned}$$

$$\delta^2 R_{\mu\nu}[h^1, h^1] \;=\; \sum_{i\ell m} \delta^2 R_{i\ell m}(r;\hat{r}) e^{-im\Omega t} Y^{i\ell m}_{\mu\nu}(r, \theta^A)$$

Second order effective source

$$\mathcal{D}_{\mu\nu}[h^{\mathrm{R1}}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S1}}]$$
$$\mathcal{D}_{\mu\nu}[h^{\mathrm{R2}}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S2}}] + \boxed{\delta^2 R_{\mu\nu}[h^1, h^1]}$$

$$\begin{aligned}\delta^2 R_{\alpha\beta}[h,h] \equiv & -\tfrac{1}{2} h^{\mu\nu}\left(2h_{\mu(\alpha;\beta)\nu} - h_{\alpha\beta;\mu\nu} - h_{\mu\nu;\alpha\beta}\right)\\ & + \tfrac{1}{4} h^{\mu\nu}{}_{;\alpha} h_{\mu\nu;\beta} + \tfrac{1}{2} h^{\mu}{}_{\beta}{}^{;\nu}\left(h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}\right)\\ & - \tfrac{1}{2} \bar{h}^{\mu\nu}{}_{;\nu}\left(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}\right)\end{aligned}$$

$$\delta^2 R_{\mu\nu}[h^1, h^1] = \sum_{i\ell m} \delta^2 R_{i\ell m}(r;\hat{r}) e^{-im\Omega t} Y^{i\ell m}_{\mu\nu}(r, \theta^A)$$

Second order Ricci tensor

$$\delta^2 R_{\alpha\beta}[h^{1\mathrm{ret}}, h^{1\mathrm{ret}}] =$$

$$\delta^2 R_{\alpha\beta}[h^{1\mathrm{R}}, h^{1\mathrm{R}}] \Longleftarrow$$ mode coupling

$$+ 2\delta^2 R_{\alpha\beta}[h^{1\mathrm{R}}, h^{1\mathrm{S}}] \Longleftarrow$$ mode coupling

$$+ \ \delta^2 R_{\alpha\beta}[h^{1\mathrm{S}}, h^{1\mathrm{S}}] \Longleftarrow$$ mode decomposition (c.f. $h^{\mathrm{S}2}$)

Mode coupling

$$\delta^2 R_{\mu\nu}[h^1, h^1] \; = \; \sum_{i\ell m} \delta^2 R_{i\ell m}(r;\hat{r}) e^{-im\Omega t} Y^{i\ell m}_{\mu\nu}(r, \theta^A)$$

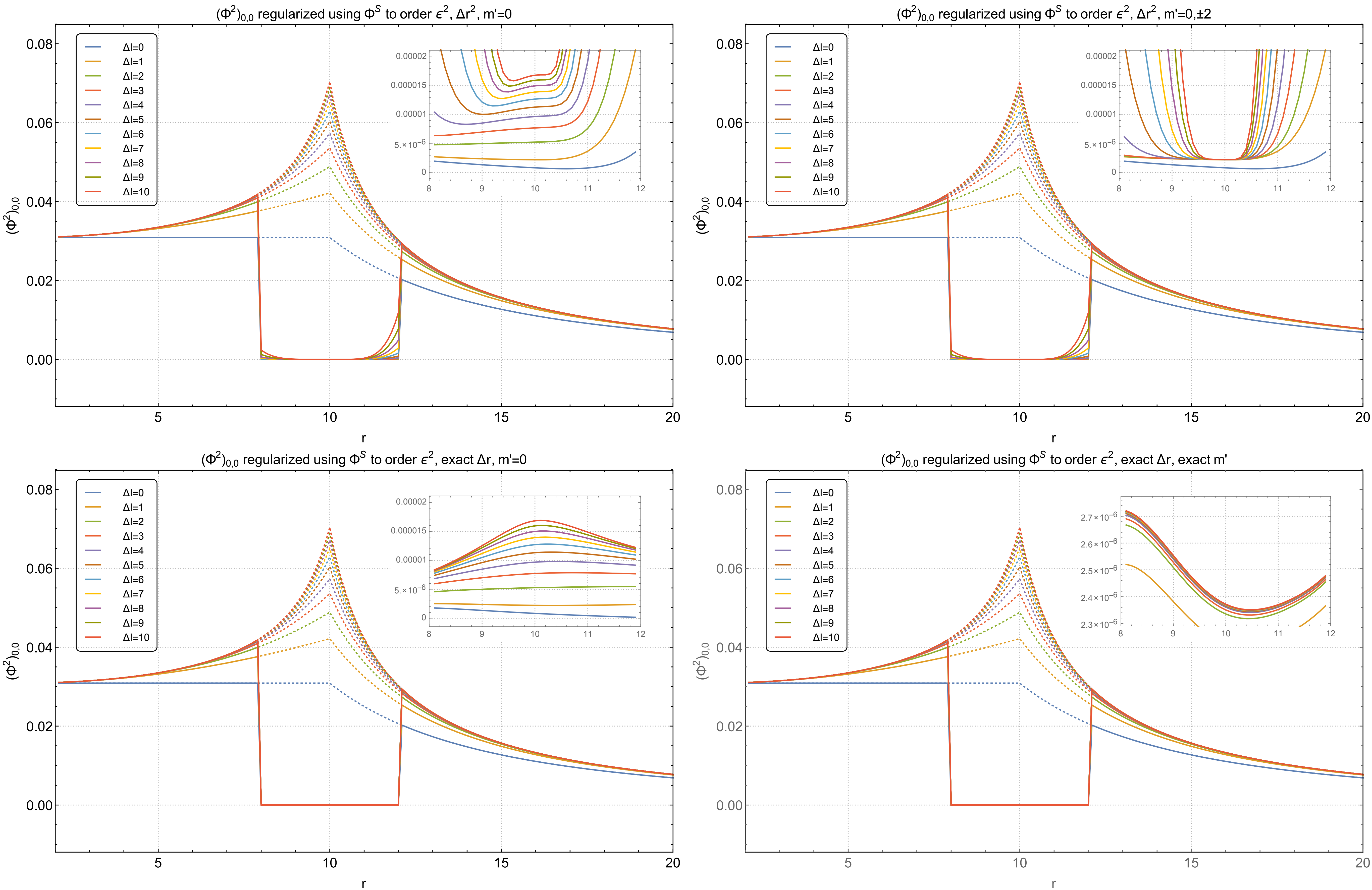
$$\delta^2 R_{i\ell m} = \sum_{\substack{i'\ell' m' \\ i''\ell'' m''}} \mathcal{D}^{i'\ell' m' i''\ell'' m''}_{i\ell m} \left[h_{1 i'\ell' m'}, h_{1 i''\ell'' m''}\right]$$

Which parts of the mode-decomposed singular field are needed?

Which parts of the mode-decomposed singular field are needed?

All of them!

[to O(ϵ)]



(Φ²)₀,₀ regularized using Φ^S to order ε², Δr², m'=0
(Φ²)₀,₀ regularized using Φ^S to order ε², Δr², m'=0,±2
(Φ²)₀,₀ regularized using Φ^S to order ε², exact Δr, m'=0
(Φ²)₀,₀ regularized using Φ^S to order ε², exact Δr, exact m'
Δl=0
Δl=1
Δl=2
Δl=3
Δl=4
Δl=5
Δl=6
Δl=7
Δl=8
Δl=9
Δl=10
(Φ²)₀,₀
r

Exact, analytic mode decomposition

For the scalar case, the modes of the singular field are given by

$$\Phi^{\rm S}_{\ell m'} = \int_0^{2\pi}\int_0^{\pi} \Phi^{\rm S}(\alpha,\beta) Y^*_{\ell m'}(\alpha,\beta)\sin\alpha \, d\alpha \, d\beta.$$

$$\Phi^{\rm S}_{\ell m} = \sum_{m'} \Phi^{\rm S}_{\ell m'} D^{\ell}_{m m'}(\pi, \pi/2, \pi/2)$$

Similar in the gravitational case

$$\bar{h}^{(i)P}_{\ell m} = \frac{r}{\mu \, a^{(i)}_\ell} \int_0^{2\pi}\int_0^{\pi} \bar{h}_{\tau\kappa}\eta^{\tau\mu}\eta^{\kappa\nu} Y^{(i)\ell m *}_{\mu\nu} \sin\alpha \, d\alpha \, d\beta.$$

$$f_{\ell m}(\theta,\varphi) = \sum_{m'=-\ell}^{\ell} D^{\ell}_{m m'}(\alpha,\beta,\gamma) f_{\ell m'}(\theta',\varphi'),$$

$$X^{\ell m}_A(\theta,\varphi) = \frac{\partial x^{A'}}{\partial x^A} \sum_{m'=-\ell}^{\ell} D^{\ell}_{m m'}(\alpha,\beta,\gamma) X^{\ell m'}_{A'}(\theta',\varphi'),$$

$$Z^{\ell m}_A(\theta,\varphi) = \frac{\partial x^{A'}}{\partial x^A} \sum_{m'=-\ell}^{\ell} D^{\ell}_{m m'}(\alpha,\beta,\gamma) Z^{\ell m'}_{A'}(\theta',\varphi'),$$

$$X^{\ell m}_{AB}(\theta,\varphi) = \frac{\partial x^{A'}}{\partial x^A}\frac{\partial x^{B'}}{\partial x^B} \sum_{m'=-\ell}^{\ell} D^{\ell}_{m m'}(\alpha,\beta,\gamma) X^{\ell m'}_{A'B'}(\theta',\varphi'),$$

$$Z^{\ell m}_{AB}(\theta,\varphi) = \frac{\partial x^{A'}}{\partial x^A}\frac{\partial x^{B'}}{\partial x^B} \sum_{m'=-\ell}^{\ell} D^{\ell}_{m m'}(\alpha,\beta,\gamma) Z^{\ell m'}_{A'B'}(\theta',\varphi'),$$

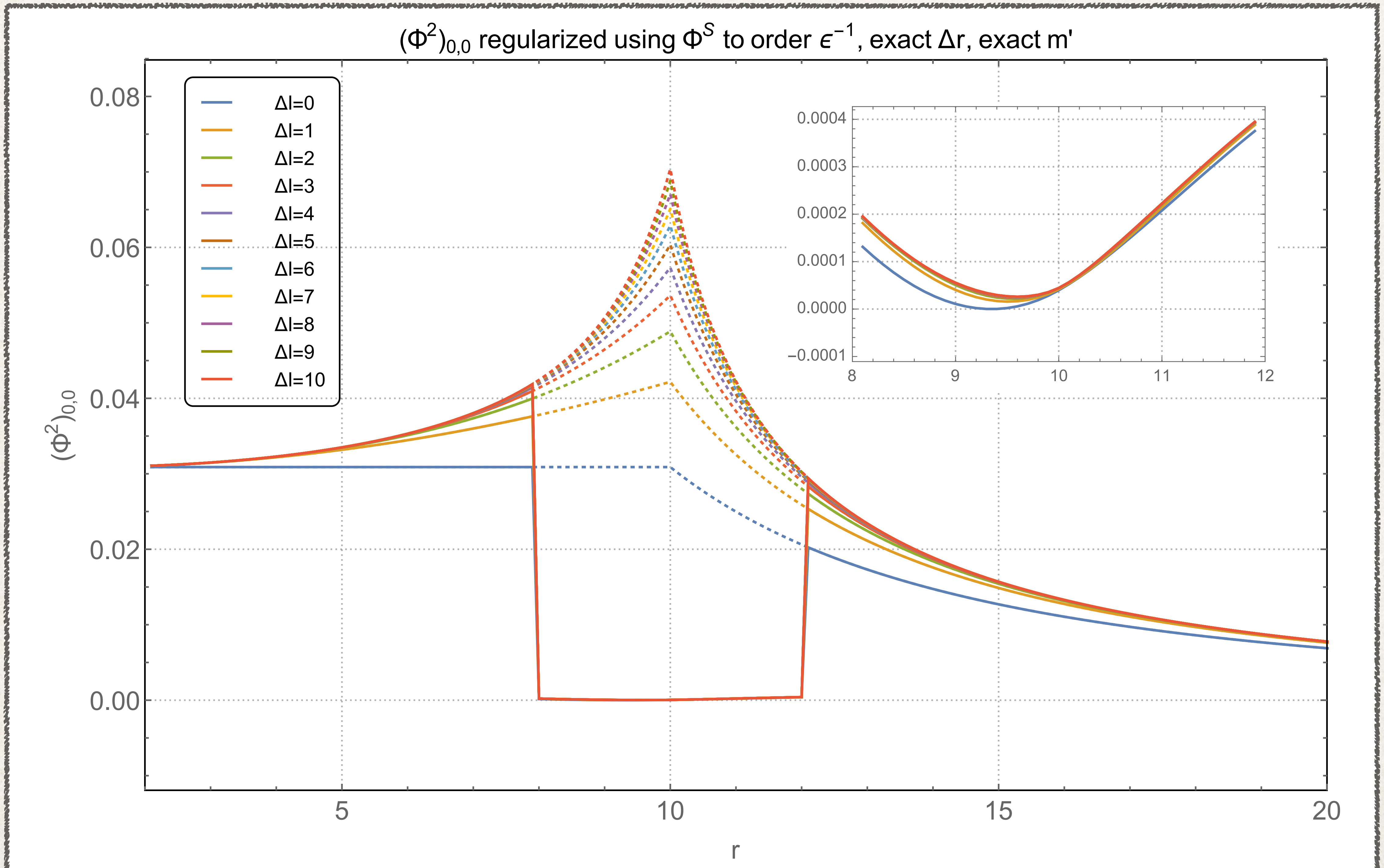
Exact, analytic mode decomposition

Scalar, m=0 mode decompositions are given analytically in terms of (finite) hypergeometric series in $\delta \sim \Delta r$.

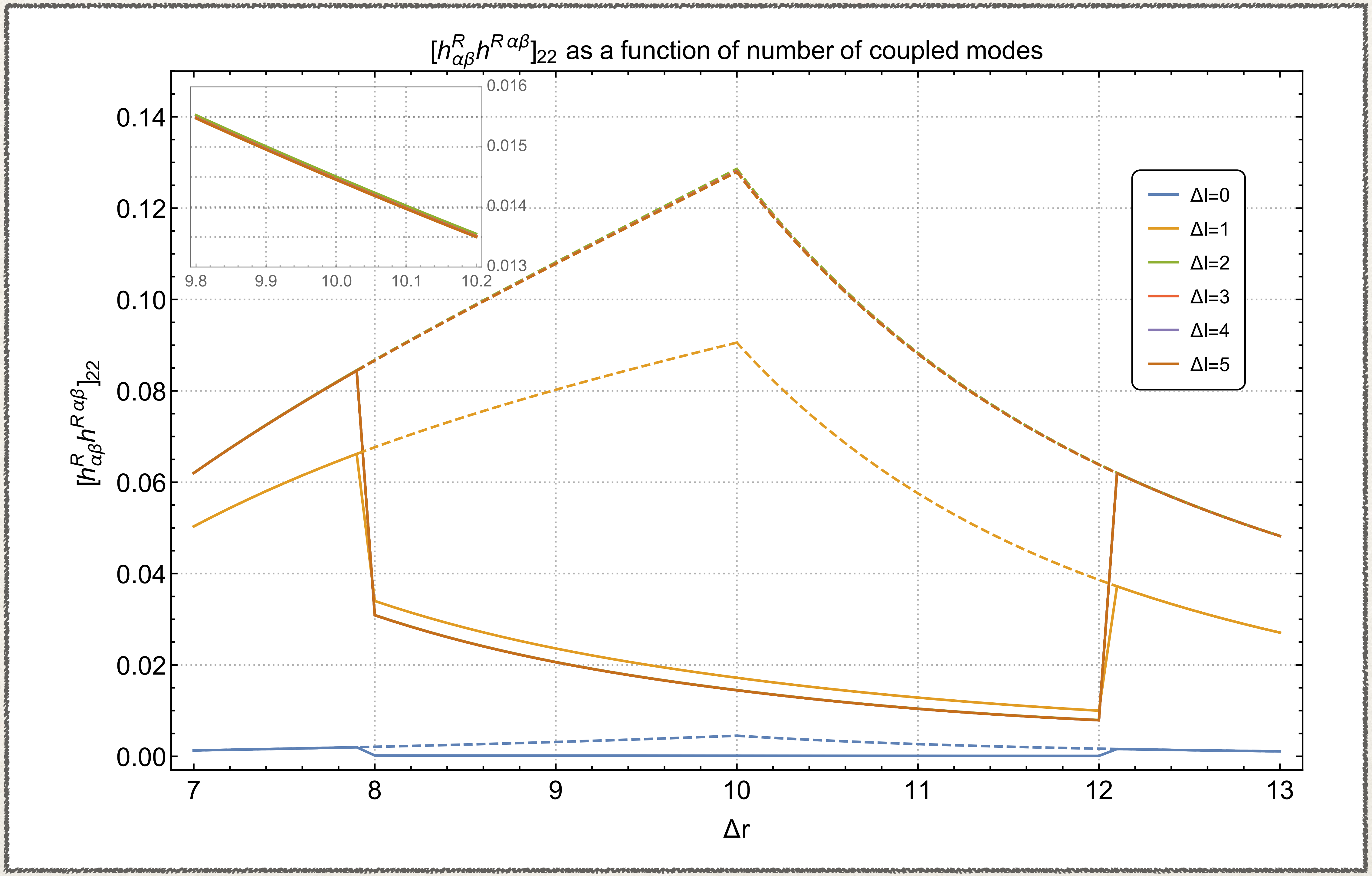
$$\int_0^\pi (\delta^2+1-\cos\alpha)^{p/2} P_\ell(\cos\alpha)\sin\alpha\, d\alpha$$
$$=\frac{(-1)^{\frac{p+1}{2}}(\delta^2+2)^{\frac{p}{2}+1}\left[\left(\frac{1}{2}\right)_{\frac{p+1}{2}}\right]^2}{\left(l-\frac{p}{2}\right)_{p+2}}\,{}_2F_1(-l,l+1;-\tfrac{p}{2};-\tfrac{\delta^2}{2})-\frac{2\,|\delta|\,\delta^{p+1}}{p+2}\,{}_2F_1(-l,l+1;\tfrac{p}{2}+2;-\tfrac{\delta^2}{2})$$
$$=\frac{(-1)^{\frac{p+1}{2}}(\delta^2+2)^{\frac{p}{2}+1}\left[\left(\frac{1}{2}\right)_{\frac{p+1}{2}}\right]^2}{\left(l-\frac{p}{2}\right)_{p+2}}\sum_{n=0}^{l}\frac{(-1)^n\delta^{2n}(l-n+1)_{2n}}{2^n n!\left(\frac{p}{2}-n+1\right)_n}-|\delta|\,\delta^{p+1}\sum_{n=0}^{l}\frac{\delta^{2n}(l-n+1)_{2n}}{2^n n!\left(\frac{p}{2}+1\right)_{n+1}}.$$

Integrate by parts - scalar, m!=0 mode decompositions can be rewritten in terms of m=0 decompositions plus a term proportional to $|\Delta r|$.
Tensor mode decompositions can be written in terms of combinations of scalar mode compositions.
Integrations over β are complete elliptic integrals of third kind.

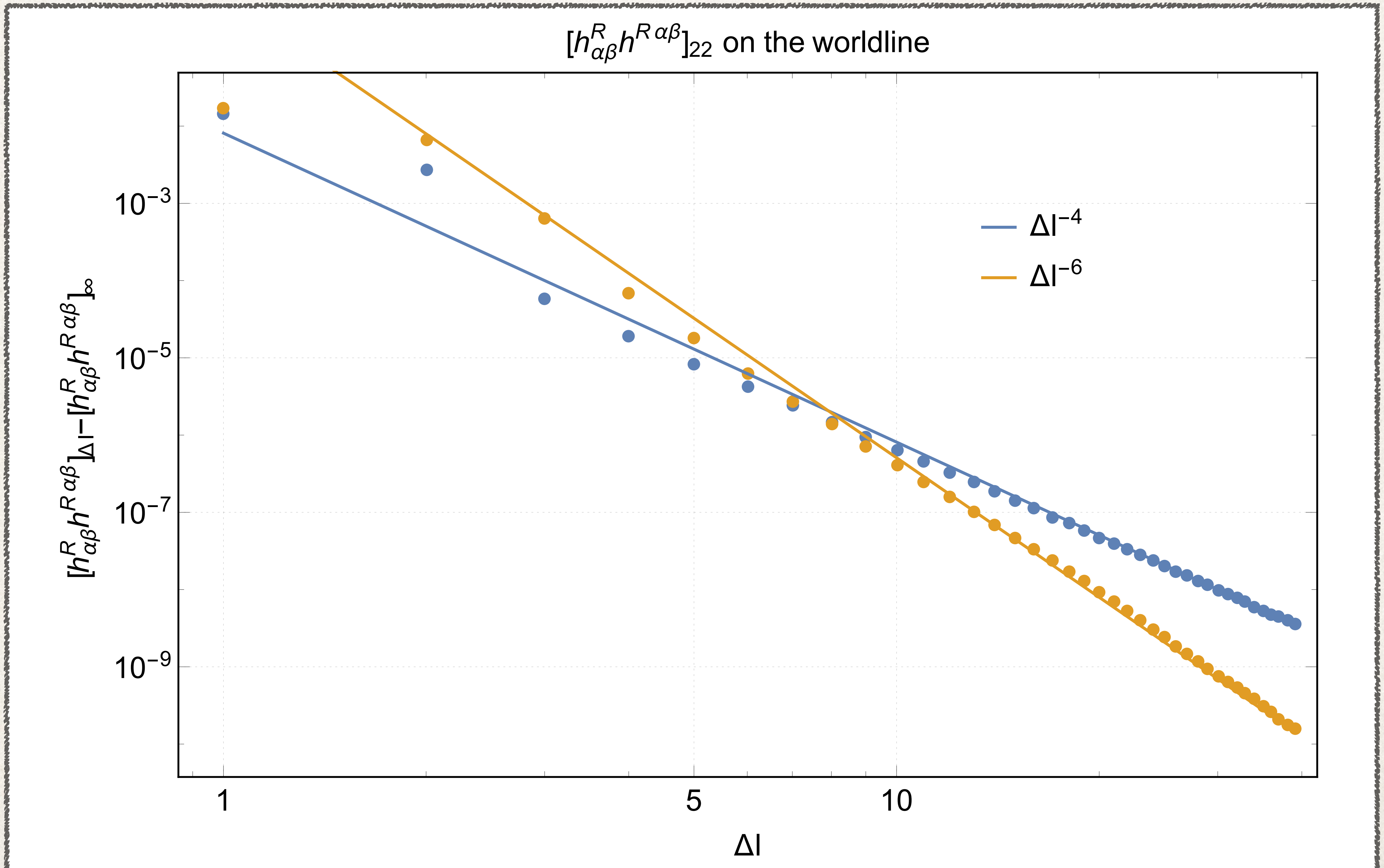
Mode coupling: scalar toy model



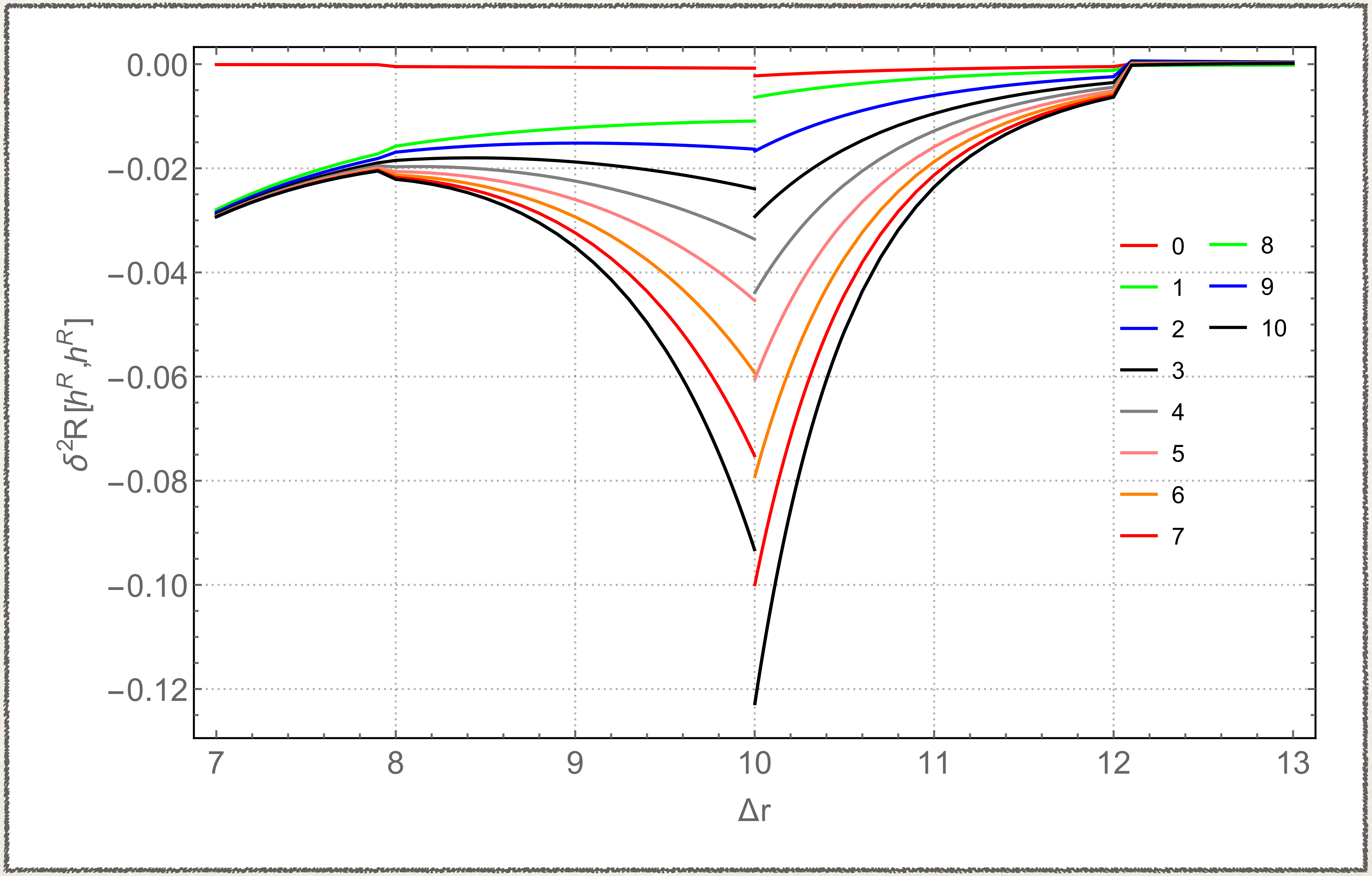
Mode coupling: metric perturbation



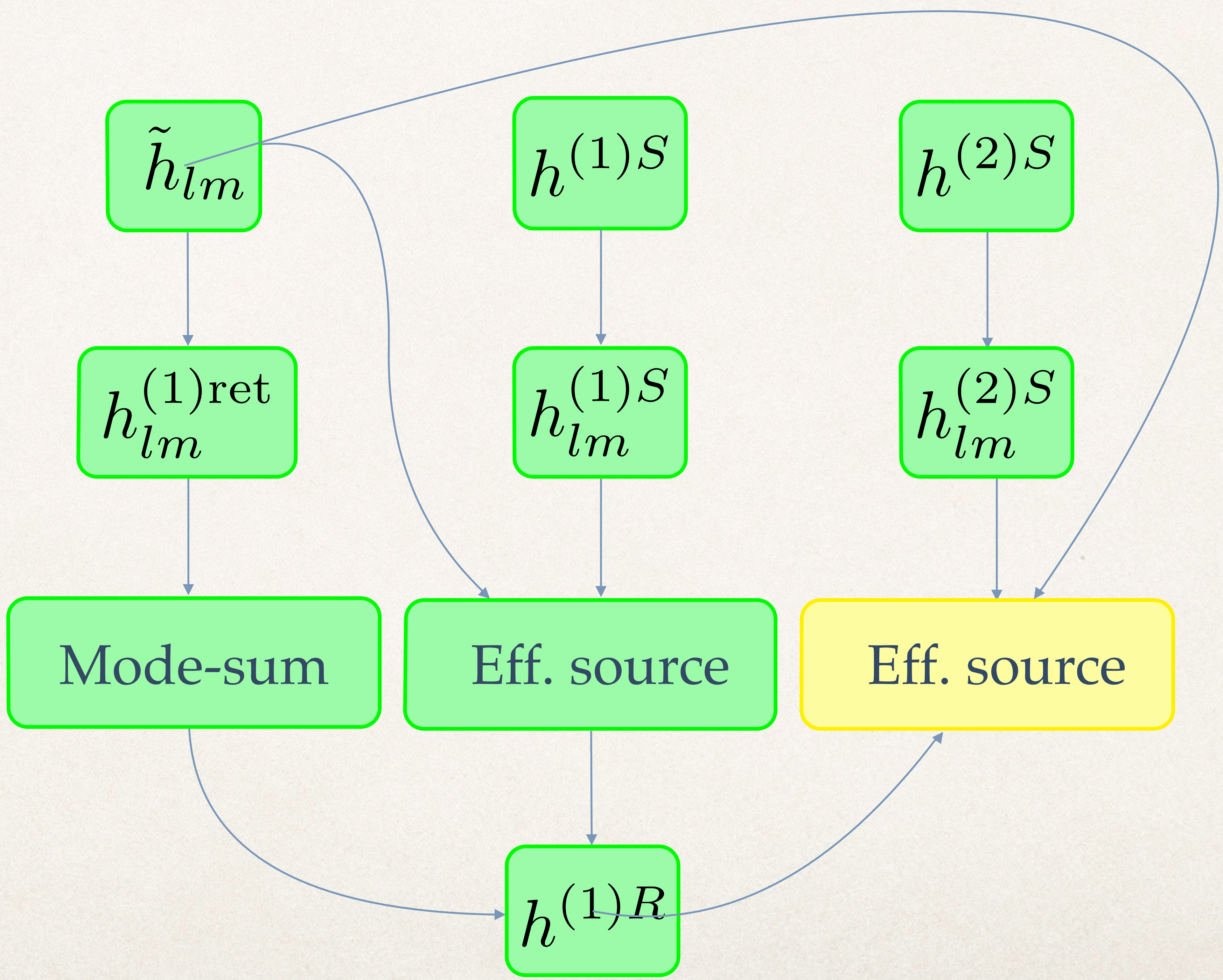
Mode coupling: metric perturbation



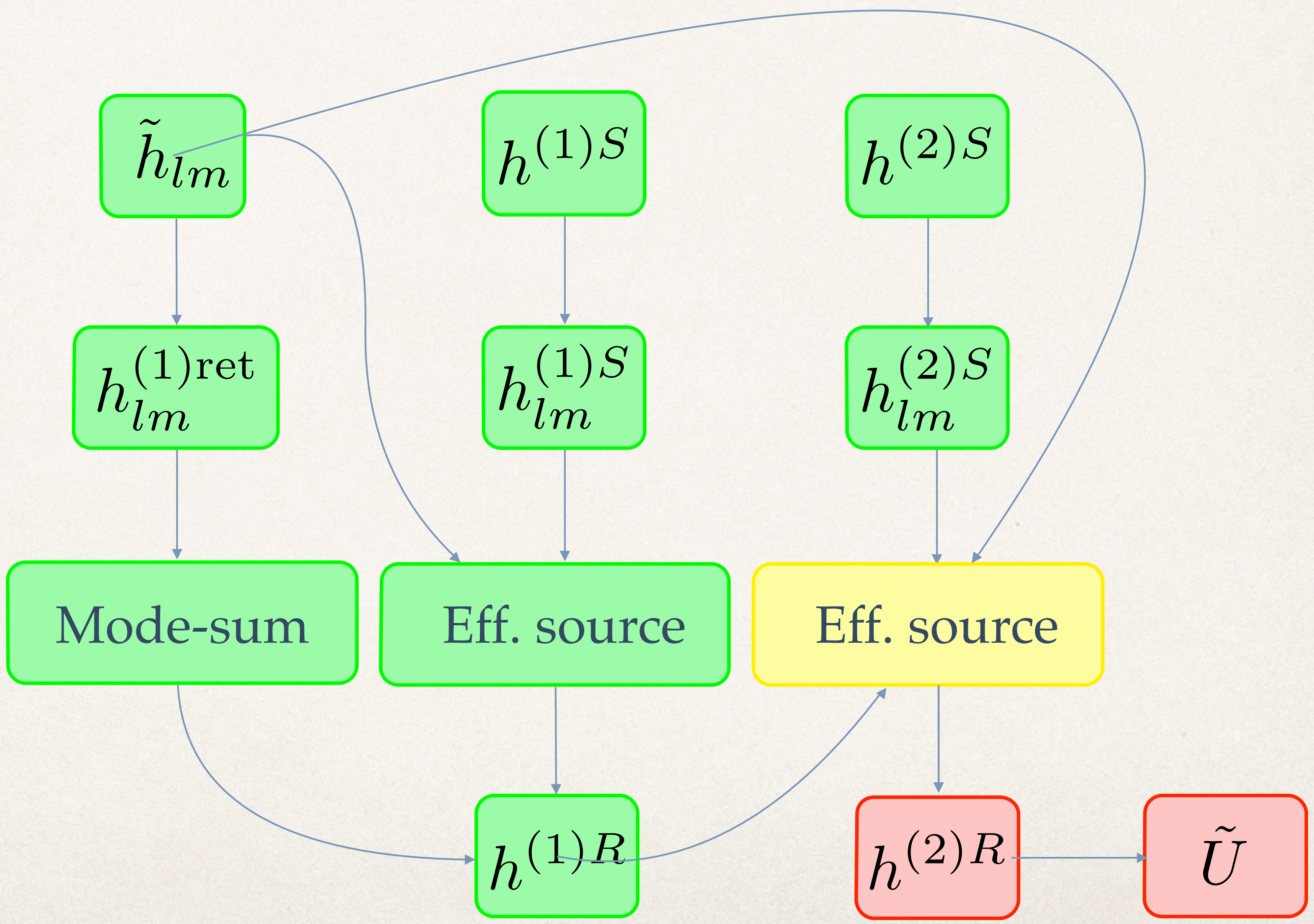
Mode coupling: $\delta^2 R[h^{\mathrm{R1}}, h^{\mathrm{R1}}]$



Towards second order self-force



Towards second order self-force



Towards second order self-force