Progress on the numerical calculation of the self-force in the time domain

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in collaboration with
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Thank you, Steve.

- If it were not for Steve, I would not be here.
- As a newcomer to the self-force field, Steve's explanations always help make physical sense of very technical talks.

The problem.

We wish to determine the self-forced motion and field (e.g. energy and angular momentum fluxes) of a particle with scalar charge

$$\Box\psi^{\mathrm{ret}} = -4\pi q \int \delta^{(4)}(x - z(\tau))\, d\tau.$$

2 general approaches:

- Compute enough "geodesic"-based self-forces and then use these to drive the motion of the particle. (Post-processing, fast, accurate self-forces, relies on slow orbit evolution)
- Compute the "true" self-force while simultaneously driving the motion. (Slow and expensive, less accurate self-forces)

Effective source approach.

... is a general approach to self-force and self-consistent orbital evolution that doesn't use any delta functions.

Key ideas

- Compute a regular field, ψ^{R}, such that the self-force is

$$F_\alpha = \nabla_\alpha \psi^{\mathsf{R}}|_{x=z},$$

  where $\psi^{\mathsf{R}} = \psi^{\mathsf{ret}} - \psi^{\mathsf{S}}$, and ψ^{S} can be approximated via local expansions: $\psi^{\mathsf{S}} = \tilde{\psi}^{\mathsf{S}} + O(\epsilon^n)$.

- The effective source, S, for the field equation for ψ^{R} is regular at the particle location.

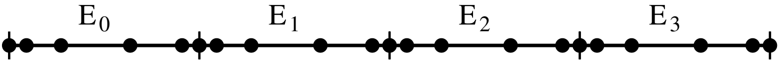
$$\Box\psi^{\mathsf{R}} = \Box\psi^{\mathsf{ret}} - \Box\psi^{\mathsf{S}} = S(x|z,u)$$

  where $\Box\psi^{\mathsf{S}} = -4\pi q \int \delta^{(4)}(x - z(\tau))\, d\tau - S$.

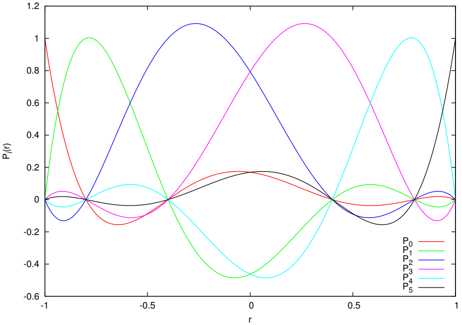
Self-consistent vs. geodesic evolutions.

- One main goal is to compare our self-consistent evolutions with Niels Warburton's geodesic evolutions.
- First attempt: 3+1 multi-patch finite difference code with a C^0 effective source.
- 3+1 accuracy limited by the non-smoothness of the source leading to high frequency noise with 2nd order convergent amplitude.
- Self-consistent evolutions agreed beautifully with geodesic evolutions within the errors (dominated by the noise).
- Next attempt: 3+1 multi-patch finite difference code with a C^2 effective source.
- Geodesic evolution agreed with the C^0 evolutions and the frequency domain result with the noise reduced by more than an order of magnitude.
- However, we found differences between C^2 and C^0 results as soon as the back-reaction was turned on.

Discontinuous Galerkin method.



- Split the domain into N nth order elements.
- Each element contains $n+1$ nodes.
- $u(t,x) \approx \sum_{i=0}^{n} \tilde{u}(t,x_i)P_i(x)$



- The numerical approximation is double valued at all element boundaries.
- Derivatives are approximated by multiplying the state vector in each element by a derivative matrix.
- Neighboring elements are glued together by numerical fluxes.

Discontinuous Galerkin method.

- Numerical fluxes can be constructed in many different ways in order to maintain numerical stability and to guarantee that the jumps in the solution at the element boundaries converge to zero.
- We use fluxes based on characteristic information.

The convergence properties of the DG method for smooth solutions are

- Exponential with the order n (with N kept fixed).
- polynomial with the element size $1/N$ (with n kept fixed).

As the DG scheme has discontinuities built in at the element boundaries, we retain these convergence properties even when the solution itself is non-smooth IF and only if, the non-smooth features can be placed at element boundaries.
(Hesthaven & Warburton, 2007)

Code description.

The code is a 1+1 dimensional code based on the spherical harmonic decomposition of the scalar wave equation in Schwarzschild tortoise coordinates $r_* = r + 2M\log(r/(2M) - 1)$ with a spherically harmonic decomposed effective source.

$$-\frac{\partial^2\psi_{\ell m}}{\partial t^2} + \frac{\partial^2\psi_{\ell m}}{\partial r_*^2} - V_\ell(r)\psi_{\ell m} = S^{\text{eff}}_{\ell m}.$$

As $r_* \in [-\infty, \infty]$ we split the domain into three regions. In the inner ($r_* \in [-\infty, T_1]$) and outer ($r_* \in [T_2, \infty]$) regions we introduce new coordinates (τ, ρ) used in Bernuzzi, Nagar & Zenginoğlu (2011).

$$\begin{aligned} t &= \tau + h(\rho) \\ r_* &= \rho/\Omega(\rho) \end{aligned}$$

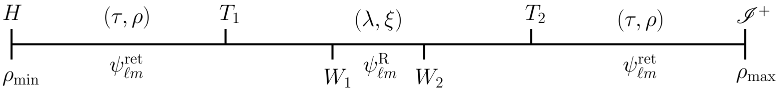
where $h(\rho)$ and $\Omega(\rho)$ are chosen suitably (hyperboloidal layers) in each region to make the inner boundary ($\rho_{\min}$) coincide with the horizon H and the outer boundary ($\rho_{\max}$) coincide with $\mathscr{I}^+$.

Code description.

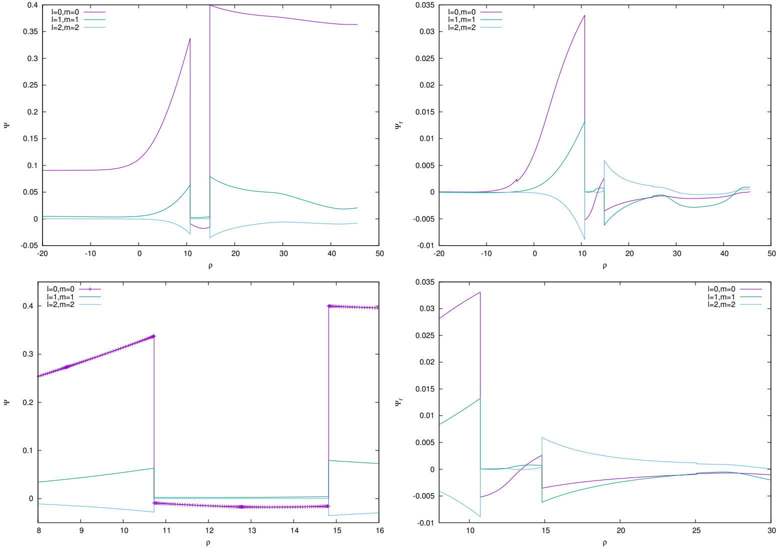
In the middle region ($r_* \in [T_1, T_2]$) we introduce a time dependent coordinate transformation (Field, Hesthaven & Lau, 2009)

$$\begin{aligned} t &= \lambda \\ r_* &= T_1 + \frac{r_*^p - T_1}{\xi^p - T_1}(\xi - T_1) + \\ & \quad \frac{(T_2 - r_*^p)(\xi^p - T_1) - (r_*^p - T_1)(T_2 - \xi^p)}{(\xi^p - T_1)(T_2 - \xi^p)(T_2 - T_1)}(\xi - T_1)(\xi - \xi^p) \end{aligned}$$

where r_*^p is the time-dependent particle location. This satisfies $r_*(\lambda, T_1) = T_1$, $r_*(\lambda, \xi^p) = r_*^p$, $r_*(\lambda, T_2) = T_2$.
In addition we use the world tube approach so that we evolve $\psi_{\ell m}^{\mathrm{R}} = \psi_{\ell m}^{\mathrm{ret}} - \psi_{\ell m}^{\mathrm{S}}$ in the region $r_* \in [W_1, W_2]$ (where typically $W_1 > T_1$ and $W_2 < T_2$), while elsewhere we evolve $\psi_{\ell m}^{\mathrm{ret}}$.
The values of T_1, W_1, W_2 and T_2 is of course chosen to coincide with element boundaries.



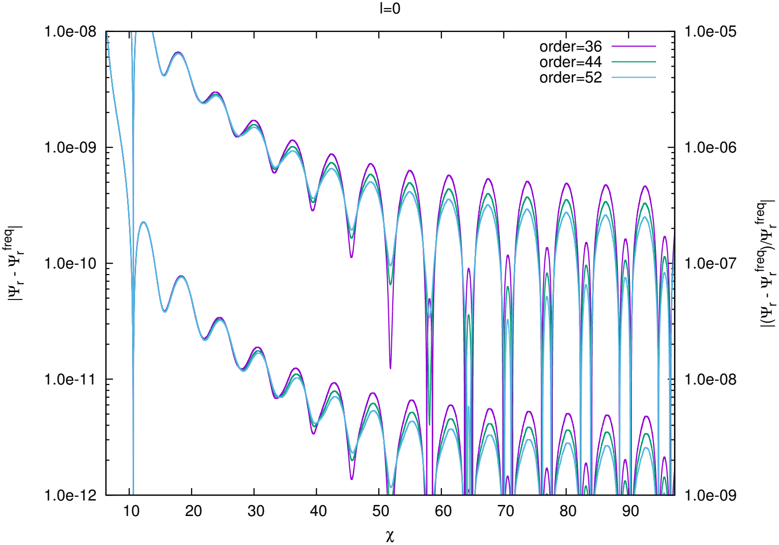
Representative snapshots ($e = 0.1, p = 9.9$).



Changes to the code since last year.

- We added the capability for the world-tube to be smaller than the region with the time dependent coordinate transformation. This allow us to use a smaller world-tube, resulting in higher accuracy.
- We added checkpoint/restart capability allowing us to restart at suitable time-steps with the option of changing some of the parameters without having to rerun.
- We added acceleration terms to the effective source resulting in the current effective source being less smooth than before. Modes now converges as $1/((2\ell-1)(2\ell+3))$ instead of $1/((2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5))$.
- We added the capability of fitting the behavior of the high ℓ-modes and correct the sum over ℓ on the fly.
- Added self-force terms to the osculating orbit equations.

Errors for geodesic eccentric orbit ($e = 0.1, p = 9.9$).



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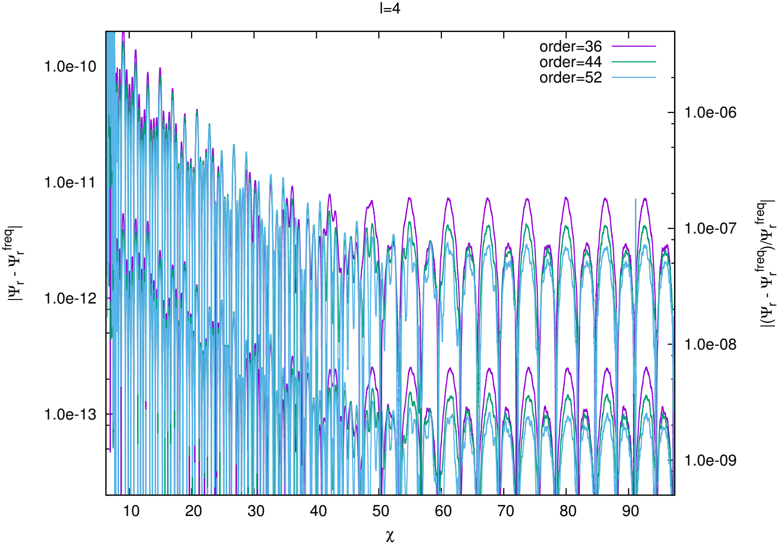

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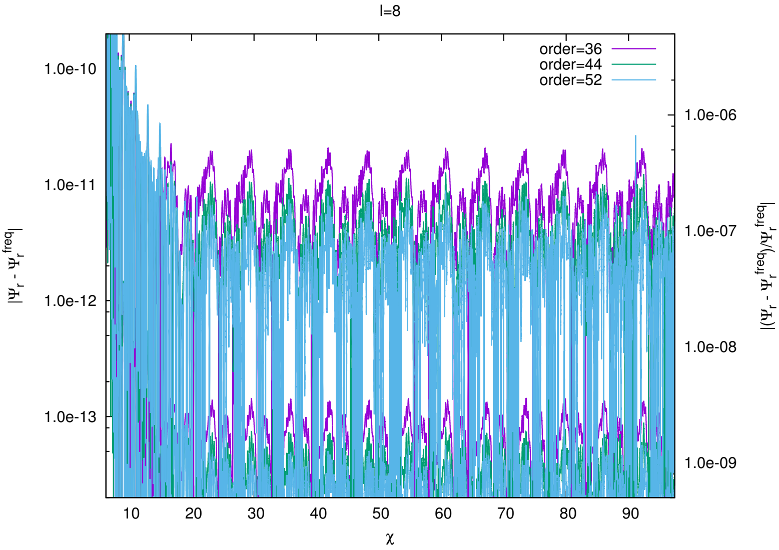

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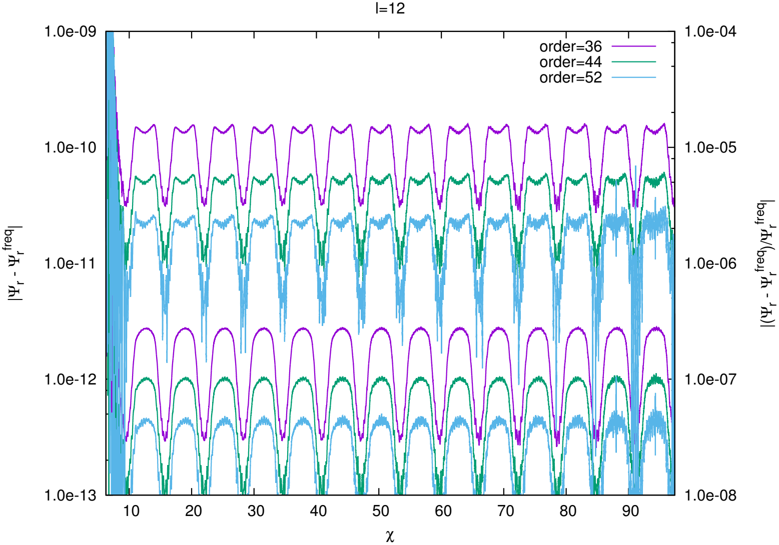

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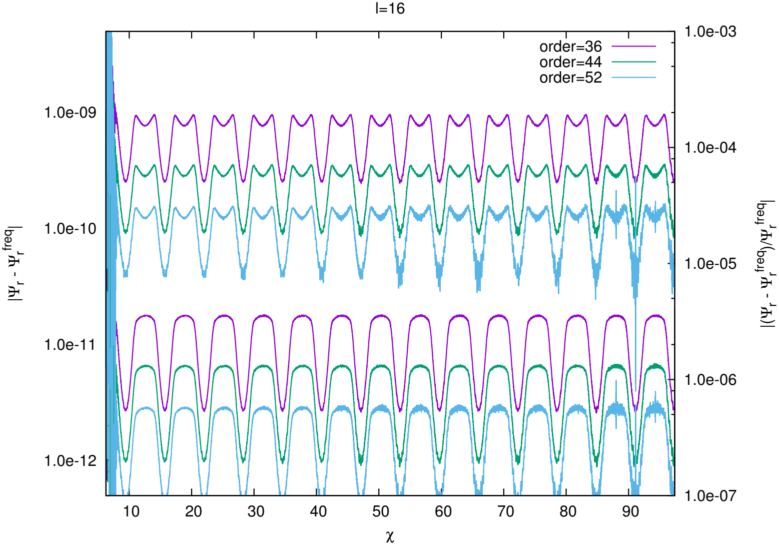

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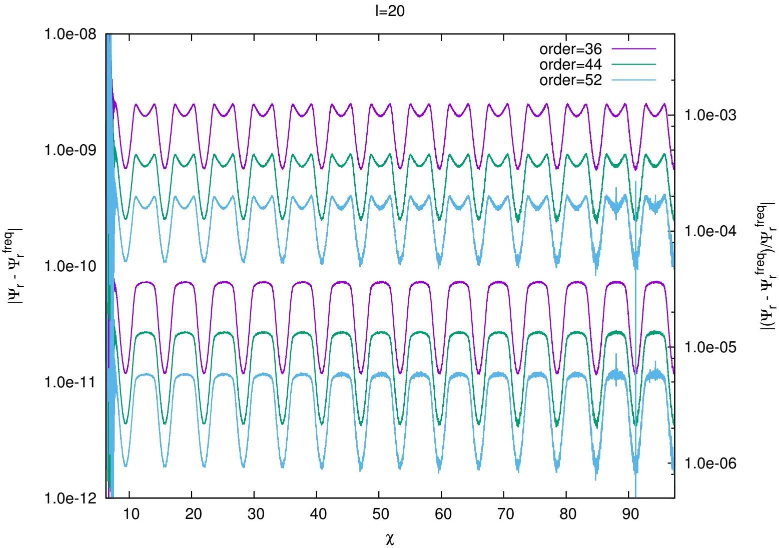
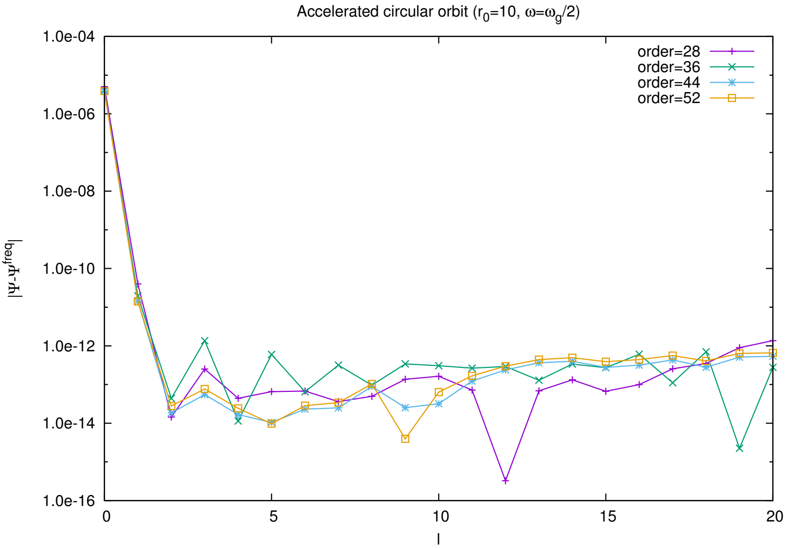


Errors for accelerated circular orbit $(r_0 = 10, \omega = \frac{1}{2}\omega_g)$.



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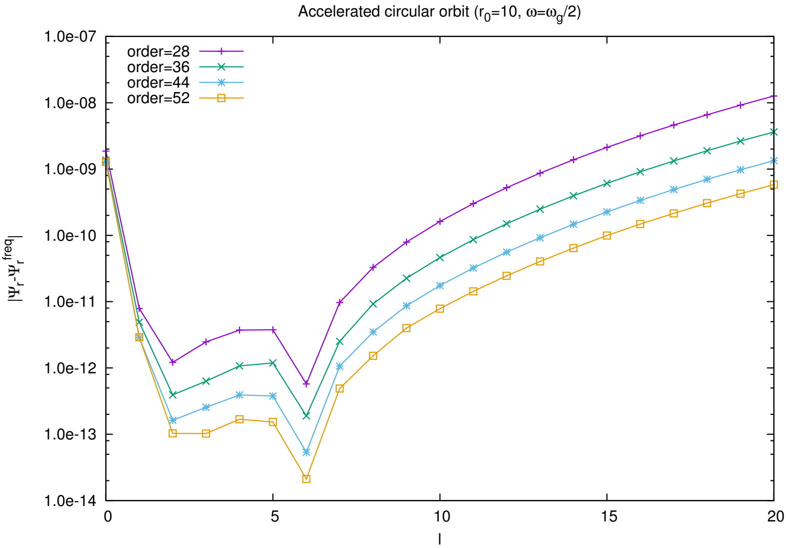

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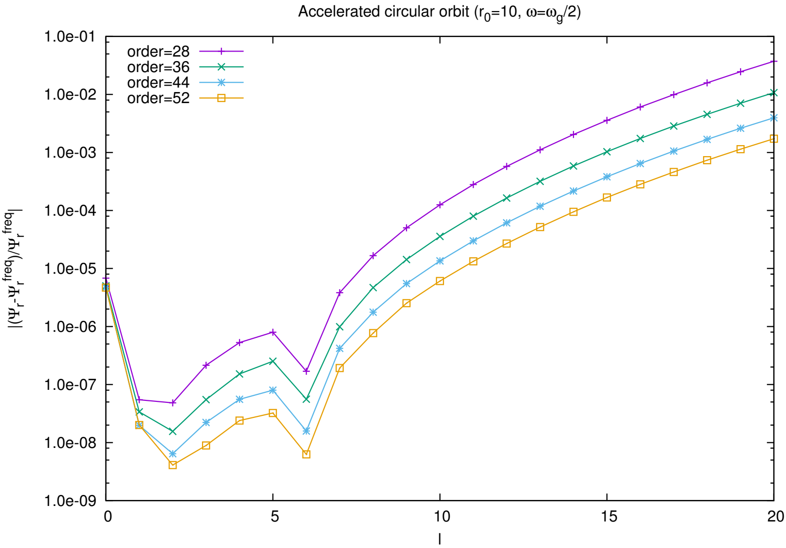
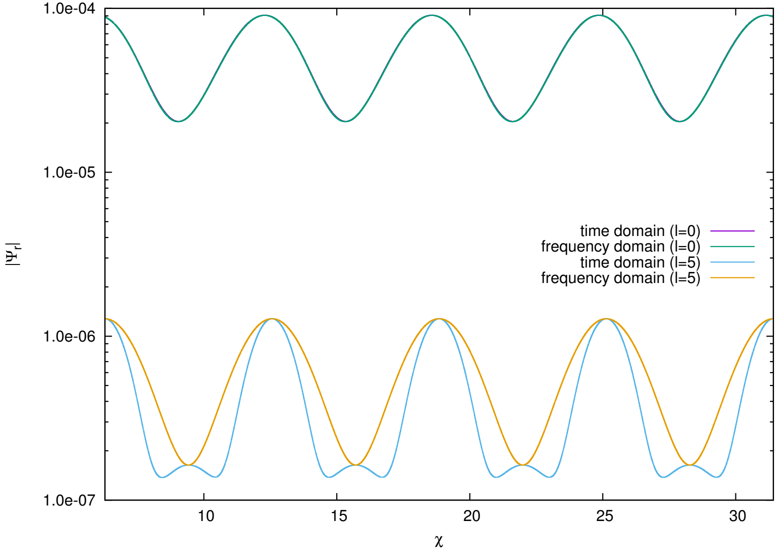
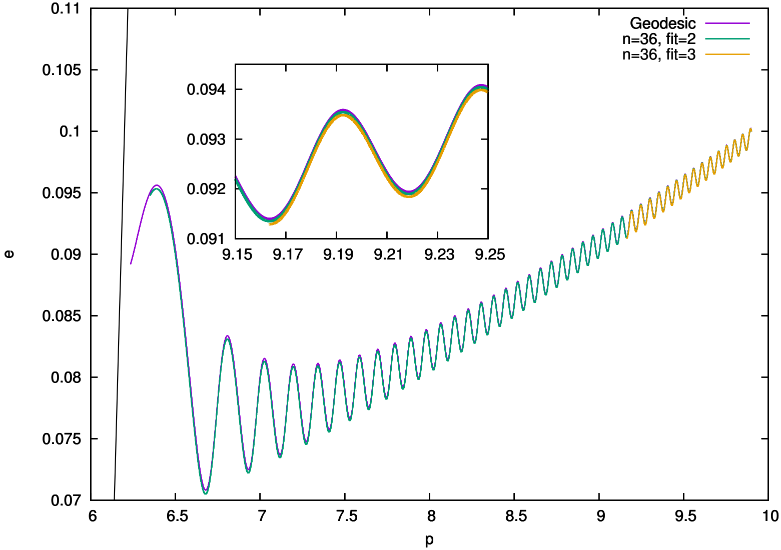


Accelerated elliptical orbit ($e_0 = 0.1, p = 8$).



Self-consistent evolution ($e_0 = 0.1, p = 9.9$).



Conclusions and Outlook.

- Discontinuous Galerkin is a powerful numerical method that allows us to overcome the non-smoothness of the effective source.
- The accuracy has been improved and computational cost reduced by at least 2 to 3 orders of magnitude.
- Eccentric geodesic orbits and constant accelerated circular orbits works very well.
- Accelerated eccentric orbits and non-constant accelerated circular orbits do not work yet (but seem to be really close).
- Self-consistent evolutions are just around the corner.
- We have a finite difference prototype of a coupled mode evolution code for scalar fields in Kerr (to be ported to DG).
- Gravitational perturbation codes (both Lorentz and Regge-Wheeler) are in various stages of development/testing.