Further high-precision comparisons between perturbation calculations and PN theory for eccentric inspirals

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Aim: Use perturbation theory to push PN knowledge

- Work at 1st-order in mass ratio μ/M

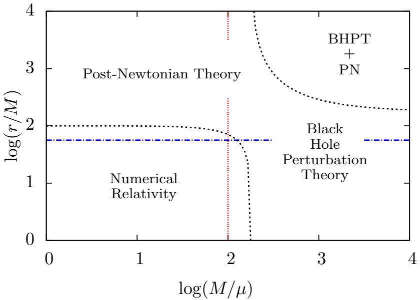
  Calculate fluxes and/or gravitational self-force

- By choice focus on wide orbits $r \gg M$

  $\rightarrow$ overlap with PN theory

  Pluck off new higher-order PN terms

  $\rightarrow$ 3.5PN, 4PN,..., 7PN



- As Barry mentioned–thriving industry! (partial list)

Poisson (1993); Poisson and Sasaki (1995); various by Sasaki, Tagoshi, Tanaka, Shibata, Takasugi, Mano (mid-1990s); Detweiler (2008); Blanchet, Detweiler, Le Tiec, and Whiting (2010,2011); Fujita (2012); Bini and Damour (2013, 2014, etc); Shah, Friedman, and Whiting (2014); Shah (2014); Fujita (2014); Johnson-McDaniel, Shah, and Whiting (2015); Sago and Fujita (2015); Kavanagh, Ottewill, and Wardell (2015a,b); Forseth, CRE, Hopper (2016)

Calculational method in a nutshell

- Analytic function expansions for $R^{\pm}_{lm\omega}$ using MST formalism (here $a = 0$)

$$\left[r^2 f \frac{d^2}{dr^2} - 2(r-M)\frac{d}{dr} + U_{l\omega}(r)\right] R^{\pm}_{lm\omega}(r) = 0,$$

- Convert $R^{\pm}_{lm\omega}$ to RWZ functions $X^{\pm}_{lm\omega}$

- Specify orbit (p, e); solve source problem with SSI
  PhysRev D92, 044048

- Find TD solution via EHS (if want self-force)

$$\Psi^{\pm}_{lm}(t,r) = \sum_n C^{\pm}_{lmn}\, \hat{X}^{\pm}_{lmn}\, e^{-i\omega_{mn}t}$$

- MST → RWZ → p,e → SSI → (EHS) → *Mathematica* → Fluxes $|C^{\pm}_{lmn}|^2$

  PhysRev D93, 064058

Why very high precision calculations?

- Calculate at large radii: $r \sim 10^{20} M$

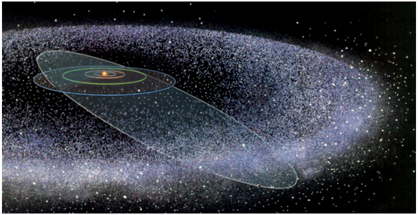
  $\rightarrow$ MST converges rapidly

  Dial-in your own 'fine-structure' constant

  $\rightarrow$ $y = (M\Omega)^{2/3} = 10^{-20}$

- Computed results at 200+ digits contain many PN orders

  Use PSLQ to pluck off exact coefficients



- Example: Analytically-known circular-orbit redshift invariant

  (Shah, Friedman, and Whiting 2014)

$$\Delta U = \frac{-1}{r} + \frac{-2}{r^2} + \frac{-5}{r^3} + \frac{-3872 + 123\pi^2}{96 r^4} + \frac{-592384 - 196608\gamma_E + 10155\pi^2 - 393216\ln(2)}{7680 r^5}$$

Numerical result at $r = 10^{10} M$ $\rightarrow$ Read off the terms

$$\Delta U = -1.000000000200000000050000000276879\cdots \times 10^{-10}$$

Reminder: Eccentricity dependent PN energy flux

Flux at infinity depends on e_t and $y = (\omega M)^{2/3} \ll 1$

e_t is PN "time eccentricity"

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{32}{5} \left(\frac{\mu}{M} \right)^2 y^5 \Big[\mathcal{I}_0(e_t) + y\, \mathcal{I}_1(e_t) + y^{3/2}\, \mathcal{K}_{3/2}(e_t) + y^2\, \mathcal{I}_2(e_t) + y^{5/2}\, \mathcal{K}_{5/2}(e_t) + y^3\, \mathcal{I}_3(e_t) + y^3 \log y\, \mathcal{I}_{3L}(e_t) + y^3\, \mathcal{K}_3(e_t) + y^{7/2}\, \mathcal{L}_{7/2}(e_t) + y^4\, \mathcal{L}_4(e_t) + y^4 \log y\, \mathcal{L}_{4L}(e_t) + \cdots \Big]$$

Enhancement functions: $\mathcal{I}_n(e_t)$ are instantaneous; $\mathcal{K}_n(e_t)$ are hereditary

$$\mathcal{I}_0(e_t) = \frac{1}{(1-e_t^2)^{7/2}} \left(1 + \frac{73}{24}\, e_t^2 + \frac{37}{96}\, e_t^4 \right)$$ Peters-Mathews (1963)

$$\mathcal{I}_1(e_t) = \frac{1}{(1-e_t^2)^{9/2}} \left(-\frac{1247}{336} + \frac{10475}{672} e_t^2 + \frac{10043}{384} e_t^4 + \frac{2179}{1792} e_t^6 \right)$$

See Arun et al. 2008a,b; 2009; Blanchet 2014 (LRR)

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Peters-Mathews (1963)

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Hereditary terms: New high-order expansions

- We had to find expansions for $\mathcal{K}_{3/2}$ (tail), $\mathcal{K}_{5/2}$ (1PN correction to tail), and $\mathcal{K}_3$ (tail-of-tail and tail^2) to arbitrary order in e_t

- Alternative quantities (see Blanchet LRR 2014): φ, ψ, χ

- For example, the 1.5PN tail has enhancement function

$$\varphi(e_t) = \frac{1}{(1-e_t^2)^5}\left(1 + \frac{1375}{192}e_t^2 + \frac{3935}{768}e_t^4 + \frac{10007}{36864}e_t^6 + \frac{2321}{884736}e_t^8 + \frac{237857}{353894400}e_t^{10} + \frac{182863}{4246732800}e_t^{12} + \frac{4987211}{6658877030400}e_t^{14} - \frac{47839147}{35514010828800}e_t^{16} + \cdots\right)$$

- Determined to order e_t^{200}

- Identified eccentricity singular factor–results in rapidly converging series with finite limit at $e_t \to 1$

Asymptotic analysis of enhancement functions

- Used asymptotic analysis to "prove" our assumed eccentricity singular factors for $\varphi(e_t)$, $\chi(e_t)$, as well as $\mathcal{I}_0(e_t)$ and $F(e_t)$

$$g(n, e_t) = \frac{1}{6} n^2 \frac{1 + x + x^2 + 3n^2 x^3}{(1-x)^2} J_n(ne_t)^2 + \frac{1}{2} n^2 \frac{x(1+n^2 x)}{1-x} J_n'(ne_t)^2 - \frac{1}{2} n^3 \frac{x(1+3x)}{(1-x)^{3/2}} J_n(ne_t) J_n'(ne_t)$$

where $x = 1 - e_t^2$. Use transition-region asymptotic expansion for Bessel functions

$$J_n(ne_t) \sim \left(\frac{4\zeta}{x}\right)^{\frac{1}{4}} \left[n^{-1/3} \mathrm{Ai}(n^{2/3}\zeta) \sum_{k=0}^{\infty} \frac{A_k}{n^{2k}} + n^{-5/3} \mathrm{Ai}'(n^{2/3}\zeta) \sum_{k=0}^{\infty} \frac{B_k}{n^{2k}} \right]$$

and asymptotic expansion of Airy functions

$$\mathrm{Ai}(n^{2/3}\zeta) \sim \frac{e^{-\xi}}{2^{5/6} 3^{1/6} \sqrt{\pi} \xi^{1/6}} \left(1 - \frac{5}{72\xi} + \frac{385}{10368\xi^2} - \frac{85085}{2239488\xi^3} + \frac{37182145}{644972544\xi^4} + \cdots \right)$$

- Approximate infinte sums over n with integrals and read off singular factors and (fairly) sharp estimates of proportionality constants

2.5PN hereditary term

- Second case and most difficult to calculate: 2.5PN tail enhancement function

$$\psi(e_t) = \frac{1}{(1-e_t^2)^6}\left(1 - \frac{72134}{8191}e_t^2 - \frac{19817891}{524224}e_t^4 - \frac{62900483}{4718016}e_t^6 - \frac{184577393}{603906048}e_t^8 + \frac{1052581}{419379200}e_t^{10} - \frac{686351417}{1159499612160}e_t^{12} + \frac{106760742311}{852232214937600}e_t^{14} + \cdots\right)$$

- Determined to order e_t^{70}

- Identified different eccentricity singular factor–remaining series has finite limit at $e_t \to 1$

3PN hereditary term

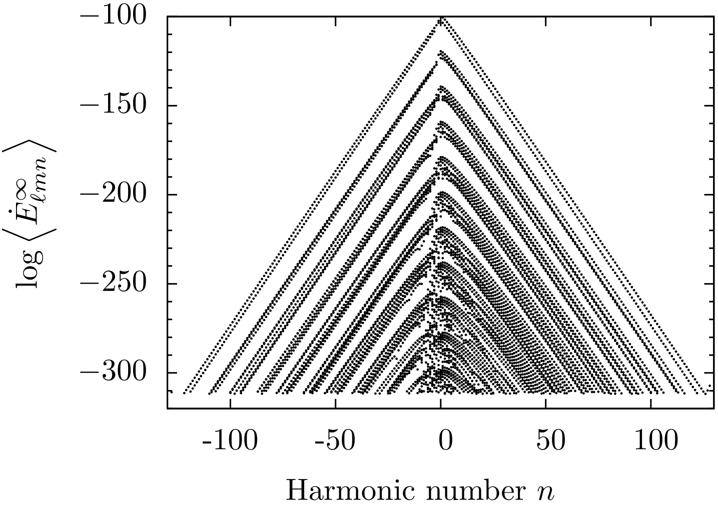
- Third case has most interesting structure: 3PN hereditary enhancement function

$$\chi(e_t) = -\frac{3}{2}\frac{\log(1-e_t^2)}{(1-e_t^2)^{13/2}}\left(1+\frac{85}{6}e_t^2+\frac{5171}{192}e_t^4+\frac{1751}{192}e_t^6+\frac{297}{1024}e_t^8\right)$$
$$+\frac{1}{(1-e_t^2)^{13/2}}\left\{\left[-\frac{3}{2}-\frac{77}{3}\log(2)+\frac{6561}{256}\log(3)\right]e_t^2+\left[-22+\frac{34855}{64}\log(2)-\frac{295245}{1024}\log(3)\right]e_t^4\right.$$
$$\left.+\left[-\frac{6595}{128}-\frac{1167467}{192}\log(2)+\frac{24247269}{16384}\log(3)+\frac{244140625}{147456}\log(5)\right]e_t^6+\cdots\right\}$$

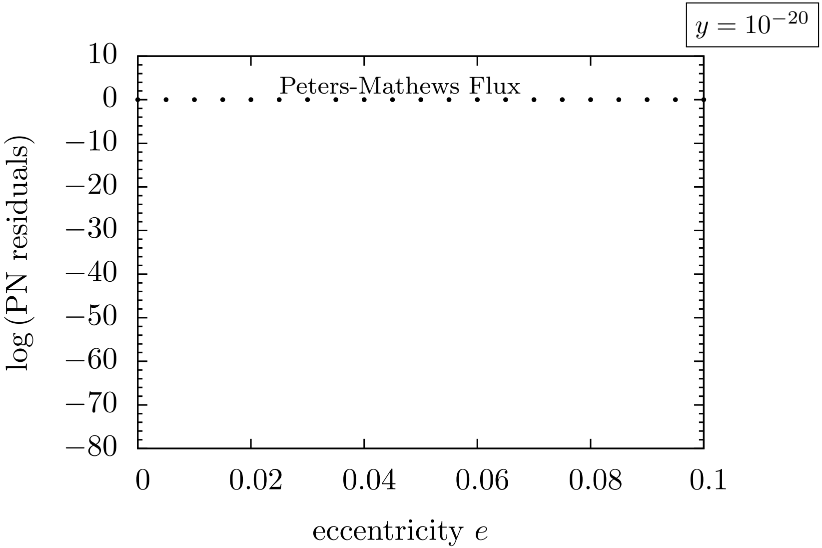
- Asymptotic analysis identified closed-form Log singular term
- Second term is also singular and just sub-dominant
- Remaining series is rapidly convergent
- Aside: Points to use of (e, p^{-1}) instead of (e, y)

Confirming 3PN: Fluxes $\dot{E}^{\infty}_{\ell mn}$ from one orbital model

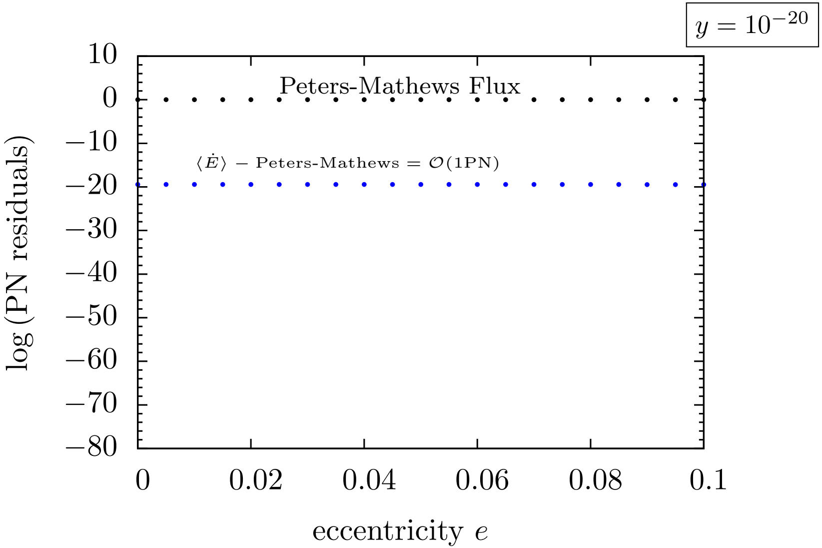
Orbit: $y = 10^{-20}$, $e = 0.1$, accuracy: 200 digits, total modes > 7000



Confirming 3PN: Normalize fluxes to Peters-Mathews

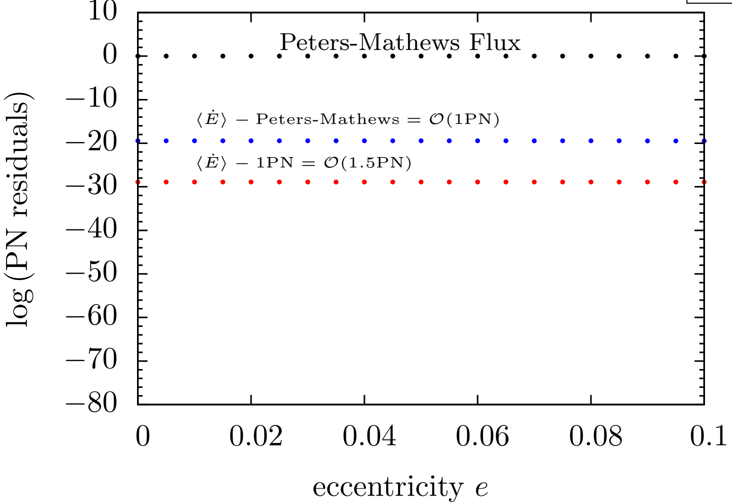


Confirming 3PN: Subtract $\mathcal{I}_0(e)$ from $\ell = 2$ flux

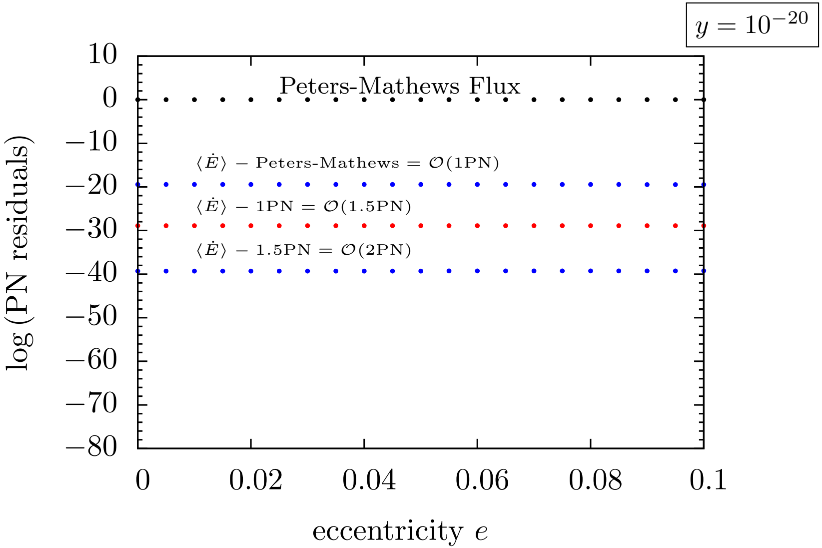


Confirming 3PN: And subtract 1PN from $\ell \leq 3$ flux

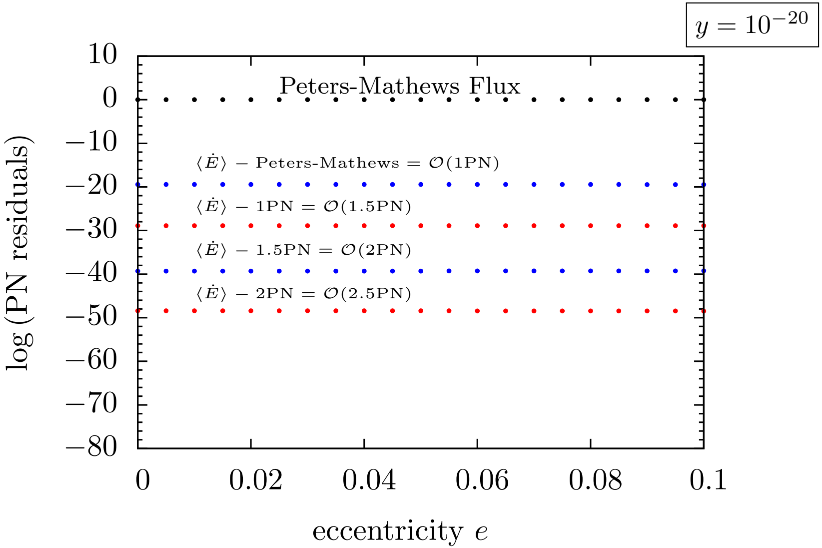
$y = 10^{-20}$



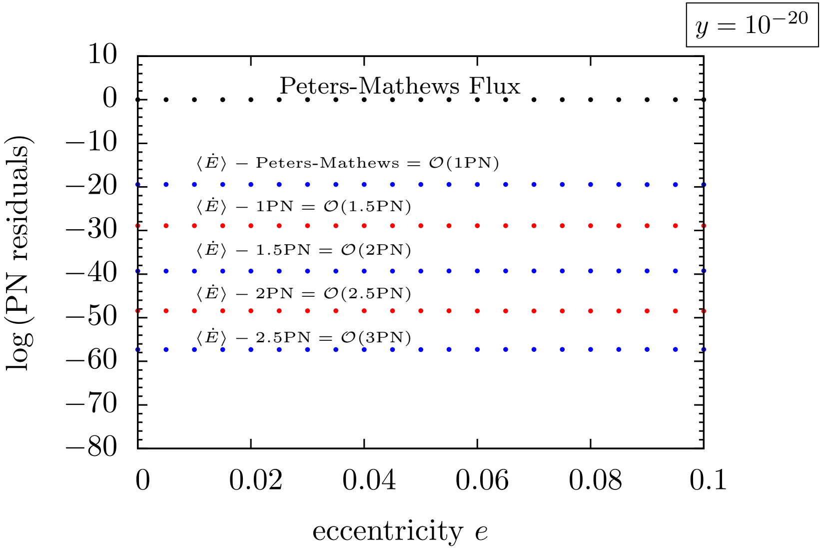
Confirming 3PN: And subtract 1.5PN from $\ell \leq 3$ flux



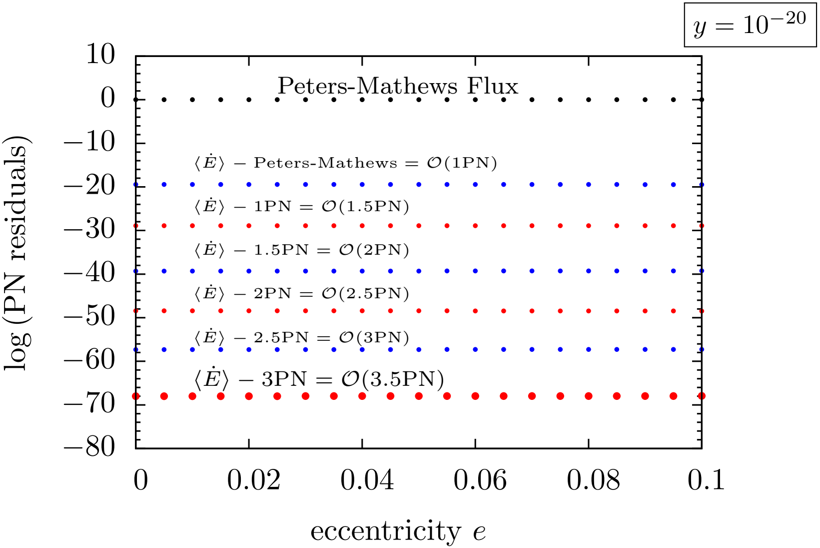
Confirming 3PN: And subtract 2PN from $\ell \leq 4$ flux



Confirming 3PN: And subtract 2.5PN from $\ell \leq 4$ flux



Confirming 3PN: And subtract 3PN from $\ell \leq 5$ flux



Reaching beyond 3PN

$$
\begin{aligned}
\langle \dot{E} \rangle = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 y^5 \Big[& \mathcal{I}_0 + y\mathcal{I}_1 + y^{3/2}\mathcal{K}_{3/2} + y^2\mathcal{I}_2 + y^{5/2}\mathcal{K}_{5/2} + y^3\mathcal{I}_3 + y^3\mathcal{K}_3 \\
& + y^{7/2}\mathcal{L}_{7/2} + y^4\left(\mathcal{L}_4 + \log(y)\mathcal{L}_{4L}\right) + y^{9/2}\left(\mathcal{L}_{9/2} + \log(y)\mathcal{L}_{9/2L}\right) \\
& + y^5\left(\mathcal{L}_5 + \log(y)\mathcal{L}_{5L}\right) + y^{11/2}\left(\mathcal{L}_{11/2} + \log(y)\mathcal{L}_{11/2L}\right) \\
& + y^6\left(\mathcal{L}_6 + \log(y)\mathcal{L}_{6L} + \log^2(y)\mathcal{L}_{6L^2}\right) \\
& + y^{13/2}\left(\mathcal{L}_{13/2} + \log(y)\mathcal{L}_{13/2L}\right) \\
& + y^7\left(\mathcal{L}_7 + \log(y)\mathcal{L}_{7L} + \log^2(y)\mathcal{L}_{7L^2}\right) + \cdots \Big]
\end{aligned}
$$

Reaching beyond 3PN

Peters and Mathews, 1963

$$\langle \dot{E} \rangle = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 y^5 \Big[\mathcal{I}_0 + y\mathcal{I}_1 + y^{3/2}\mathcal{K}_{3/2} + y^2\mathcal{I}_2 + y^{5/2}\mathcal{K}_{5/2} + y^3\mathcal{I}_3 + y^3\mathcal{K}_3$$
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Reaching beyond 3PN

Wagoner and Will, 1976

↓

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Reaching beyond 3PN

Blanchet and Schäfer; Wiseman; Poisson (1993)

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Reaching beyond 3PN

Gopakumar and Iyer, 1997

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Reaching beyond 3PN

Arun, et. al., 2008

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Forseth, CRE, Hopper, PhysRev D93, 064058 (2016)

(lowest order in mass ratio)

Experimental mathematics

- Find exact rationals (and/or transcendentals) from very accurate numbers

  See e.g., Johnson-McDaniel, Shah, Whiting (2015)

- For 3.5PN, we expect the form

$$\mathcal{L}_{7/2} = -\frac{16285\pi}{504(1-e^2)^7}\left(1 + a_2 e^2 + a_4 e^4 + \cdots\right)$$

- Fit gives

$$a_2 = 13.75256306928666461979326578651110428819977484392590318\\ 28881383686418994985160167843598422334113482417705119\\ 883034494121793$$

- Integer relation algorithm (PSLQ; see `FindIntegerNullVector`) finds

$$a_2 = \frac{21500207}{1563360}$$

  to 108 digits

- Likelihood of coincidence $\sim 10^{-93}$

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Experimental mathematics

- Find exact rationals (and/or transcendentals) from very accurate numbers

  See e.g., Johnson-McDaniel, Shah, Whiting (2015)

- For 3.5PN, we expect the form

$$\mathcal{L}_{7/2} = -\frac{16285\pi}{504(1-e^2)^7}\left(1 + a_2 e^2 + a_4 e^4 + \cdots\right)$$

- Fit gives

$$\begin{aligned} a_2 = 13.&75256306928666461979326578651110428819977484392590318 \\ &288813836864189949851601678435984223341134824177051 19 \\ &883034494121793 \end{aligned}$$

- Integer relation algorithm (PSLQ; see `FindIntegerNullVector`) finds

$$a_2 = \frac{21500207}{1563360}$$

  to 108 digits

- Likelihood of coincidence $\sim 10^{-93}$

Example: 3.5PN exact to high order in e

- We find singular factor and exact coefficients to e^{24}:

$$
\begin{aligned}
\mathcal{L}_{7/2} = -\frac{16285\pi}{504(1-e^2)^7}\bigg(1 &+ \frac{21500207}{1563360}e^2 + \frac{3345329}{1563360}e^4 - \frac{111594754909}{1350743040}e^6 \\
&- \frac{82936785623}{1800990720}e^8 - \frac{11764982139179}{3457902182400}e^{10} \\
&- \frac{216868426237103}{311211196416000}e^{12} - \frac{30182578123501193}{813298593300480000}e^{14} \\
&- \frac{351410391437739607}{1561533299136921600}e^{16} - \frac{10065633193333377521717}{674582385227150131200 0}e^{18} \\
&- \frac{1384335564976030365 91}{1317543721146777600000}e^{20} - \frac{168362170547496099724064 21}{673646213172736032768 0000}e^{22} \\
&- \frac{2077866815397007172515220959}{1091306865339832373084160000}e^{24} + \cdots\bigg)
\end{aligned}
$$

... plus much higher orders in e^2 in decimal form

Example: 4PN Log term has exact closed form

- Next consider 4PN Log term

$$\langle \dot{E} \rangle = \frac{32}{5} \left(\frac{\mu}{M} \right)^2 y^5 \left(\mathcal{I}_0 + \cdots + y^{7/2} \mathcal{L}_{7/2} + y^4 \log(y) \mathcal{L}_{4L} + \cdots \right)$$

- The $\mathcal{L}_{4L}$ enhancement function is:

$$\mathcal{L}_{4L} = \frac{232597}{8820(1-e^2)^{15/2}} \left(1 + \frac{14770533}{465194} e^2 + \frac{142278179}{930388} e^4 + \frac{318425291}{1860776} e^6 + \frac{1256401651}{29772416} e^8 + \frac{64986219}{59544832} e^{10} \right)$$

- Which is an exact, closed-form expression!

- Nice touchstone for current work on 4PN

Example: 4PN Log term has exact closed form

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- Which is an exact, closed-form expression!

- Nice touchstone for current work on 4PN

New: Angular momentum flux at infinity

- Confirm and extend results for angular momentum flux at infinity

$$\langle \dot{J} \rangle = \frac{32}{5}\frac{\mu^2}{M} y^{3/2} \Big[\mathcal{J}_0 + y\mathcal{J}_1 + y^{3/2}\mathcal{J}_{3/2} + y^2\mathcal{J}_2 + y^{5/2}\mathcal{J}_{5/2} + y^3\mathcal{J}_3 + y^{7/2}\mathcal{J}_{7/2}$$
$$+ y^4\Big(\mathcal{J}_4 + \mathcal{J}_{4L}\log(y)\Big) + y^{9/2}\Big(\mathcal{J}_{9/2} + \mathcal{J}_{9/2L}\log(y)\Big)$$
$$+ y^5\Big(\mathcal{J}_5 + \mathcal{J}_{5L}\log(y)\Big) + y^{11/2}\Big(\mathcal{J}_{11/2} + \mathcal{J}_{11/2L}\log(y)\Big)$$
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$$+ y^7\Big(\mathcal{J}_7 + \mathcal{J}_{7L}\log(y) + \mathcal{J}_{7L^2}\log^2(y)\Big) + \cdots \Big]$$

New: Angular momentum flux at infinity

- Confirm and extend results for angular momentum flux at infinity

Peters, 1964

$$\langle \dot{J} \rangle = \frac{32}{5}\frac{\mu^2}{M} y^{3/2} \Big[\mathcal{J}_0 + y\mathcal{J}_1 + y^{3/2}\mathcal{J}_{3/2} + y^2\mathcal{J}_2 + y^{5/2}\mathcal{J}_{5/2} + y^3\mathcal{J}_3 + y^{7/2}\mathcal{J}_{7/2}$$
$$+ y^4\Big(\mathcal{J}_4 + \mathcal{J}_{4L}\log(y)\Big) + y^{9/2}\Big(\mathcal{J}_{9/2} + \mathcal{J}_{9/2L}\log(y)\Big)$$
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New: Angular momentum flux at infinity

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Junker and Schäfer, 1992

$$\langle \dot{J} \rangle = \frac{32}{5}\frac{\mu^2}{M}y^{3/2}\Big[\mathcal{J}_0 + y\mathcal{J}_1 + y^{3/2}\mathcal{J}_{3/2} + y^2\mathcal{J}_2 + y^{5/2}\mathcal{J}_{5/2} + y^3\mathcal{J}_3 + y^{7/2}\mathcal{J}_{7/2}$$
$$+ y^4\Big(\mathcal{J}_4 + \mathcal{J}_{4L}\log(y)\Big) + y^{9/2}\Big(\mathcal{J}_{9/2} + \mathcal{J}_{9/2L}\log(y)\Big)$$
$$+ y^5\Big(\mathcal{J}_5 + \mathcal{J}_{5L}\log(y)\Big) + y^{11/2}\Big(\mathcal{J}_{11/2} + \mathcal{J}_{11/2L}\log(y)\Big)$$
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New: Angular momentum flux at infinity

- Confirm and extend results for angular momentum flux at infinity

Rieth and Schäfer, 1997

$$
\begin{aligned}
\langle \dot{J} \rangle = \frac{32}{5}\frac{\mu^2}{M} y^{3/2} \Big[& \mathcal{J}_0 + y\mathcal{J}_1 + y^{3/2}\mathcal{J}_{3/2} + y^2\mathcal{J}_2 + y^{5/2}\mathcal{J}_{5/2} + y^3\mathcal{J}_3 + y^{7/2}\mathcal{J}_{7/2} \\
& + y^4\left(\mathcal{J}_4 + \mathcal{J}_{4L}\log(y)\right) + y^{9/2}\left(\mathcal{J}_{9/2} + \mathcal{J}_{9/2L}\log(y)\right) \\
& + y^5\left(\mathcal{J}_5 + \mathcal{J}_{5L}\log(y)\right) + y^{11/2}\left(\mathcal{J}_{11/2} + \mathcal{J}_{11/2L}\log(y)\right) \\
& + y^6\left(\mathcal{J}_6 + \mathcal{J}_{6L}\log(y) + \mathcal{J}_{6L^2}\log^2(y)\right) \\
& + y^{13/2}\left(\mathcal{J}_{13/2} + \mathcal{J}_{13/2L}\log(y)\right) \\
& + y^7\left(\mathcal{J}_7 + \mathcal{J}_{7L}\log(y) + \mathcal{J}_{7L^2}\log^2(y)\right) + \cdots \Big]
\end{aligned}
$$

New: Angular momentum flux at infinity

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Gopakumar and Iyer, 1997

$$\langle \dot{J} \rangle = \frac{32}{5}\frac{\mu^2}{M} y^{3/2} \Big[\mathcal{J}_0 + y\mathcal{J}_1 + y^{3/2}\mathcal{J}_{3/2} + y^2\mathcal{J}_2 + y^{5/2}\mathcal{J}_{5/2} + y^3\mathcal{J}_3 + y^{7/2}\mathcal{J}_{7/2}$$
$$+ y^4\left(\mathcal{J}_4 + \mathcal{J}_{4L}\log(y)\right) + y^{9/2}\left(\mathcal{J}_{9/2} + \mathcal{J}_{9/2L}\log(y)\right)$$
$$+ y^5\left(\mathcal{J}_5 + \mathcal{J}_{5L}\log(y)\right) + y^{11/2}\left(\mathcal{J}_{11/2} + \mathcal{J}_{11/2L}\log(y)\right)$$
$$+ y^6\left(\mathcal{J}_6 + \mathcal{J}_{6L}\log(y) + \mathcal{J}_{6L^2}\log^2(y)\right)$$
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$$+ y^7\left(\mathcal{J}_7 + \mathcal{J}_{7L}\log(y) + \mathcal{J}_{7L^2}\log^2(y)\right) + \cdots \Big]$$

New: Angular momentum flux at infinity

- Confirm and extend results for angular momentum flux at infinity

Arun, et. al., 2009

$$\langle \dot{J} \rangle = \frac{32}{5}\frac{\mu^2}{M} y^{3/2}\Big[\mathcal{J}_0 + y\mathcal{J}_1 + y^{3/2}\mathcal{J}_{3/2} + y^2\mathcal{J}_2 + y^{5/2}\mathcal{J}_{5/2} + y^3\mathcal{J}_3 + y^{7/2}\mathcal{J}_{7/2}$$
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$$+ y^5\Big(\mathcal{J}_5 + \mathcal{J}_{5L}\log(y)\Big) + y^{11/2}\Big(\mathcal{J}_{11/2} + \mathcal{J}_{11/2L}\log(y)\Big)$$
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New: Angular momentum flux at infinity

- Confirm and extend results for angular momentum flux at infinity

$$\langle \dot{J} \rangle = \frac{32}{5}\frac{\mu^2}{M} y^{3/2} \Big[\mathcal{J}_0 + y\mathcal{J}_1 + y^{3/2}\mathcal{J}_{3/2} + y^2\mathcal{J}_2 + y^{5/2}\mathcal{J}_{5/2} + y^3\mathcal{J}_3 + y^{7/2}\mathcal{J}_{7/2}$$
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Forseth, CRE, Hopper, in prep (lowest order in mass ratio)

Examples: 3.5PN term and 4PN log term

$$\mathcal{J}_{7/2} = \frac{\pi}{(1-e^2)^{11/2}}\left(-\frac{16285}{504} - \frac{370255}{1008}e^2 - \frac{11888119}{48384}e^4 + \frac{6476904953}{20901888}e^6 + \frac{8877357035}{167215104}e^8 \right.$$
$$+ \frac{186159455101}{26754416640}e^{10} + \frac{34468729921289}{9631589990400}e^{12} + \frac{7790078031395741}{3775583276236800}e^{14}$$
$$+ \frac{3195497183505568 99}{2416373296791552 00}e^{16} + \frac{23734478429688515533}{26096831605348761600}e^{18} + \frac{857752832161057782787}{1304841580267438080000}e^{20}$$
$$\left. + \frac{16622935591075529599945669}{33682310658636801638400 00}e^{22} + \cdots\right),$$

$$\mathcal{J}_{4L} = \frac{1}{(1-e^2)^6}\left(\frac{232597}{8820} + \frac{3482879e^2}{8820} + \frac{34971299e^4}{35280} + \frac{6578731e^6}{14112} + \frac{2503623e^8}{125440}\right)$$

Again, the 4PN log term has an exact, closed form

New: Energy and angular momentum flux at horizon

- Horizon energy flux (Poisson & Sasaki 1995, $e = 0$; Sago & Fujita 2015, $\mathcal{B}_0$ to e^6)

$$\langle \dot{E} \rangle = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 y^9 \Big[\mathcal{B}_0 + y\mathcal{B}_1 + y^2\mathcal{B}_2 + y^3\left(\mathcal{B}_3 + \mathcal{B}_{3L}\log(y)\right) + y^4\left(\mathcal{B}_4 + \mathcal{B}_{4L}\log(y)\right) \\ + y^5\left(\mathcal{B}_5 + \mathcal{B}_{5L}\log(y)\right) + y^{11/2}\mathcal{B}_{11/2} \\ + y^6\left(\mathcal{B}_6 + \mathcal{B}_{6L}\log(y) + \mathcal{B}_{6L^2}\log^2(y)\right) + y^{13/2}\mathcal{B}_{13/2} \\ + y^7\left(\mathcal{B}_7 + \mathcal{B}_{7L}\log(y) + \mathcal{B}_{7L^2}\log^2(y)\right) + \cdots\Big]$$

- Horizon angular momentum flux (Sago & Fujita 2015, $\mathcal{D}_0$ to e^6)

$$\langle \dot{J} \rangle = \frac{32}{5}\frac{\mu^2}{M} y^{15/2} \Big[\mathcal{D}_0 + y\mathcal{D}_1 + y^2\mathcal{D}_2 + y^3\left(\mathcal{D}_3 + \mathcal{D}_{3L}\log(y)\right) + y^4\left(\mathcal{D}_4 + \mathcal{D}_{4L}\log(y)\right) \\ + y^5\left(\mathcal{D}_5 + \mathcal{D}_{5L}\log(y)\right) + y^{11/2}\mathcal{D}_{11/2} \\ + y^6\left(\mathcal{D}_6 + \mathcal{D}_{6L}\log(y) + \mathcal{D}_{6L^2}\log^2(y)\right) + y^{13/2}\mathcal{D}_{13/2} \\ + y^7\left(\mathcal{D}_7 + \mathcal{D}_{7L}\log(y) + \mathcal{D}_{7L^2}\log^2(y)\right) + \cdots\Big]$$

Sample terms: energy flux at horizon

- First three PN orders have closed forms

$$\mathcal{B}_0 = \frac{1}{(1-e^2)^{15/2}} \left(1 + \frac{31}{2}e^2 + \frac{255}{8}e^4 + \frac{185}{16}e^6 + \frac{25}{64}e^8\right),$$

$$\mathcal{B}_1 = \frac{1}{(1-e^2)^{17/2}} \left(4 + \frac{147}{2}e^2 + \frac{799}{8}e^4 - \frac{2635}{16}e^6 - \frac{13515}{128}e^8 - \frac{275}{64}e^{10}\right),$$

$$\mathcal{B}_2 = \frac{1}{\left(1-e^2\right)^{19/2}} \left(-\frac{181}{14} + \frac{1336}{21}e^2 + \frac{25097}{24}e^4 + \frac{42743}{48}e^6 + \frac{489245}{768}e^8 + \frac{360197}{768}e^{10} + \frac{6025}{256}e^{12}\right) + \frac{1}{\left(1-e^2\right)^9}\left(\frac{75}{2} + \frac{2175}{4}e^2 + \frac{9825}{16}e^4 - \frac{24375}{32}e^6 - \frac{53625}{128}e^8 - \frac{1875}{128}e^{10}\right),$$

$$\vdots$$

$$\mathcal{B}_{4L} = \frac{1}{(1-e^2)^{23/2}} \left(-\frac{9148}{105} - \frac{11348}{3}e^2 - \frac{2650657}{105}e^4 - \frac{412167e}{20}e^6 + \frac{9681067}{160}e^8 + \frac{4810141}{80}e^{10} + \frac{1698271}{160}e^{12} + \frac{99085}{512}e^{14}\right),$$

- As does the 4PN Log term

Sample terms: angular momentum flux at horizon

- Same for angular momentum: First three PN orders have closed forms

$$\mathcal{D}_0 = \frac{1}{(1-e^2)^6}\left(1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6\right),$$

$$\mathcal{D}_1 = \frac{1}{(1-e^2)^7}\left(4 + 42e^2 + \frac{15}{4}e^4 - 40e^6 - \frac{195}{64}e^8\right),$$

$$\mathcal{D}_2 = \frac{1}{(1-e^2)^8}\left(-\frac{38}{7} + 197e^2 + \frac{7965}{16}e^4 + \frac{1175}{16}e^6 + \frac{37825}{256}e^8 + \frac{495}{32}e^{10}\right) + \frac{1}{(1-e^2)^{15/2}}\left(30 + 195e^2 - \frac{225}{4}e^4 - \frac{1275}{8}e^6 - \frac{75}{8}e^8\right),$$

$$\vdots$$

$$\mathcal{D}_{4L} = \frac{1}{(1-e^2)^{10}}\left(-\frac{9148}{105} - \frac{18240}{7}e^2 - \frac{356711}{35}e^4 - \frac{3973}{5}e^6 + \frac{388523}{32}e^8 + \frac{304913}{80}e^{10} + \frac{6415}{64}e^{12}\right),$$

- As does the 4PN Log term again

How far can we push finding exact coefficients?

- At 200 decimal places, finding mixtures of several transcendentals is hard
- May be able to do much better by factoring the tail terms (Damour and Nagar; Damour, Iyer and Nagar; Johnson-McDaniel)
- Works for circular orbits on l, m mode basis
- *May* work for eccentric orbits on an l, m, n mode basis

Conclusions

Use BH perturbations to probe PN limit in eccentric orbits

- Combined MST with EHS and SSI in *Mathematica*
- Achieved >200 decimal places accuracy
- "See" $\sim$ 10 PN orders

New understanding of 1.5, 2.5, and 3PN hereditary terms

- Determined e singular factors in y, e representation
- Confirmed with asymptotic analysis
- Found high order (e^2) expansions: $\varphi(e)$, $\psi(e)$, and $\chi(e)$
- Found new closed form, log singular part of $\chi(e)$

First application: $\dot{E}^{\infty}(e)$

- Confirmed known PN theory to 3PN
- Determined new $\dot{E}(e)$ terms at 3.5PN, 4PN, etc

Second application: $\dot{J}^{\infty}(e)$

Third: $\dot{E}^{H}(e)$ and $\dot{J}^{H}(e)$ down horizon