Fast and accurate evaluation of black hole Green's functions using surrogate models

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with Barry Wardell (University College Dublin)

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Caltech

Green's functions

$$\Box_x G(x, x') = \frac{\delta^4(x - x')}{\sqrt{-g(x)}} \quad \Longrightarrow \quad \phi(x) = \int d^4x' \sqrt{-g(x')}\, G(x, x') J(x')$$

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What are the advantages of using Green's functions?

- Compute only once
- Nearly all physical quantities of interest are calculated via convolution integrals
- Arbitrary motion for self-force
- Geometric interpretation *(see also J. Thornburg's talk)*
- Higher-order self-force
- Self-consistent (higher-order) self-forced evolution
- Self-consistent inspiral waveforms
- Arguably straightforward to implement once known

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- Computationally expensive
- Large data sets (*G* is a bitensor!)
- Gravitational perturbations: Instabilities? Metric reconstruction?

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Goal:

Find a way for Green’s functions to be efficient and accurate to use for practical self-force and related computations.

Numerical Green's functions

$$\Box_x G(x,x') = \frac{\delta^4(x-x')}{\sqrt{-g(x)}} \quad \longrightarrow$$ Narrow Gaussian

Zenginoglu & CRG (12)

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Numerical Green's functions are globally valid approximations but utilizing analytic approximations at early and late times is extremely helpful

- Quasi-local expansions *Ottewill & Wardell (08); Wardell's thesis*
- Pade approximants *Casals et al (09)*
- Method of matched expansions *Anderson & Wiseman (05); Casals et al (13)*

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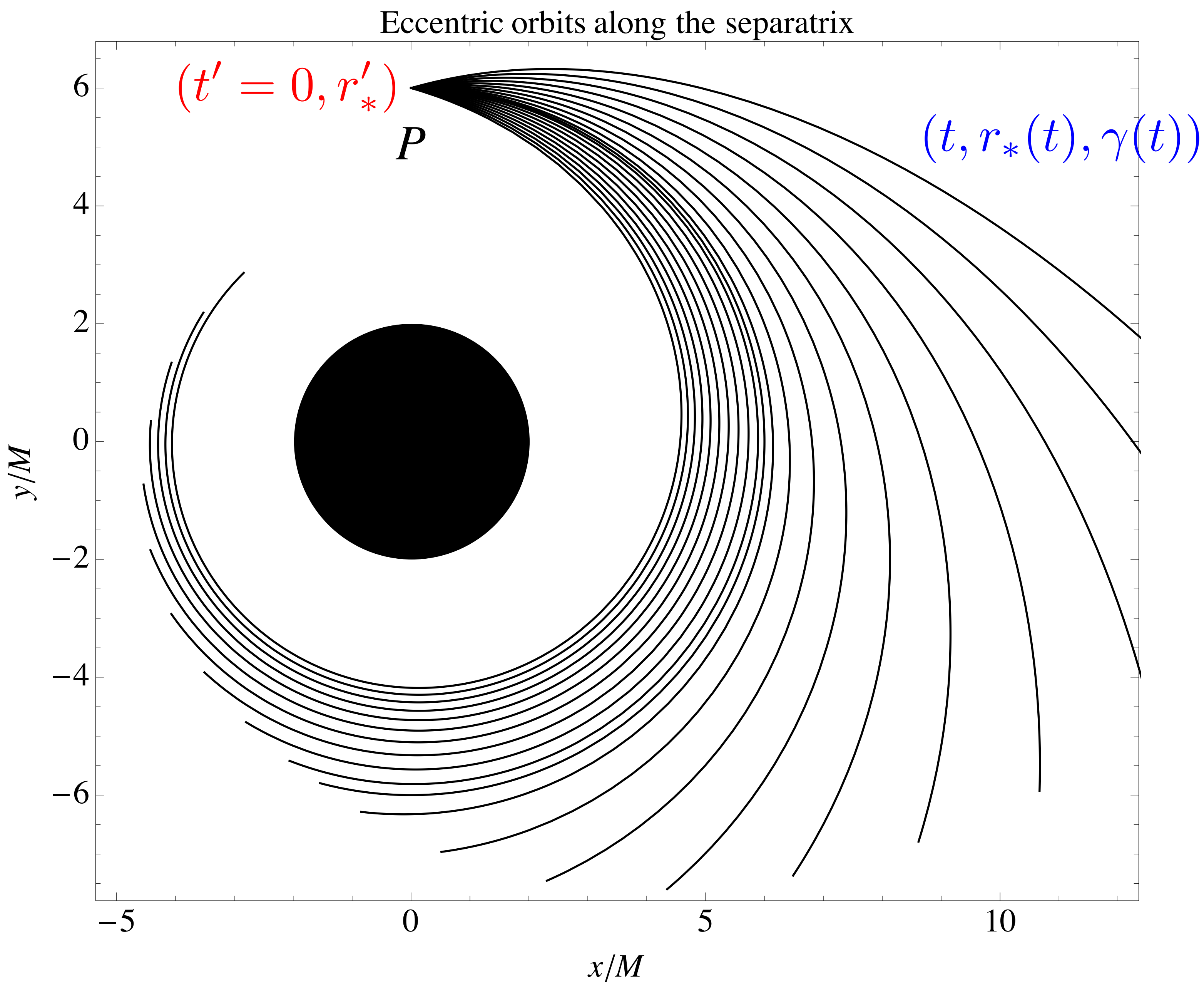
When these analytical approximations (e.g., in Schwarzschild) are available we use numerical Green's functions for intermediate times



Wardell, CRG et al (14)

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To quickly predict accurate solutions to the Green's function wave equation that are otherwise too slow and too large for practical use

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- *Reduced basis (compression in source & field points)*
- *Empirical interpolation (compression in time)*
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The result

An accurate *surrogate model* to generate new Green's function data on demand

Surrogate models for gravitational waveforms have been used successfully for:

- Non-spinning Effective One-Body (*EOBNRv2*)
  Field, CRG, et al PRX (14)
- Spin-aligned Effective One-Body (*SEOBNRv2*)
  Purrer (15)
- Non-spinning Numerical Relativity (*SpEC*)
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However, the steps for building a Green's function surrogate are necessarily a little different than for waveforms

- Provides one with dynamics, field content, and waveforms
- Source and field points are time-dependent for worldline convolutions

Surrogate building: Initial stuff

see Wardell, CRG et al (14)

- Coordinates: (t, r_*, θ, ϕ)

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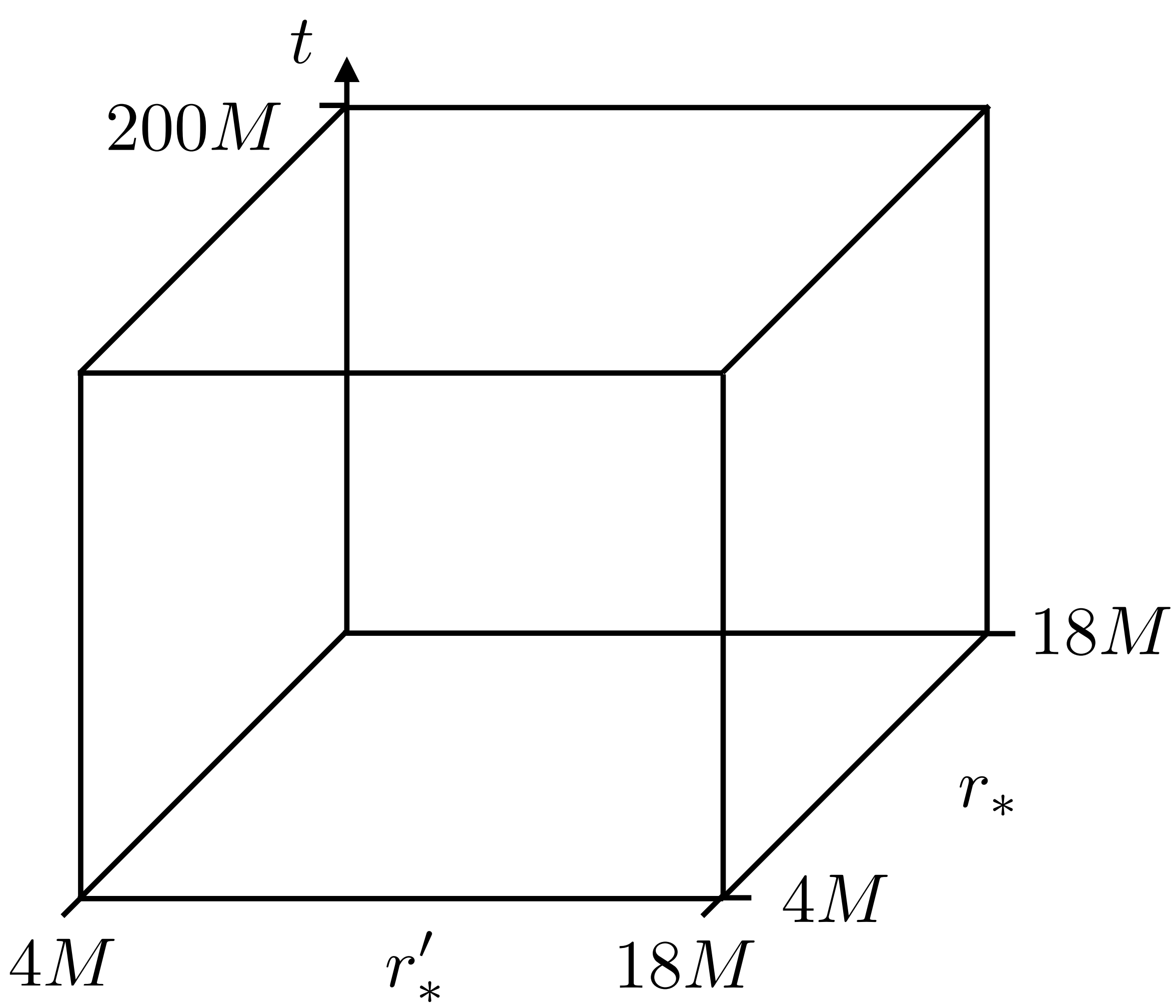
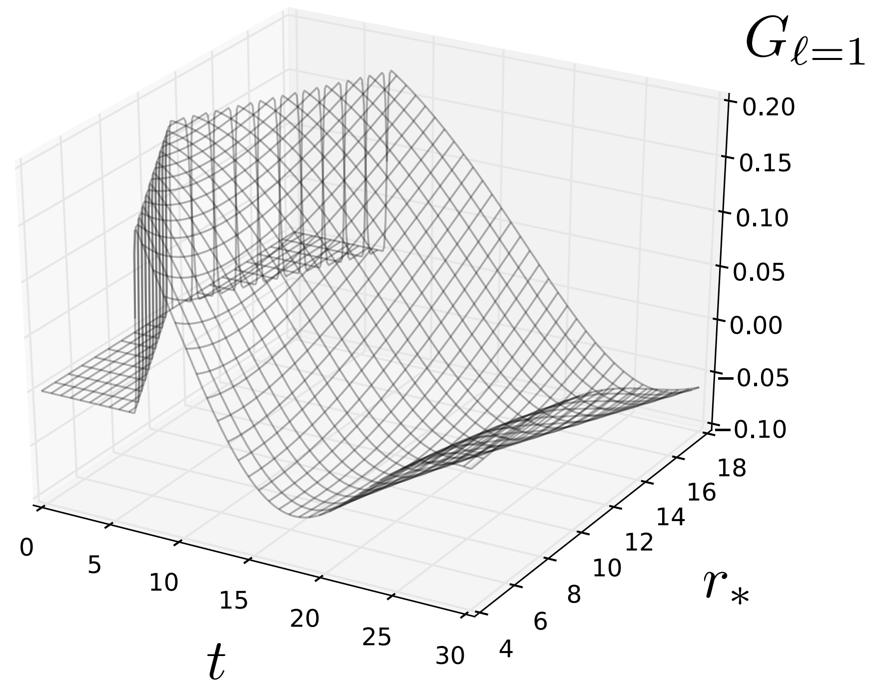


- Reduce known features by analytically time-shifting each series by light travel time from source point to field point, $t \to t - |r_* - r'_*|$



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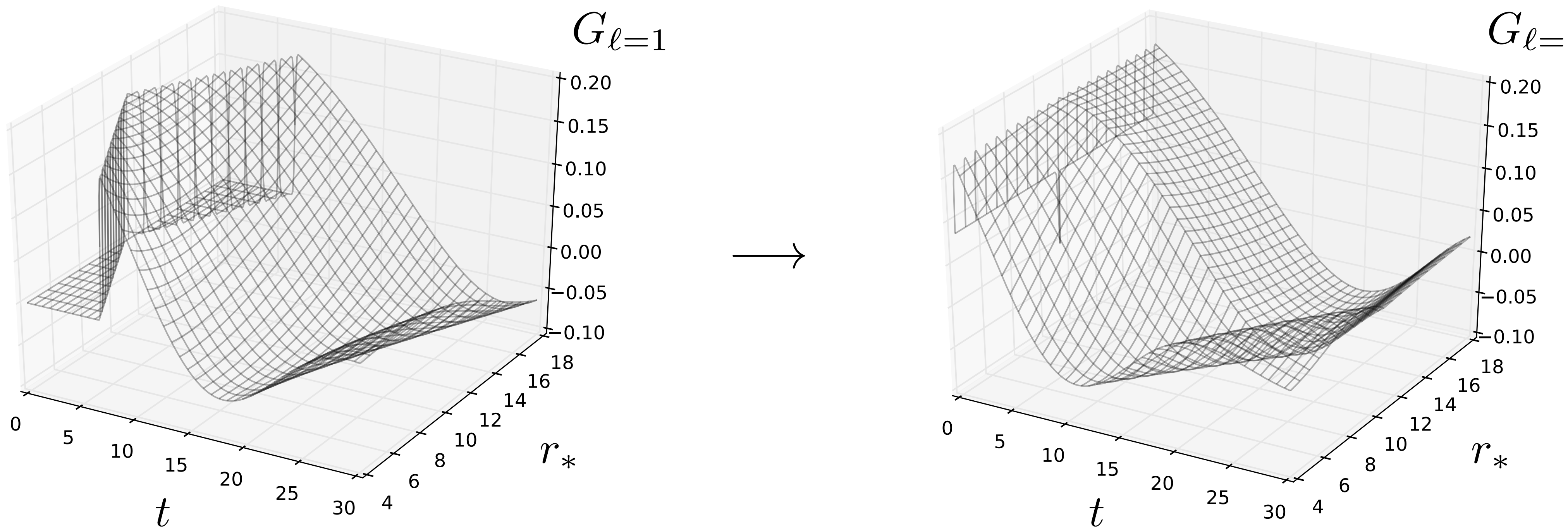

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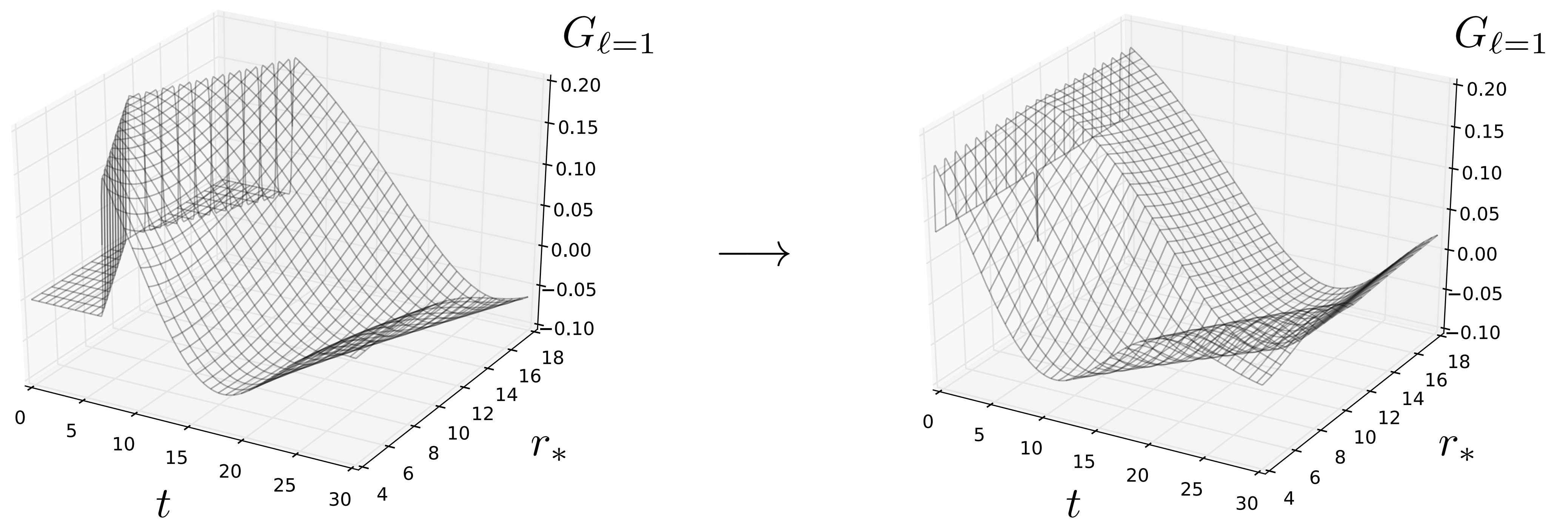


- In addition, because only a finite number of modes can be computed we introduce a smoothing factor *Wardell, CRG et al (14)*

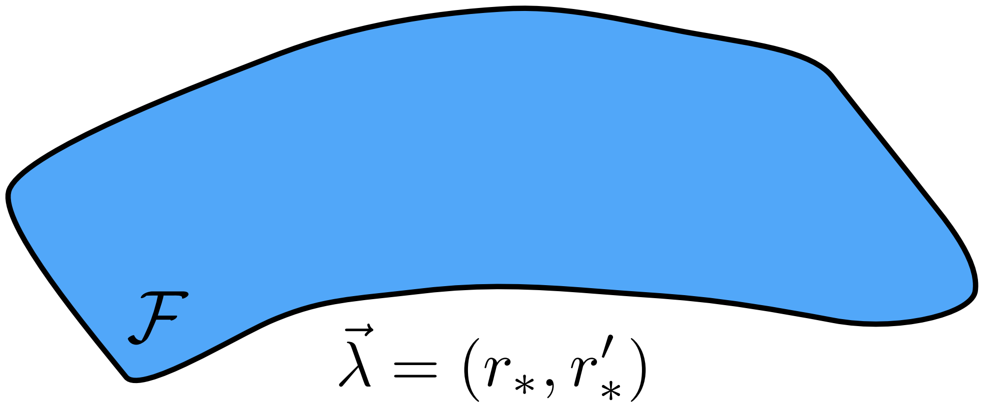
$$G(x^\alpha, x'^\alpha) \approx \frac{1}{rr'} \sum_{\ell=0}^{\ell_{\max}} P_\ell(\cos\theta)(2\ell+1)e^{-\ell^2/2\ell_{\rm cut}^2} G_\ell(t-t'; r_*, r'_*)$$

$\ell_{\max} = 100$ $\qquad$ $\ell_{\rm cut} = \ell_{\max}/5$

1) Reduced basis via greedy algorithm

Can find a linear approximation space that is nearly optimal

Set of functions $\mathcal{F}$



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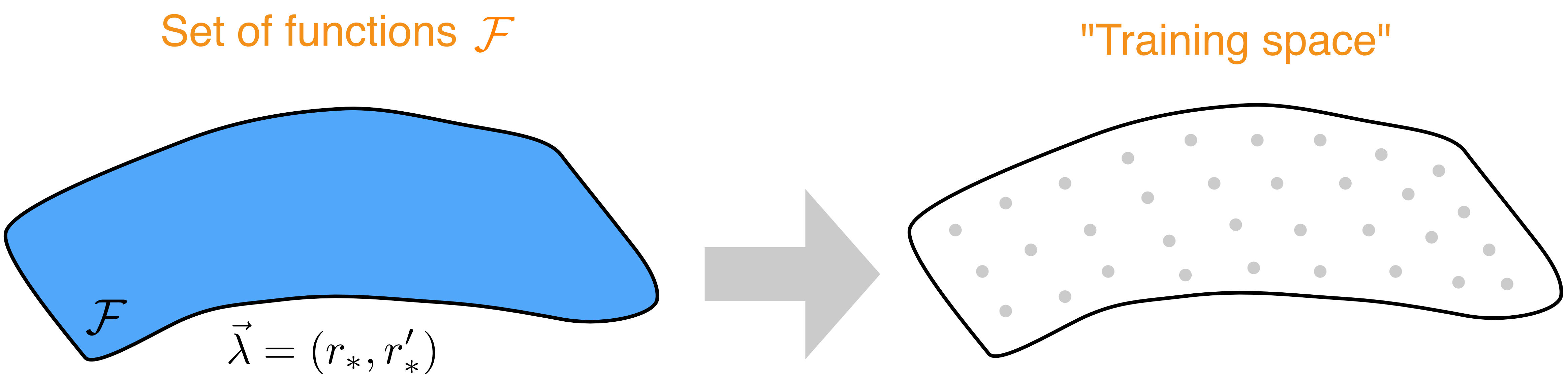

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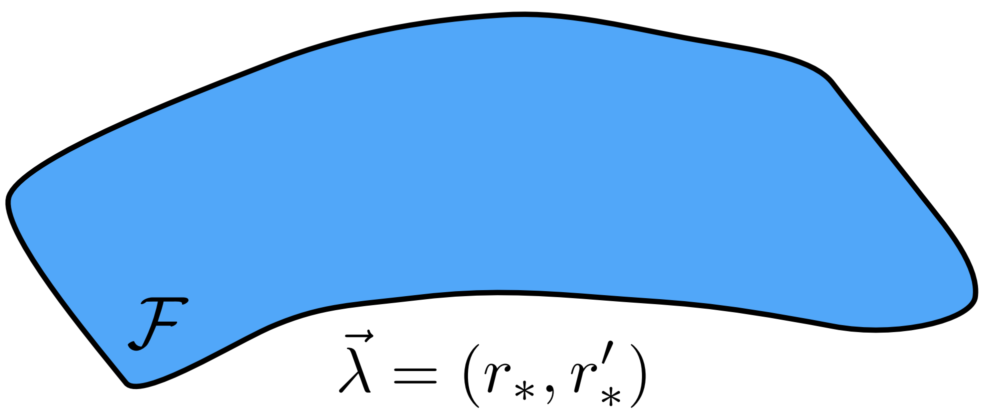

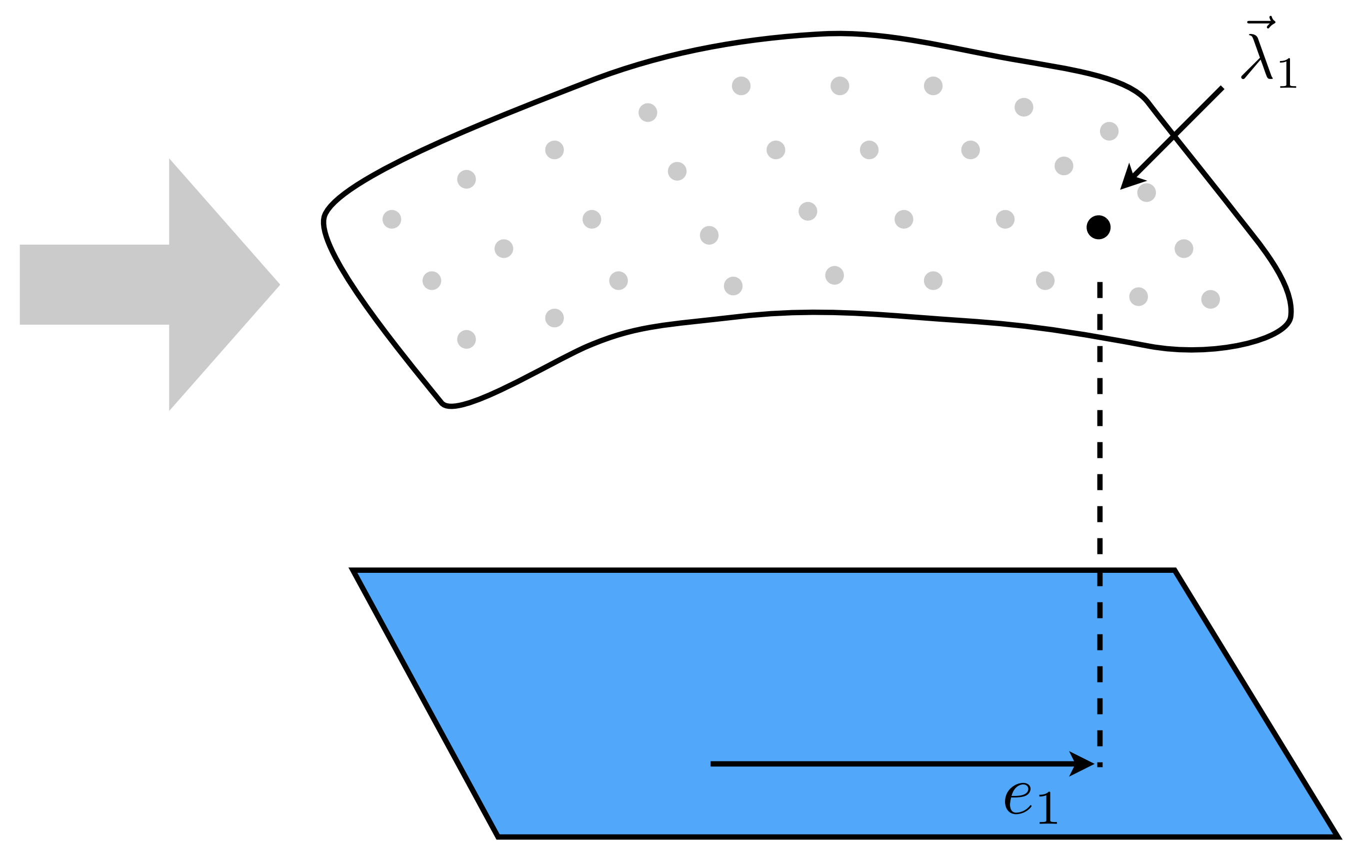

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$$e_1 = G_\ell(t; \vec{\lambda}_1), \; C_1 = \{e_1\}$$

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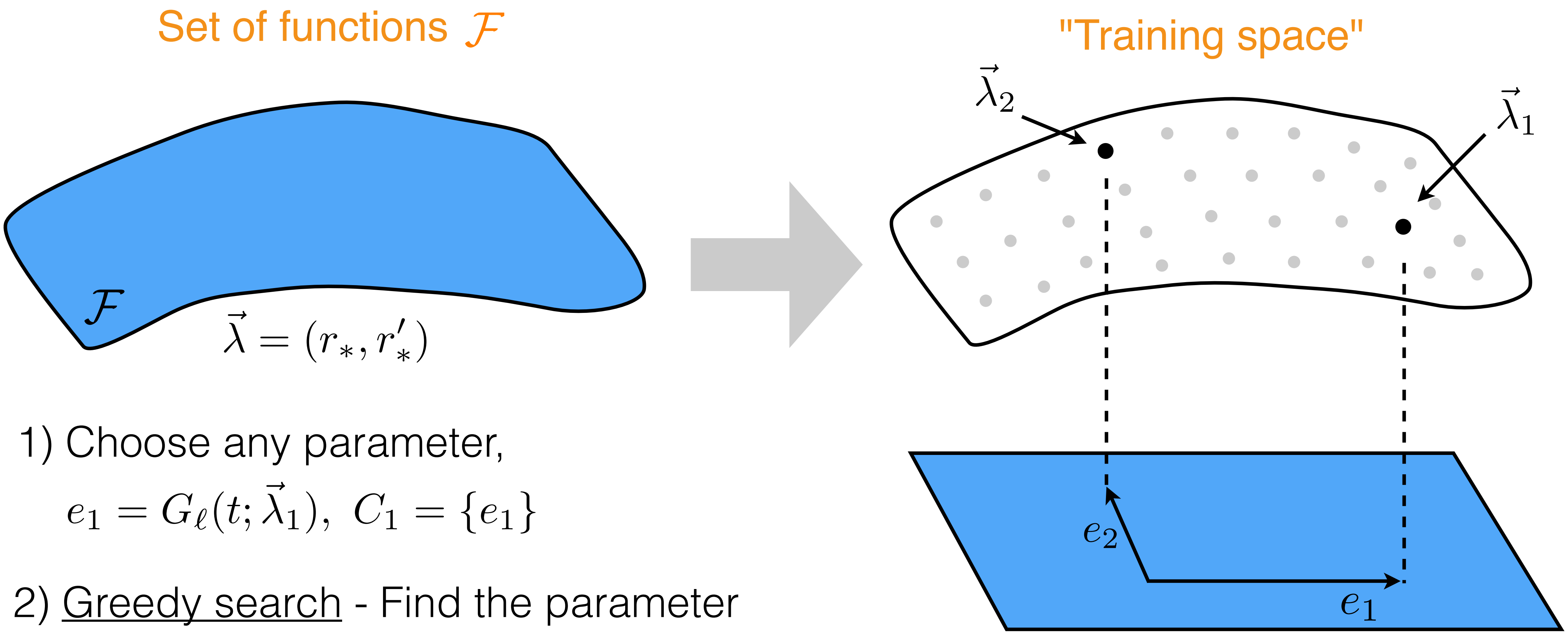

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$$\max_t \left| G_\ell(t; \vec{\lambda}) - P_1[G_\ell(t; \vec{\lambda})] \right|, \; P_1[\cdot] = e_1 \langle e_1, \cdot \rangle$$

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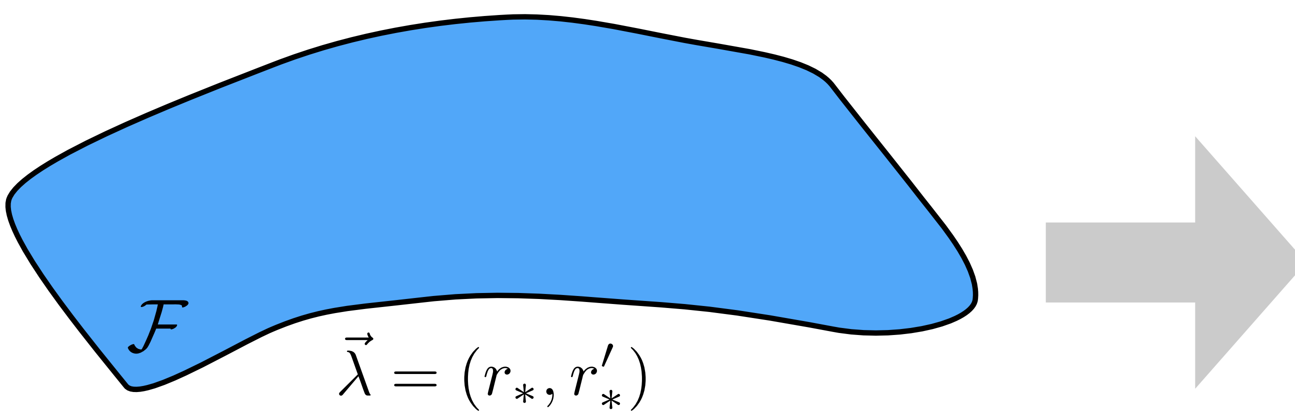

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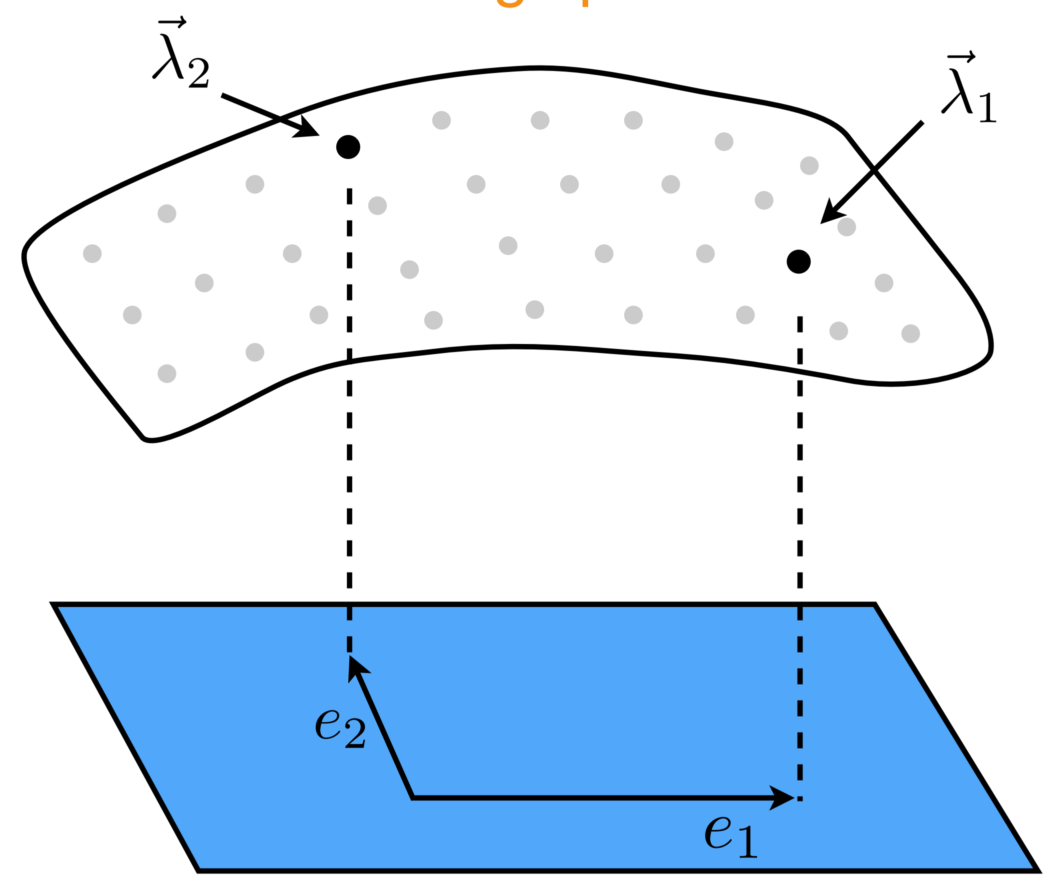

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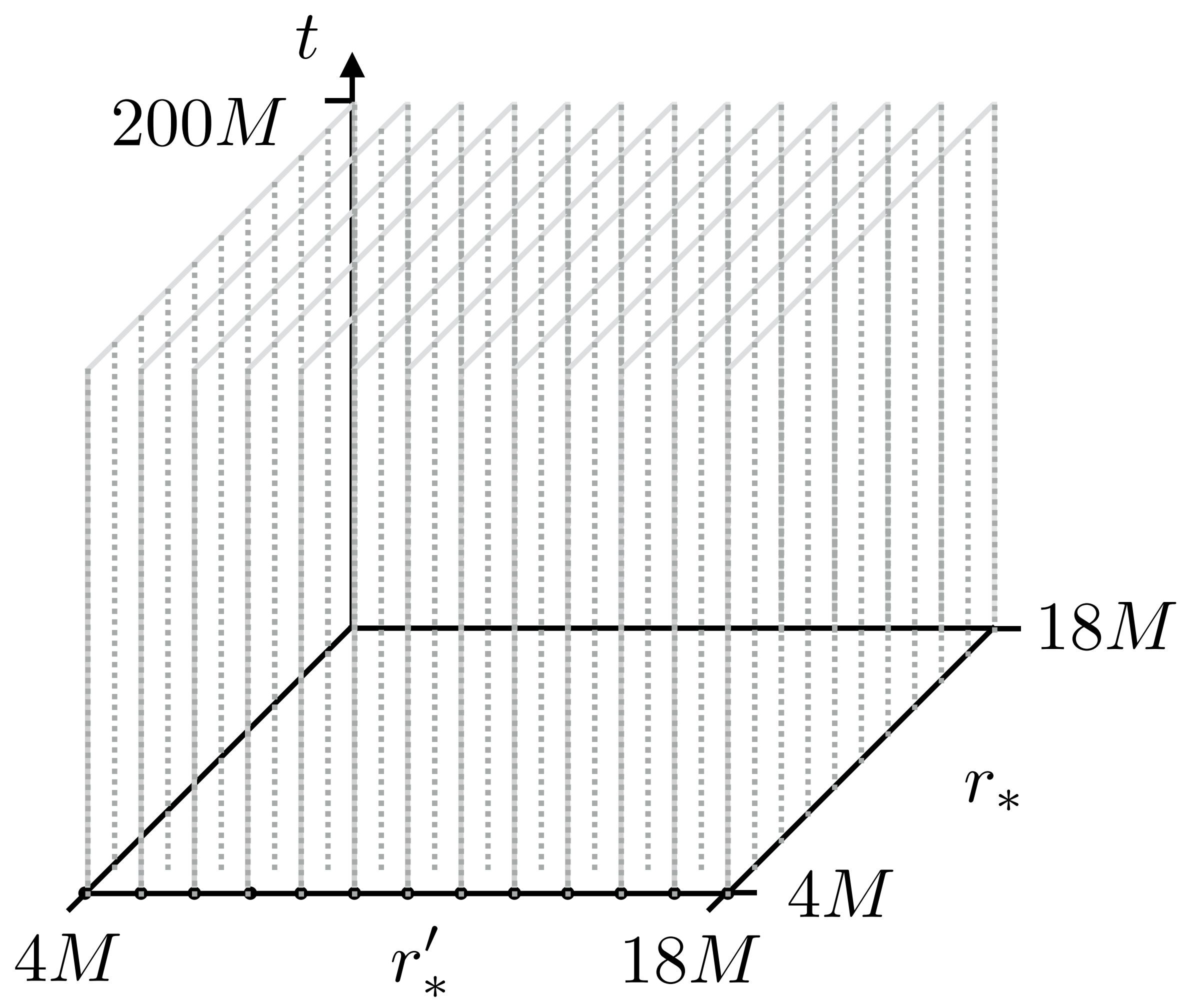
3) Orthogonalization to get basis vector e_2

$$C_2 = \{e_1, e_2\},\ C_1 \subset C_2$$



t
$200M$
$18M$
r_*
$4M$
$4M$
r'_*
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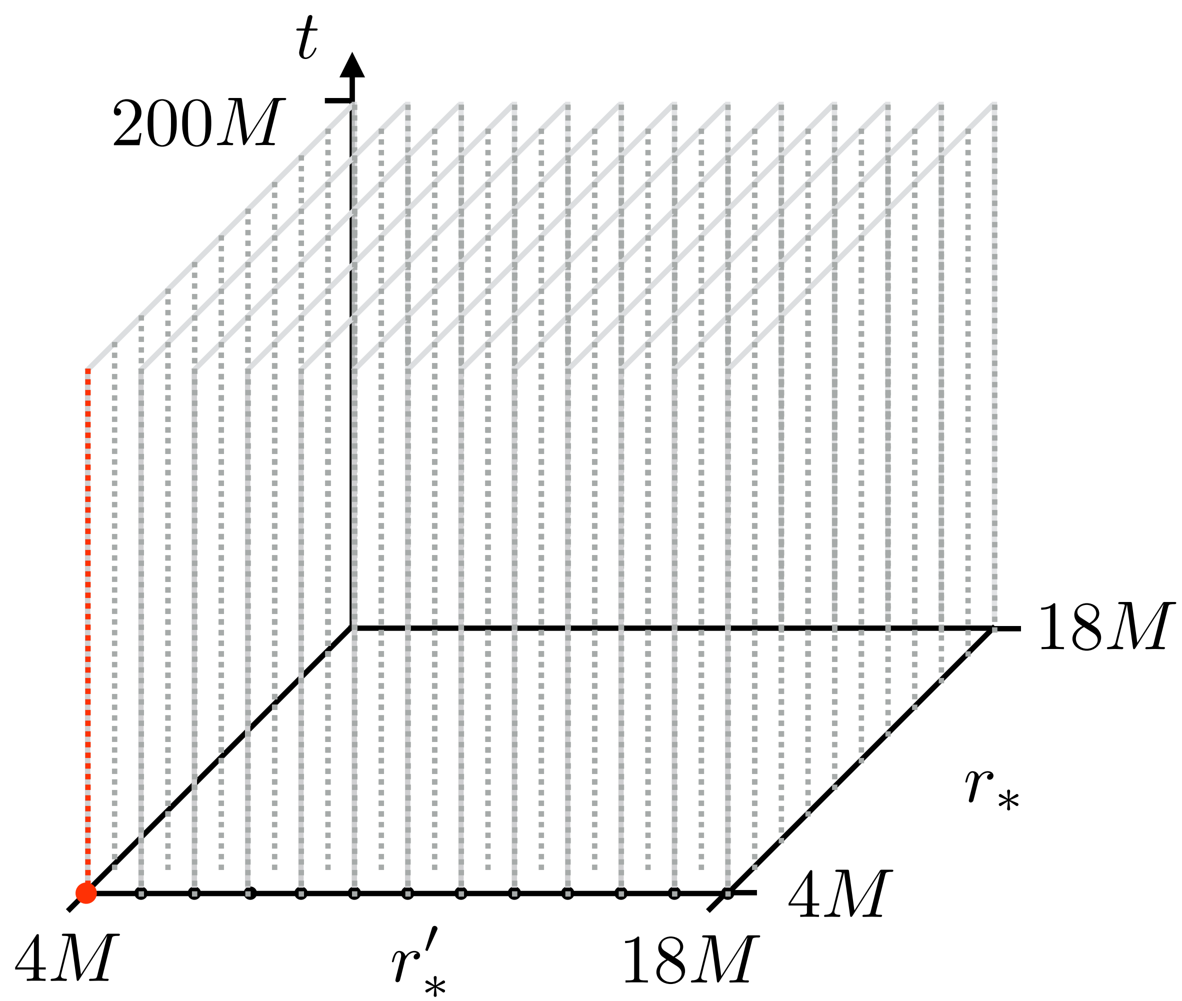

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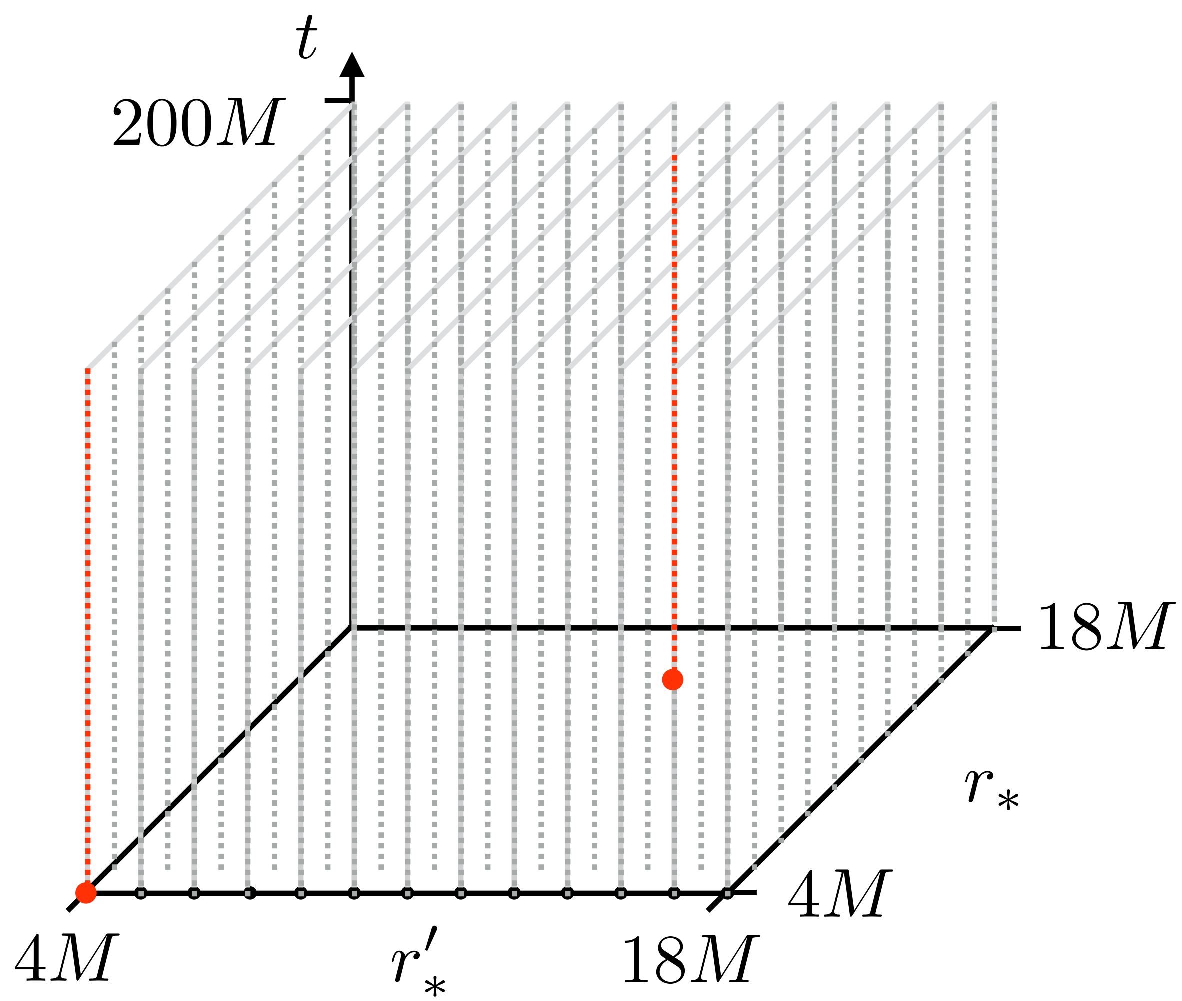

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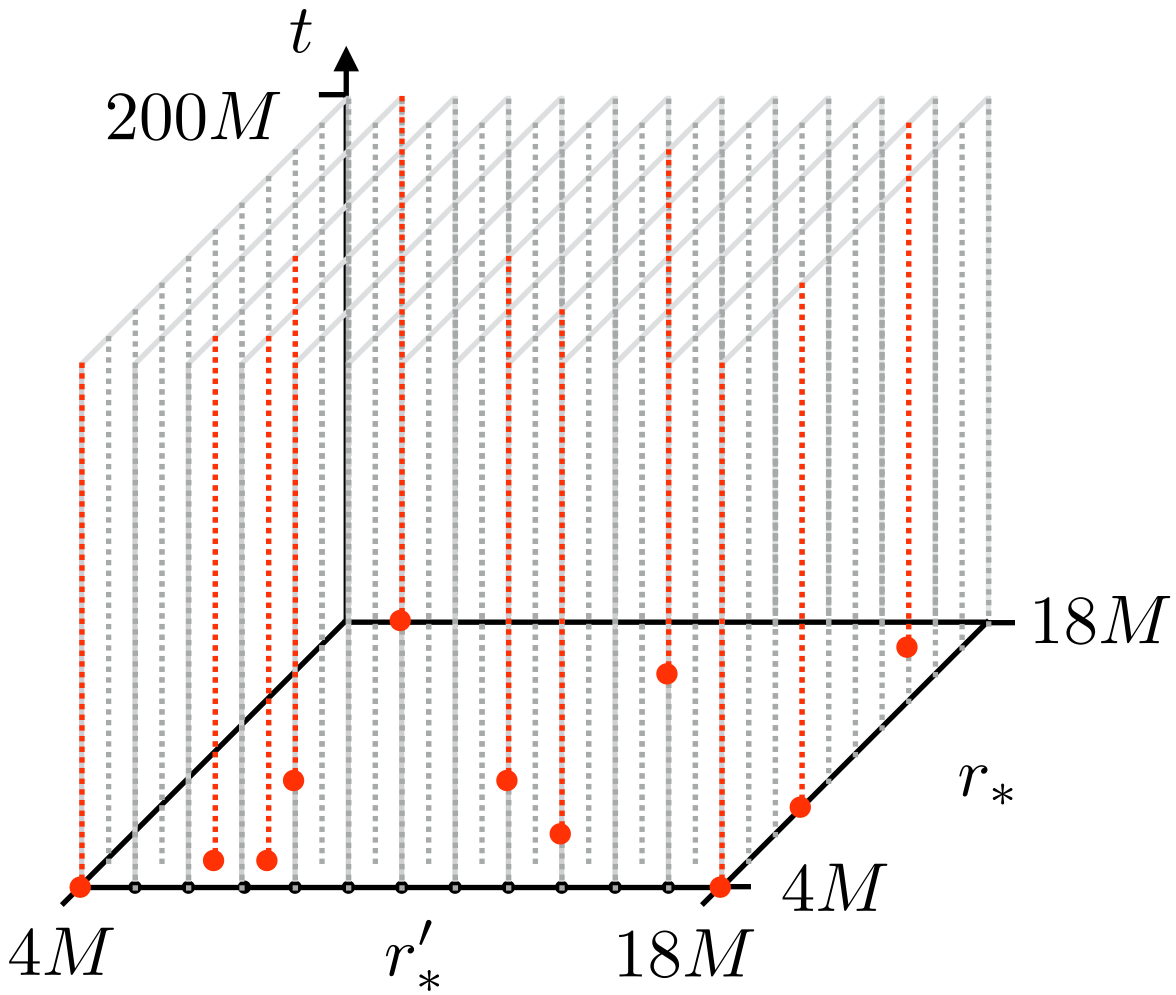

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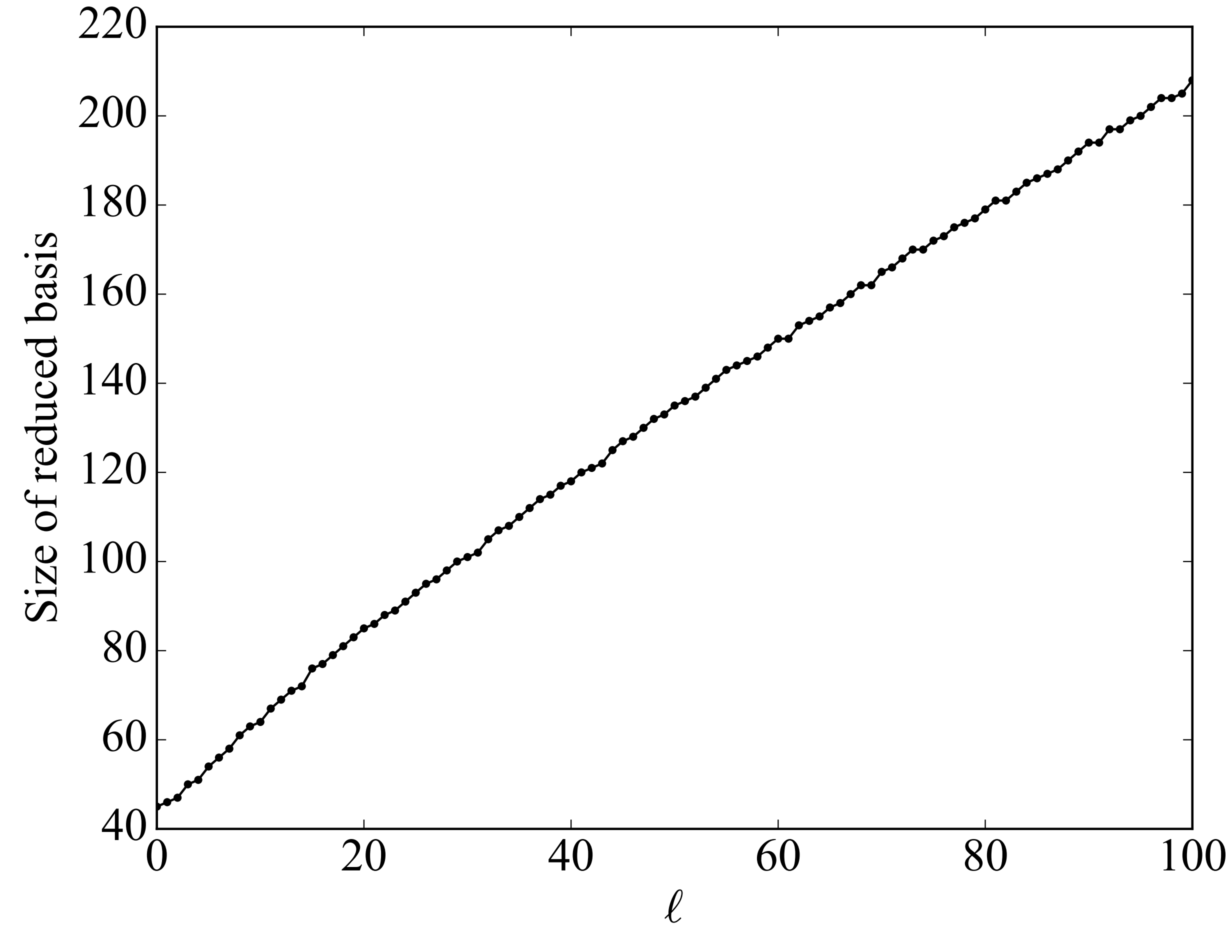
Basis size grows nearly linearly
mode number



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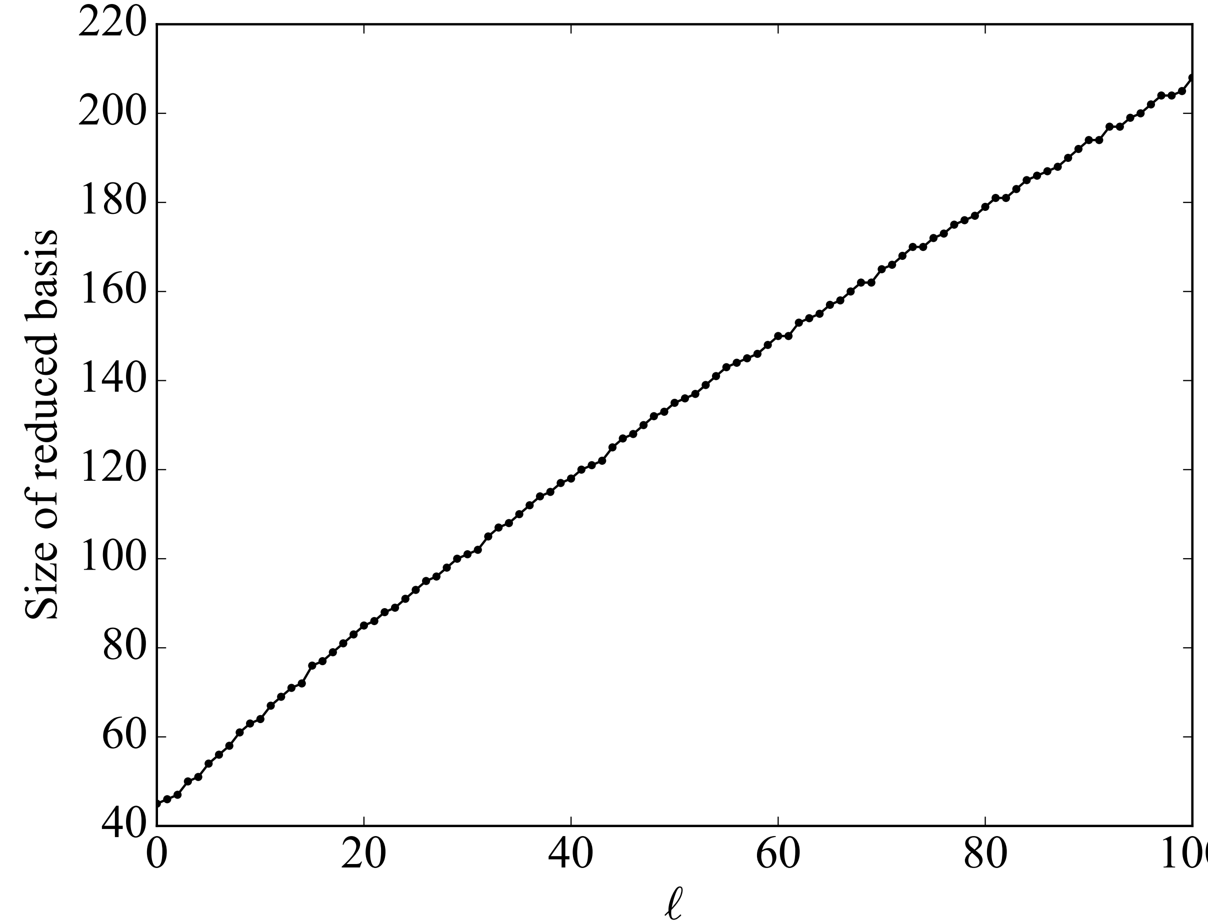

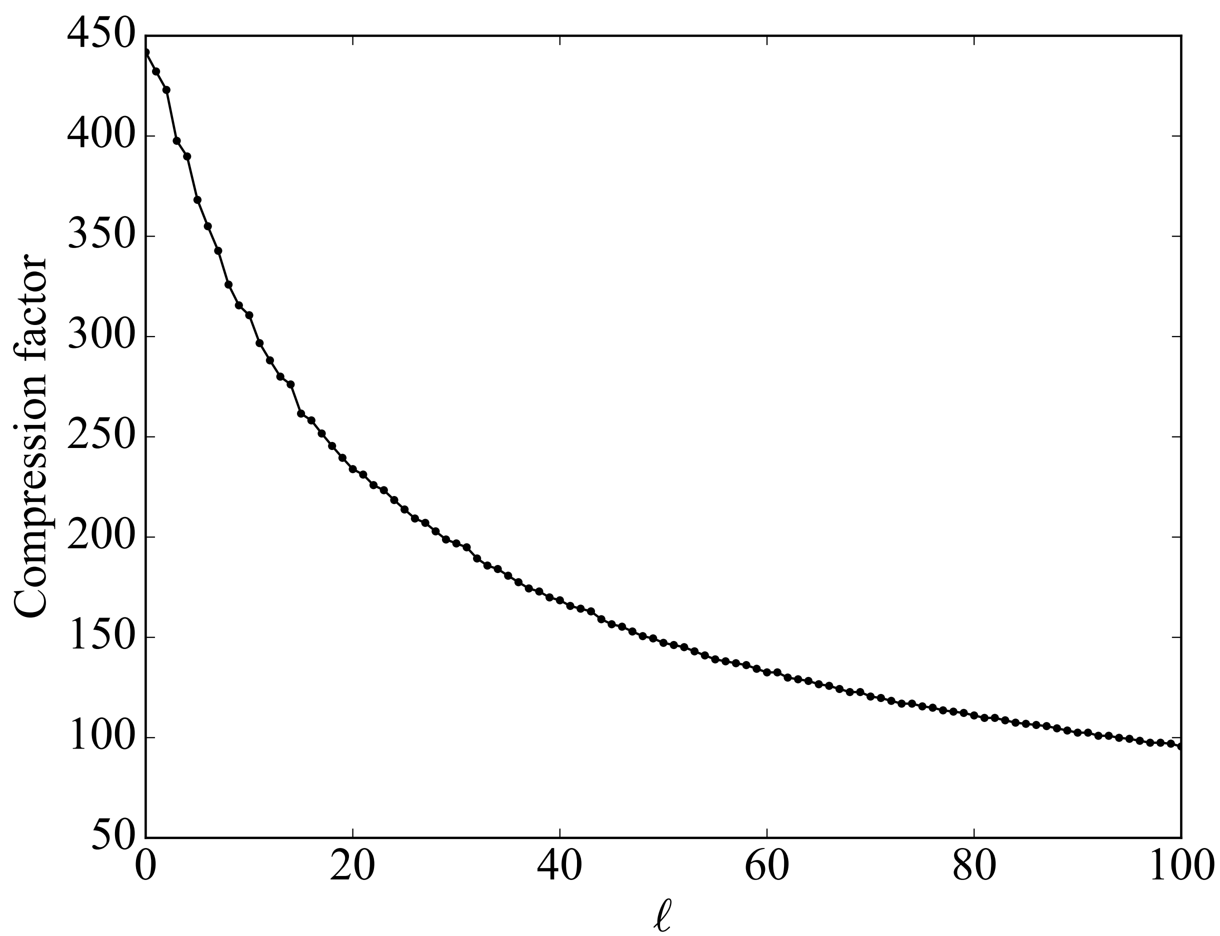

Total compression factor:

$$C_{\text{total}} = (\ell_{\max} + 1) \left(\sum_{\ell=0}^{\ell_{\max}} \frac{1}{C_\ell} \right)^{-1}$$

$$\approx 151$$

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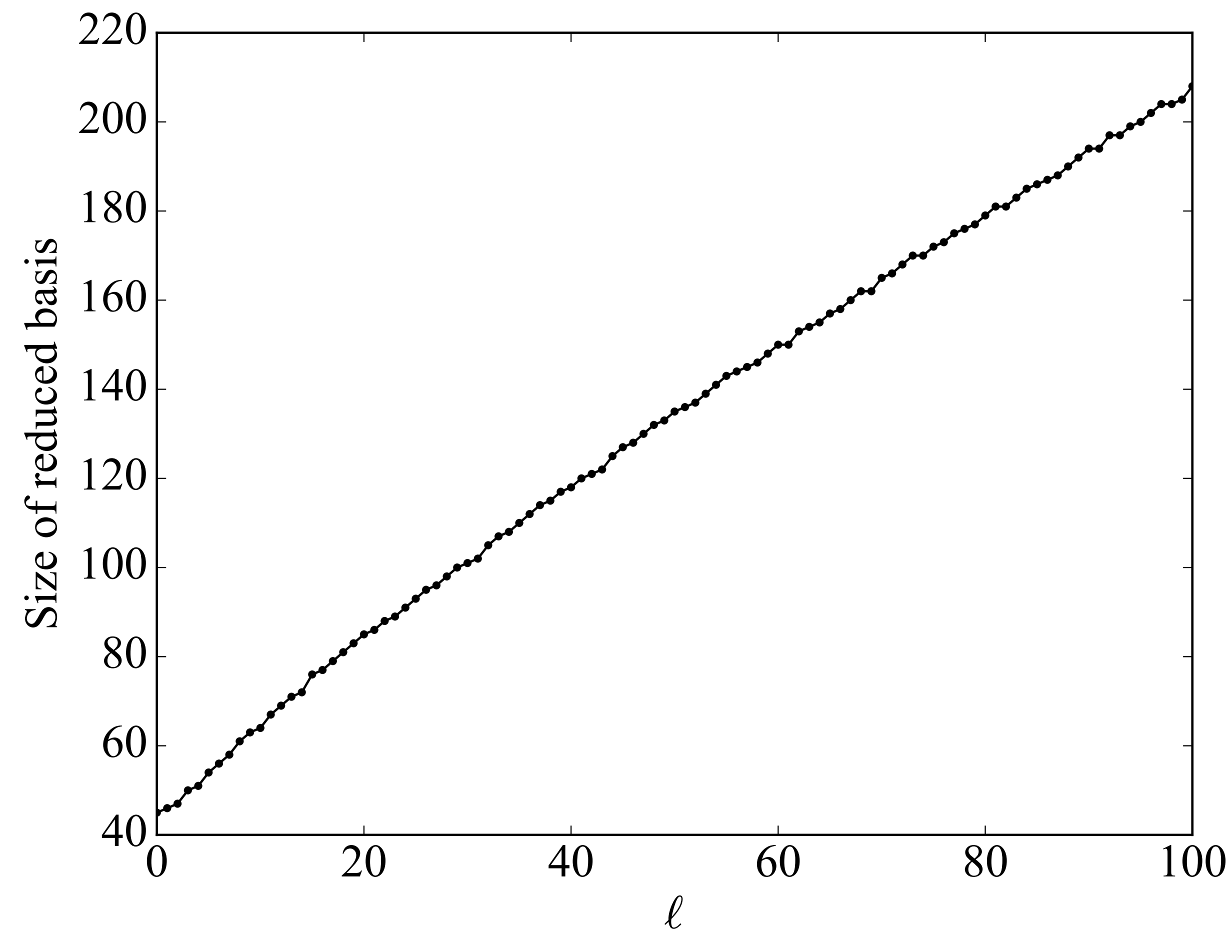

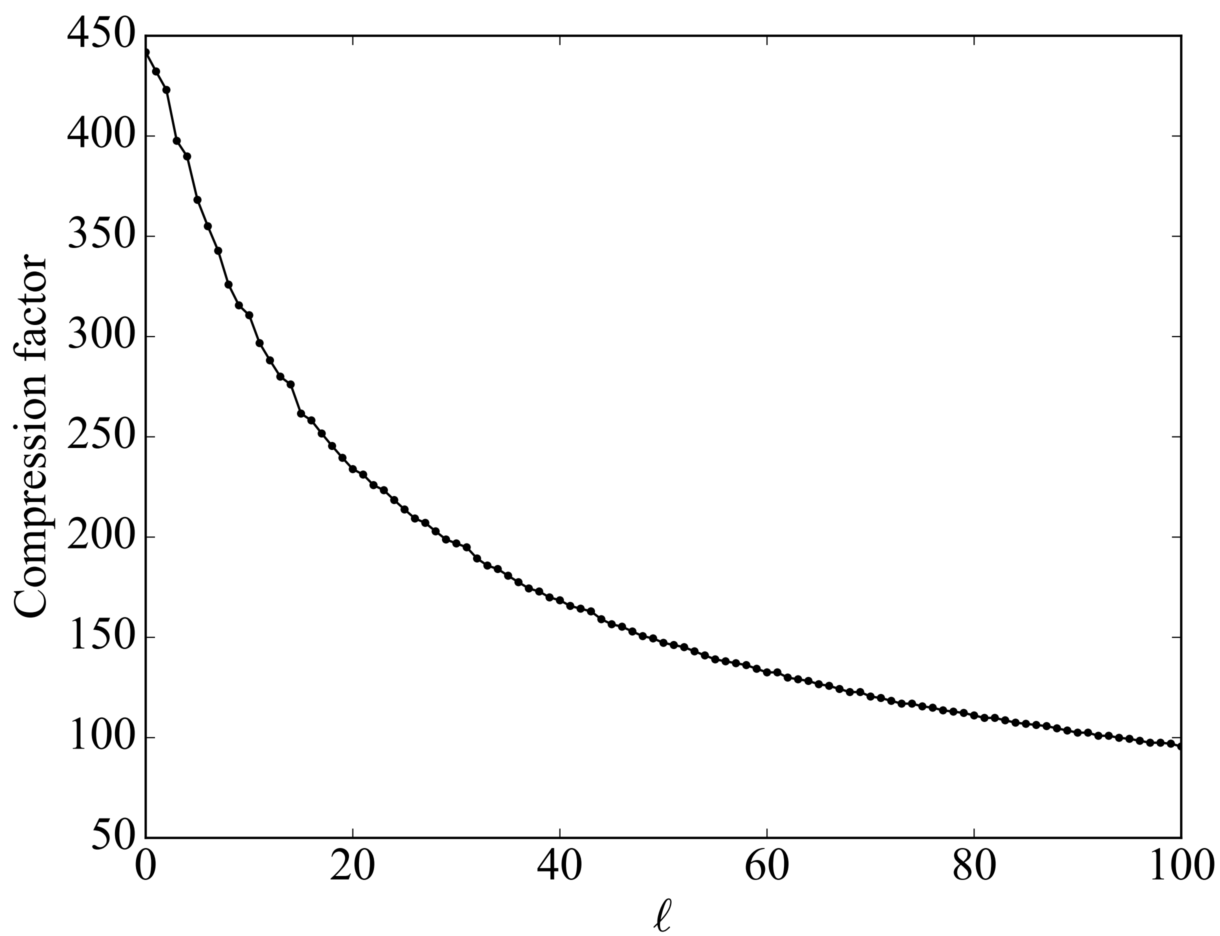

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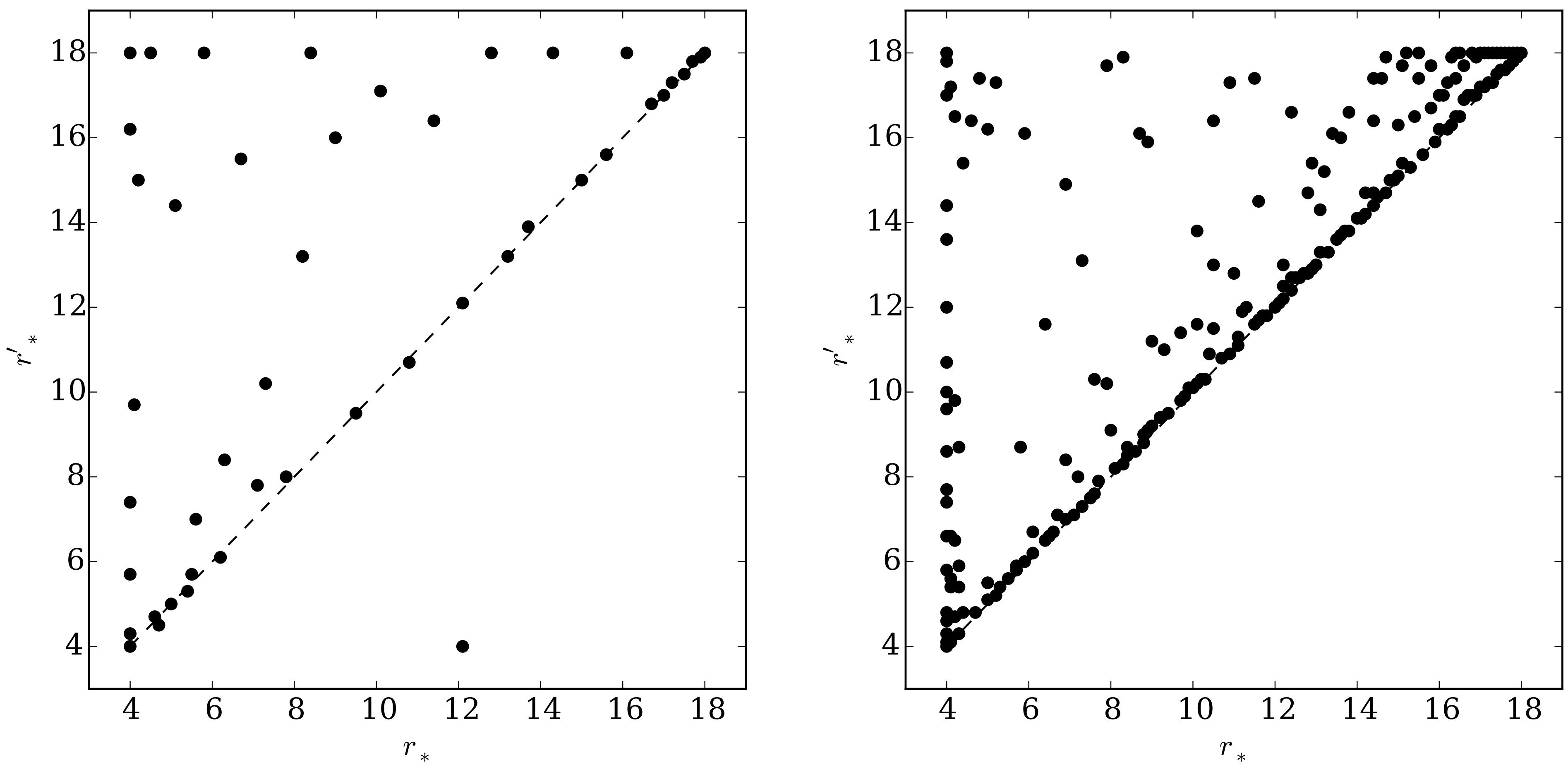
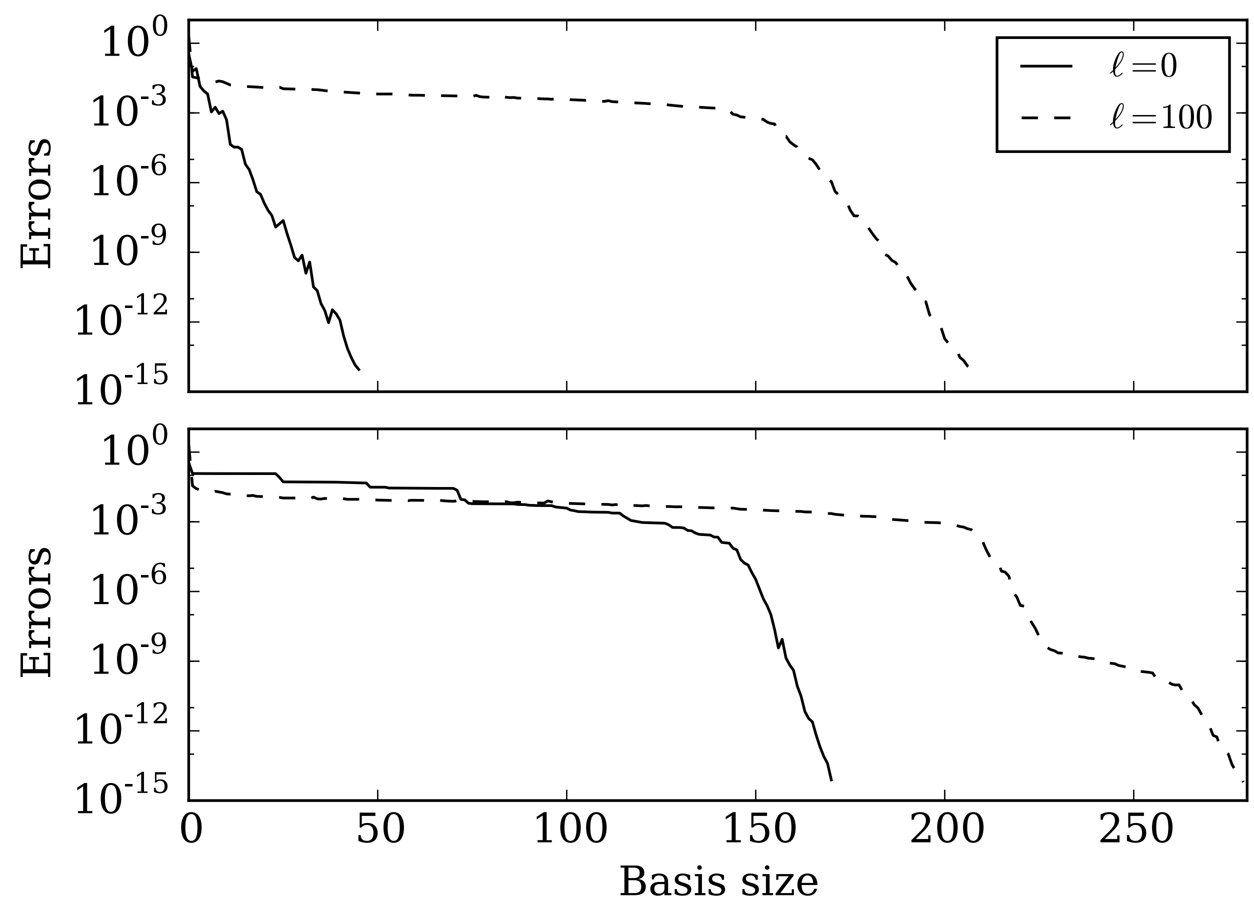
Less than 1% of the data is needed to capture all features up to numerical round-off errors



Shifting by time-of-arrival:

Not shifting by time-of-arrival:

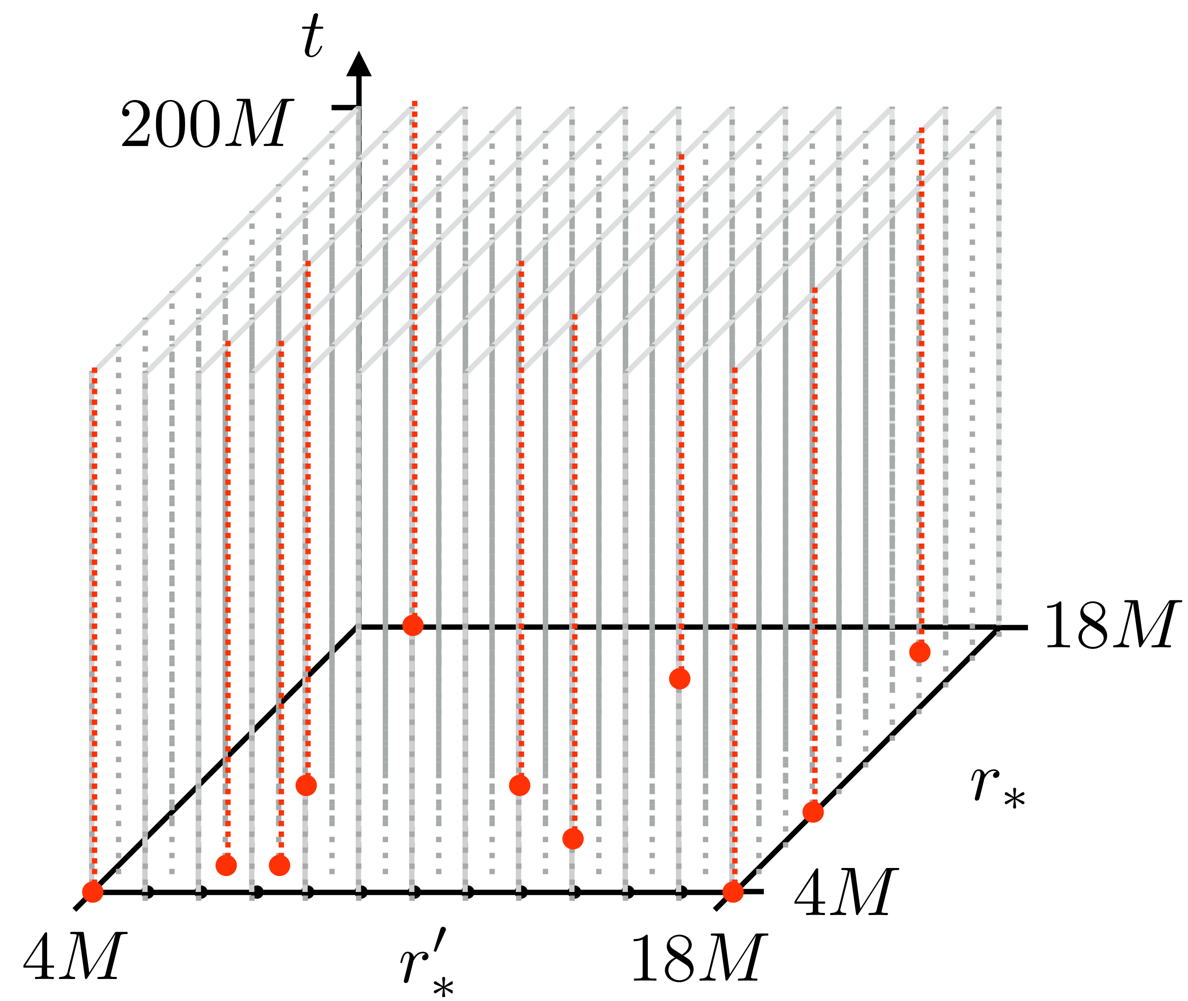


2) Empirical interpolation

Barrault+ (04)
Maday+ (09)

RB approximation:

$$G_\ell(t;\vec{\lambda}) \approx \sum_{i=1}^{N_\ell} C_i^\ell(\vec{\lambda}) e_i^\ell(t)$$



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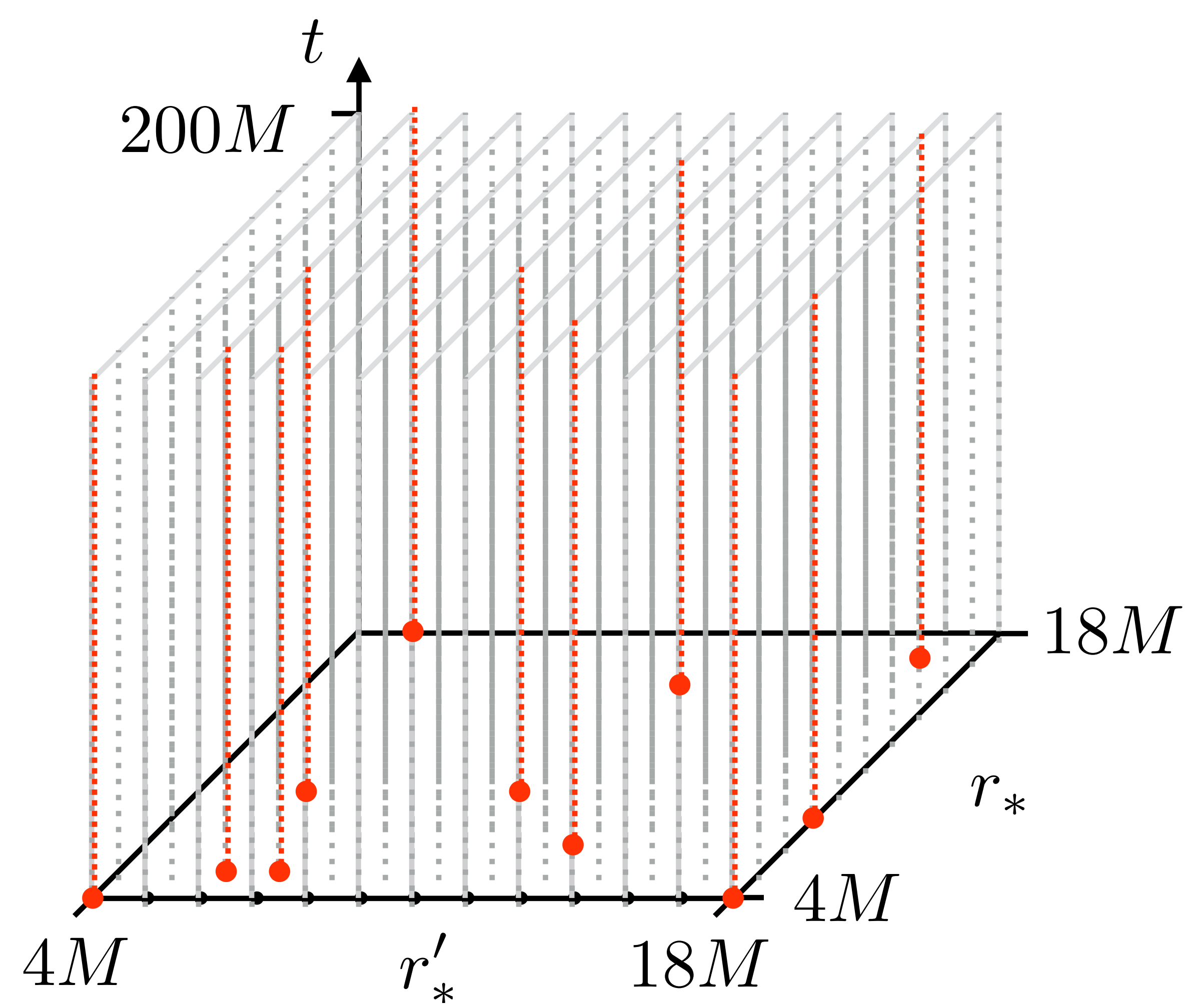

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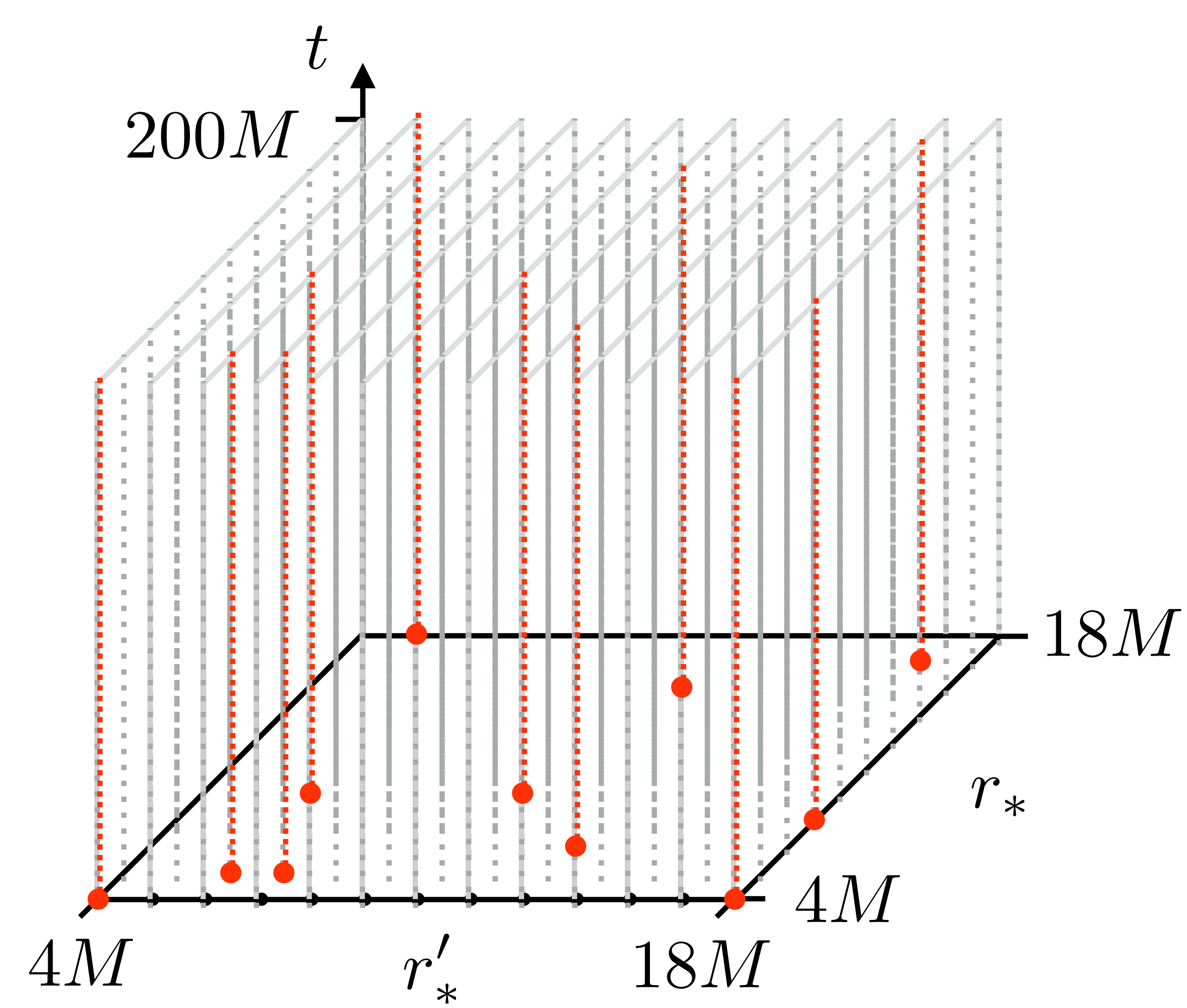

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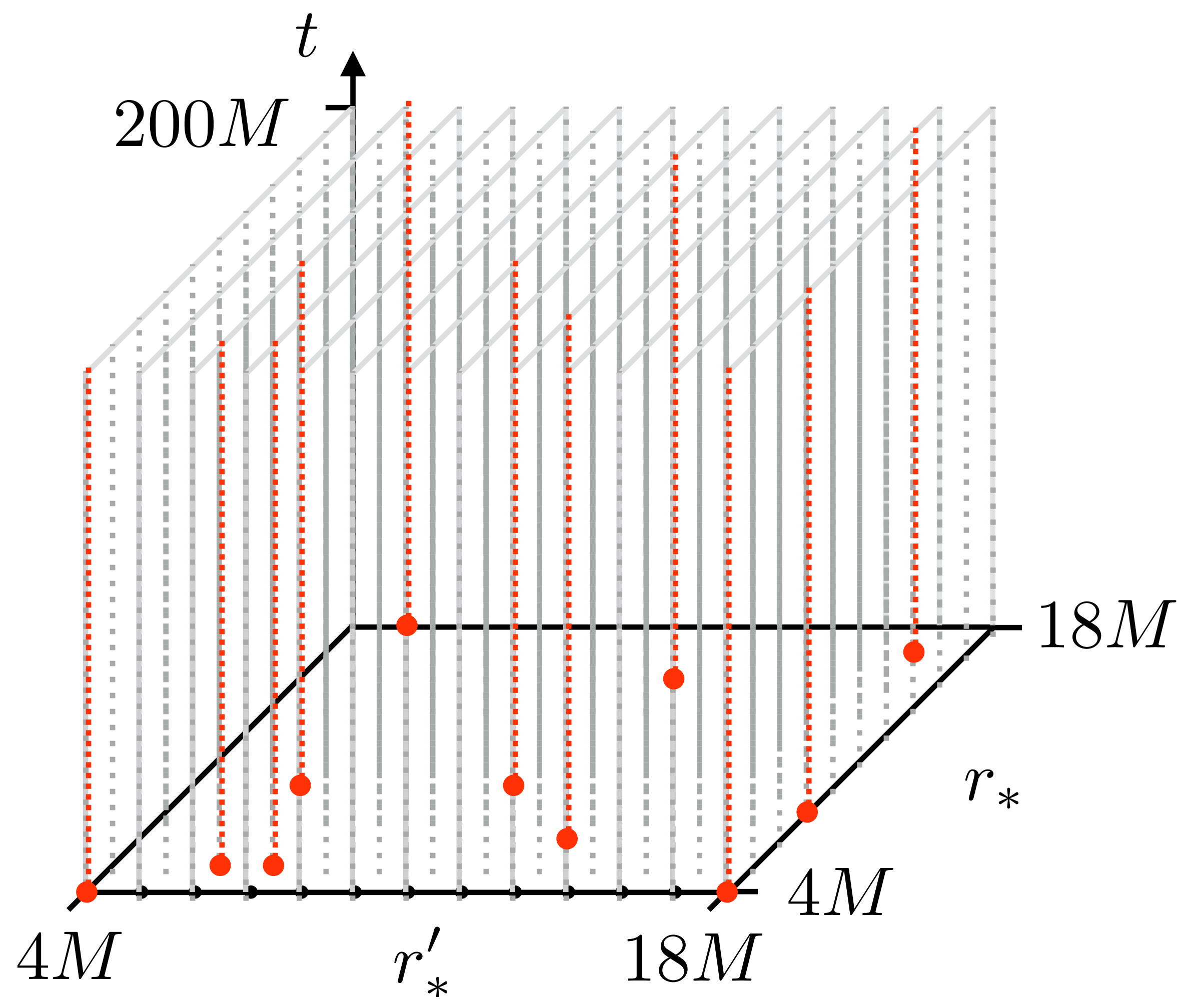
Find the interpolation nodes through another greedy algorithm that minimizes the interpolation error



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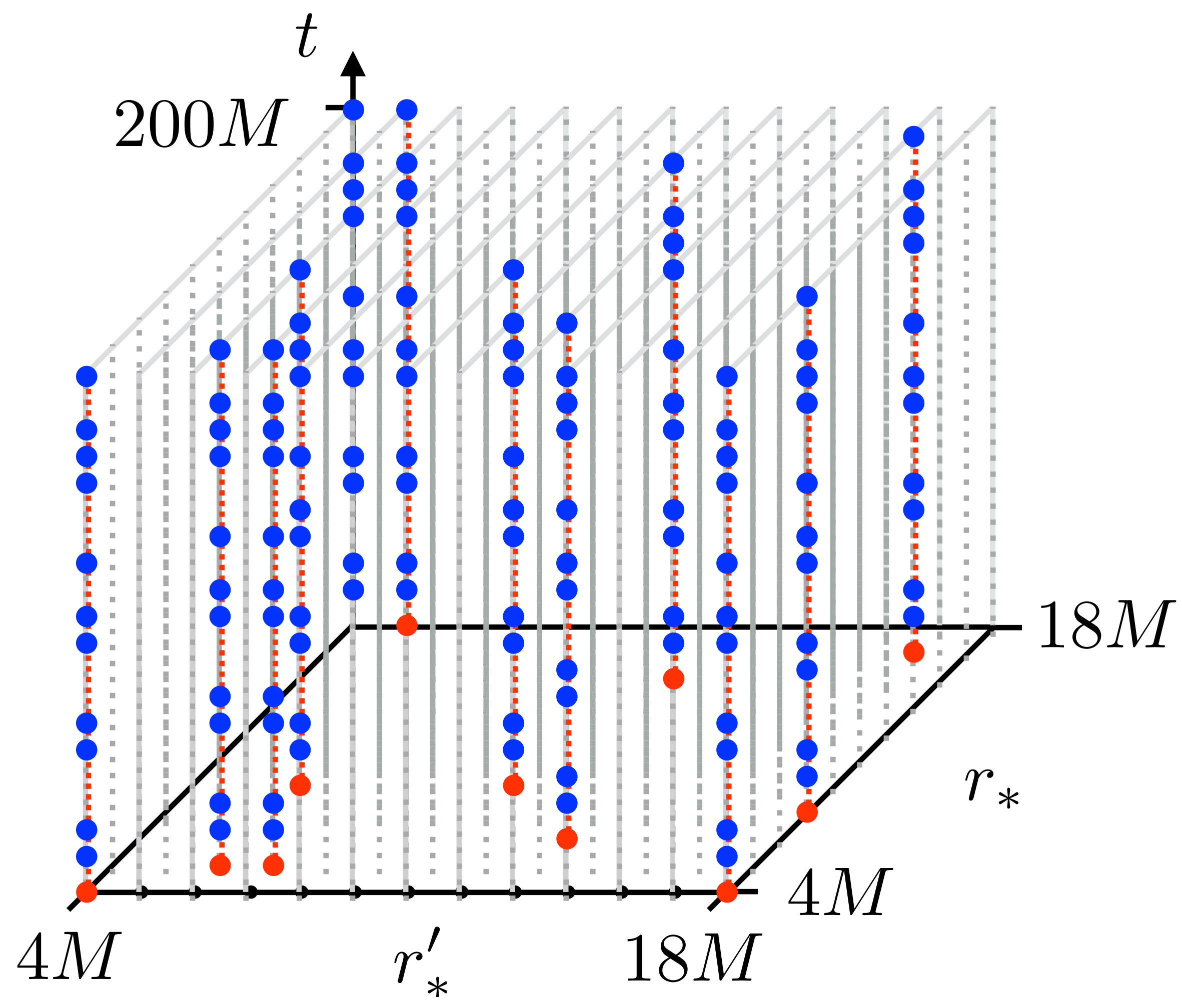
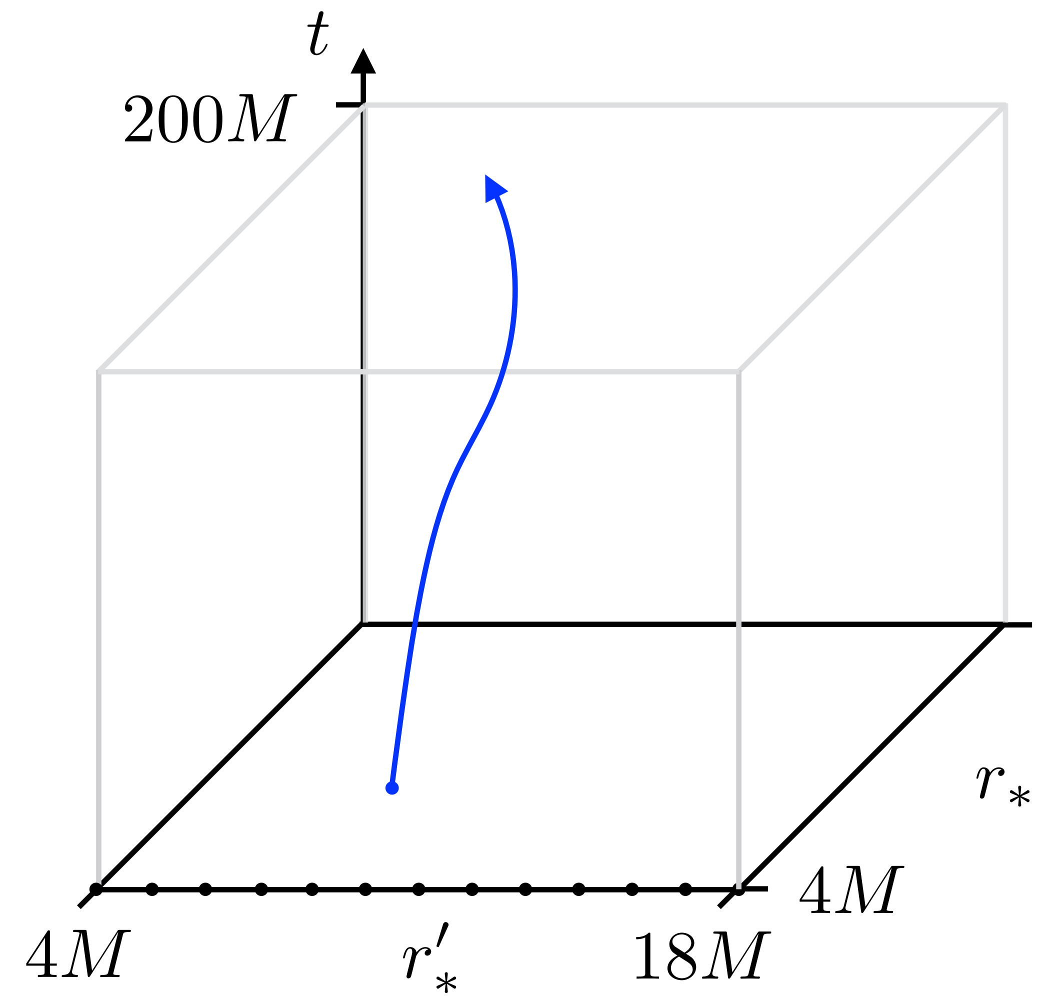


3) Parametric fitting

Mostly interested in finding Green's function on worldlines

$$z^\mu(t) = \big(t, r(t), \pi/2, \gamma(t)\big)$$



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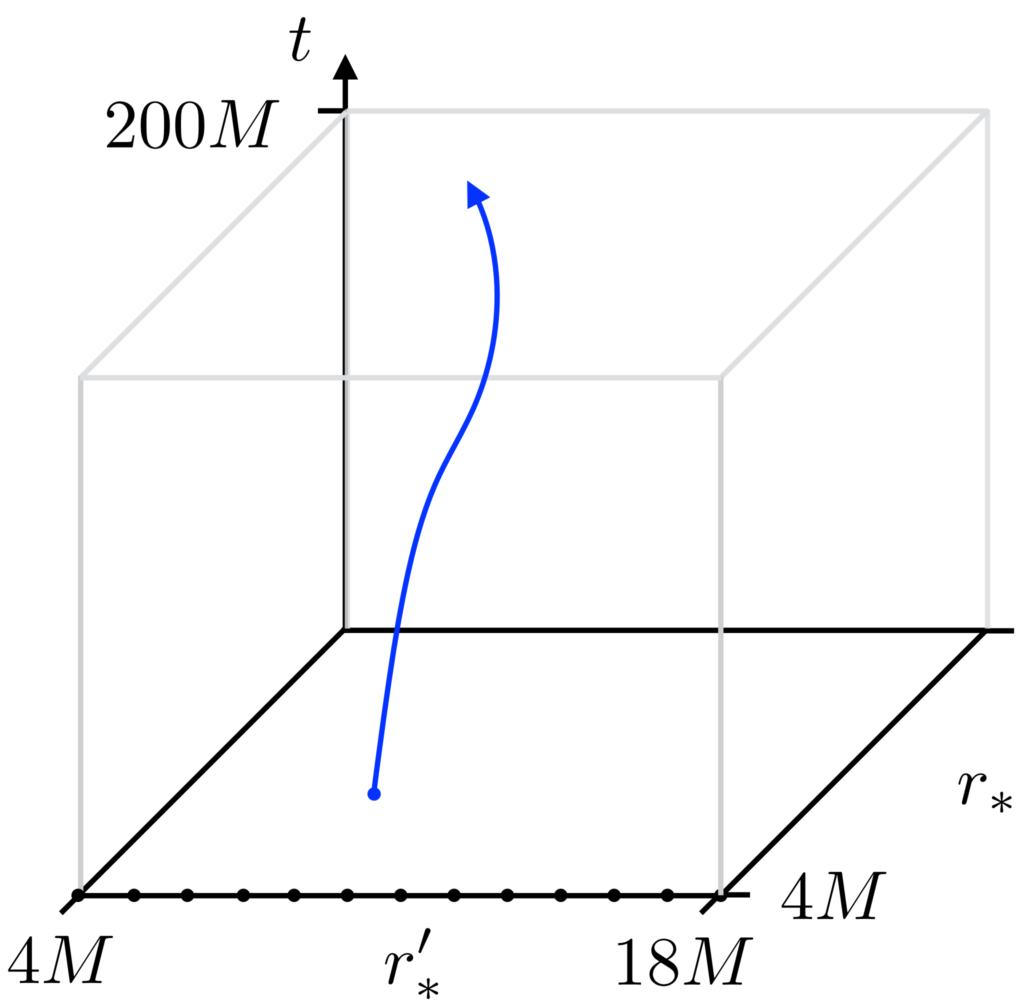
$$G_\ell(t; r_*(t), r'_*) \approx \sum_{j=1}^{N_\ell} B_j^\ell(t) G_\ell(T_j; r_*(t), r'_*)$$

As a result, fitting is done on-the-fly on a worldline



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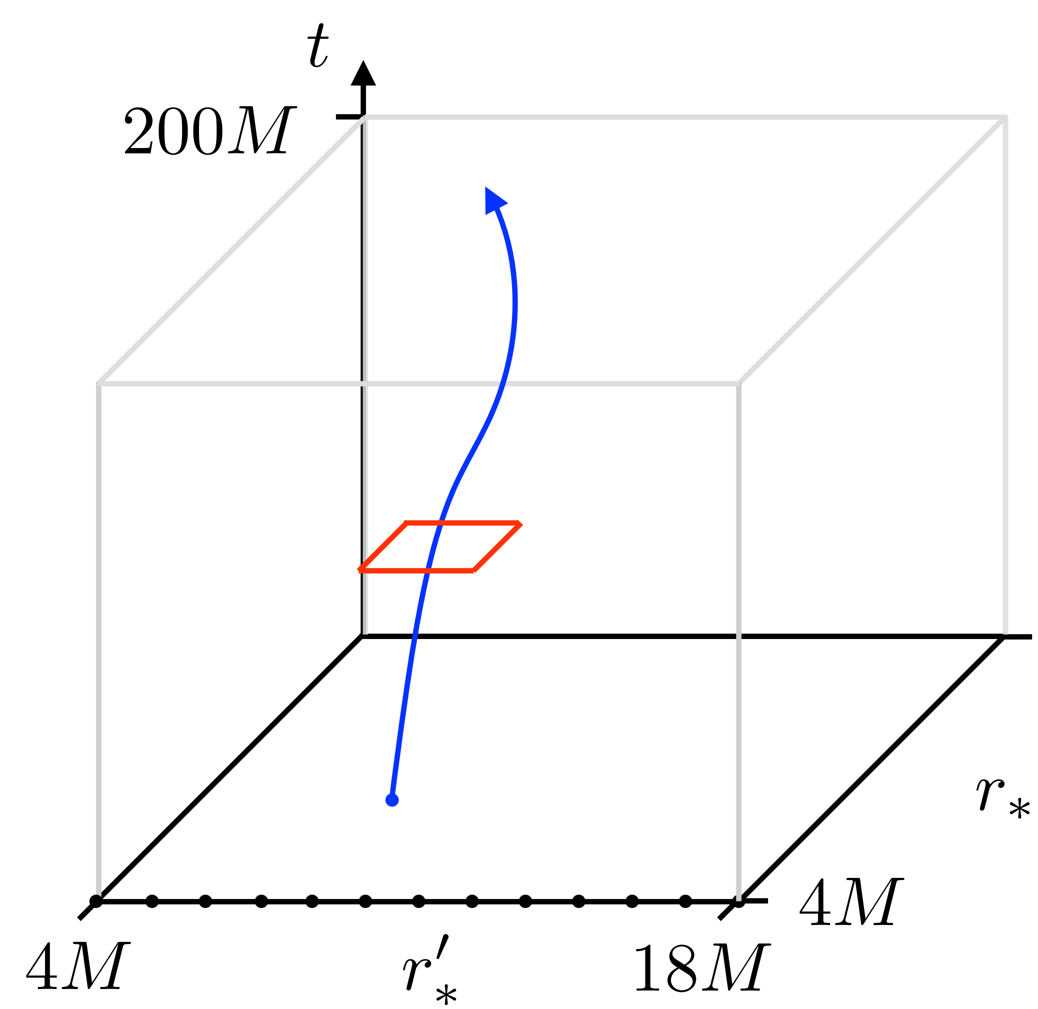
At a given time step, t_k, reconstruct the Green's function data in a small patch around the worldline

Fit with spline and interpolate to $r_*(t_k)$

Repeat for all time steps



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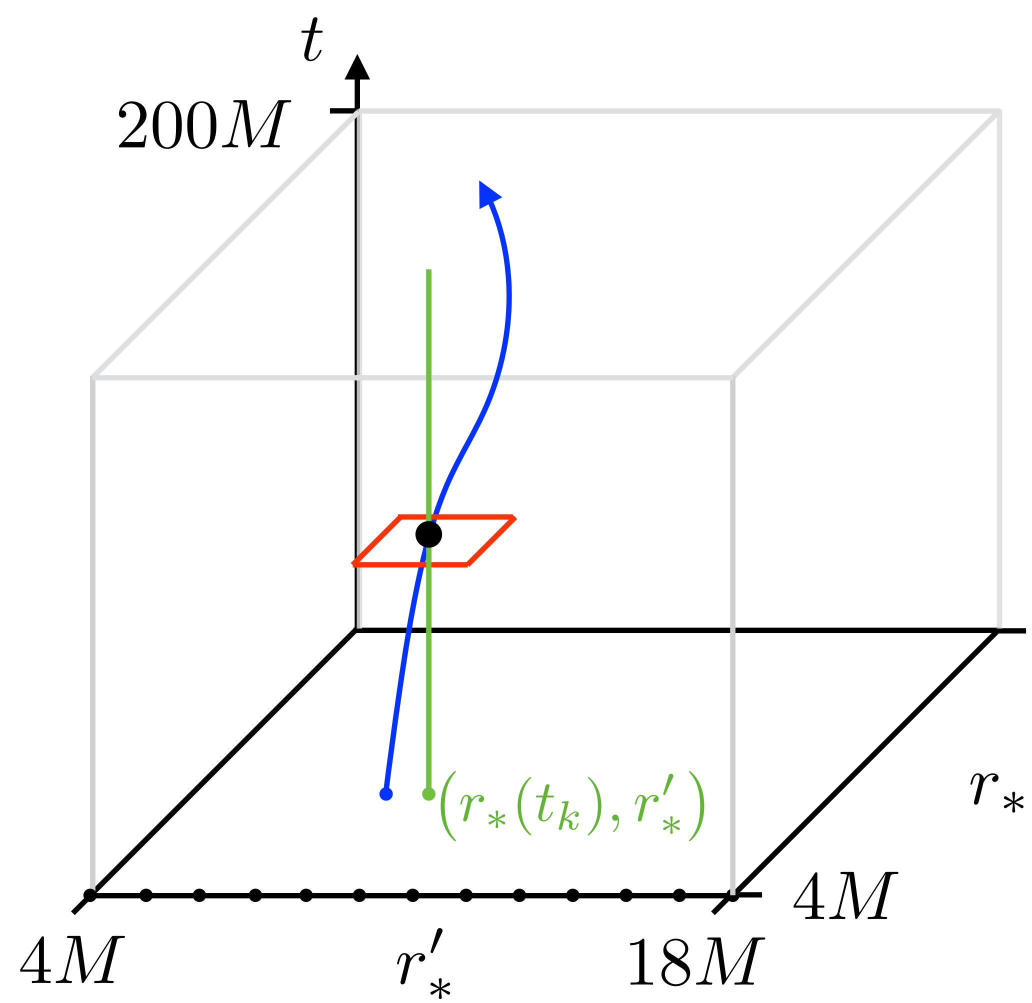

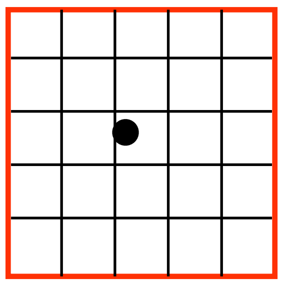

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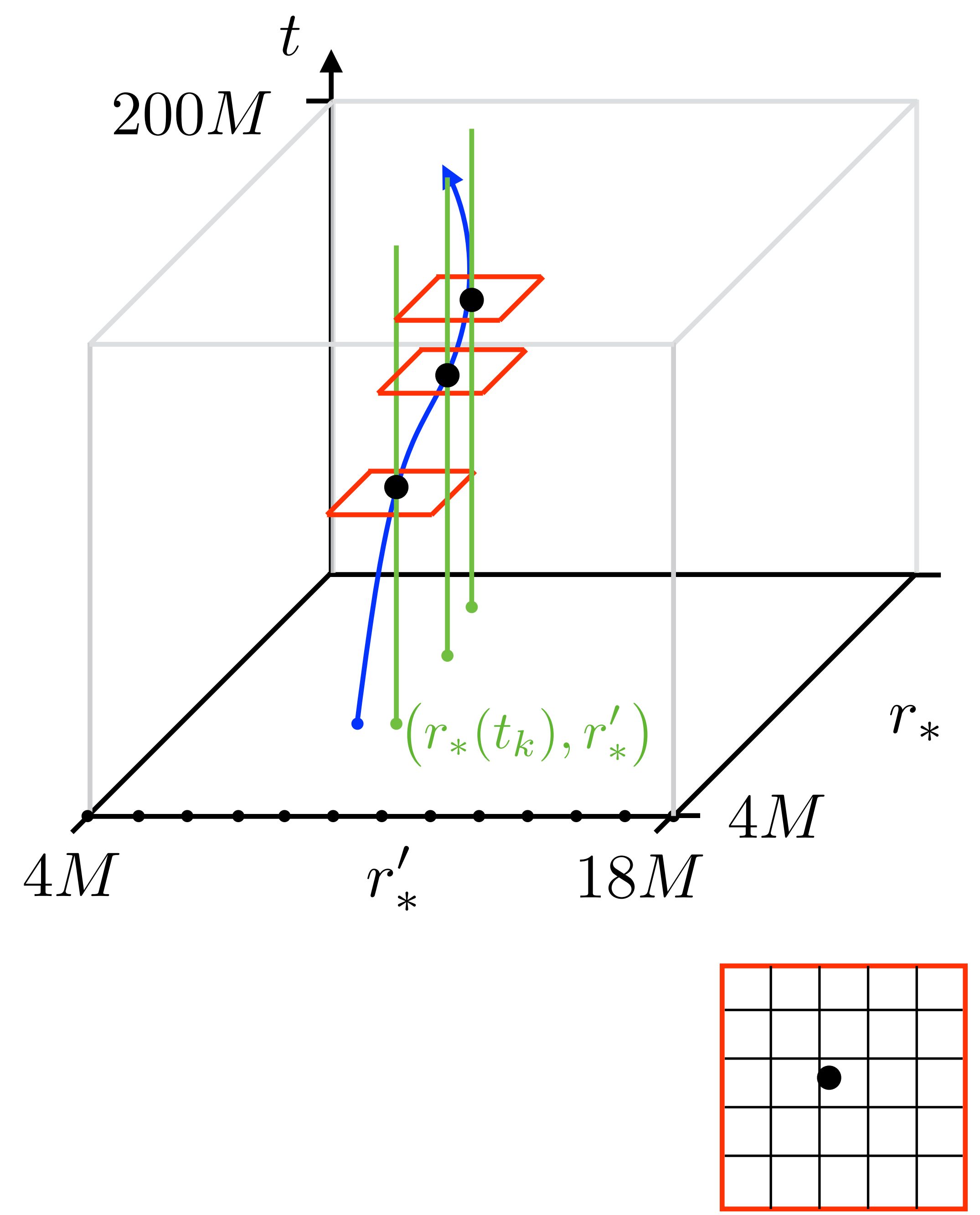

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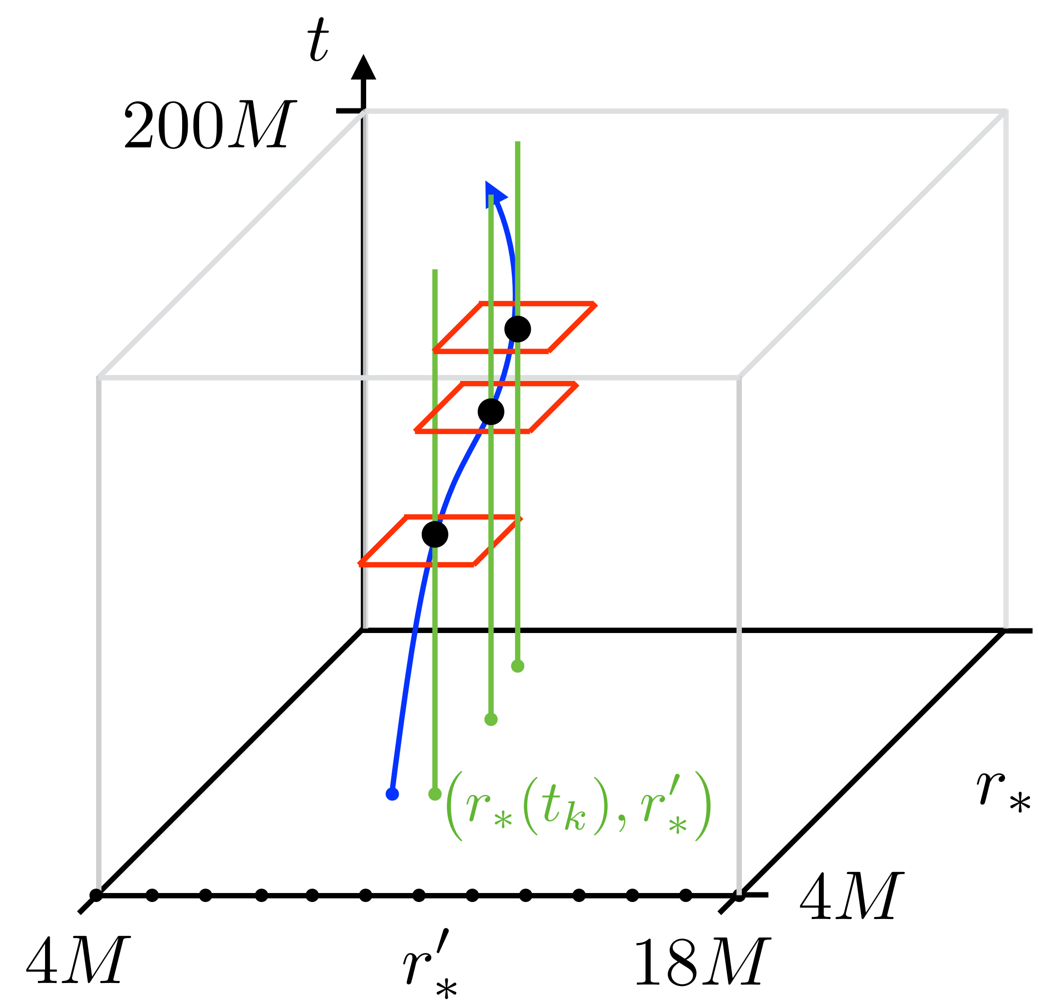

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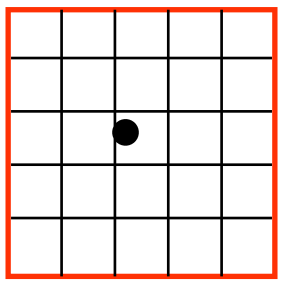

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But have to store to disk all data at each T_j…

Repeat for all time steps

Surrogate accuracy, speed-up, and size

Eccentric geodesic orbit ($e = 0.5$, $p = 7.2$)

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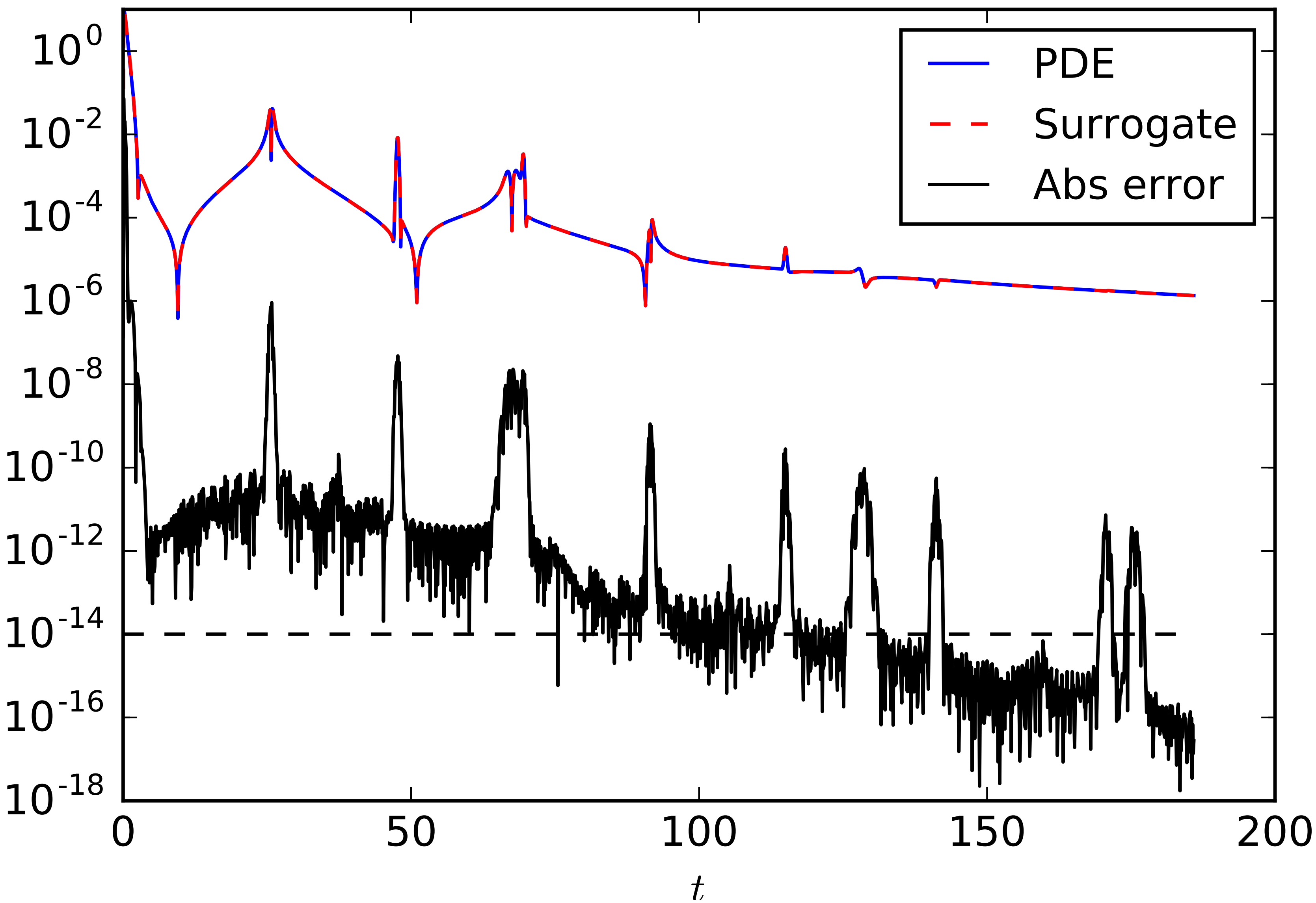

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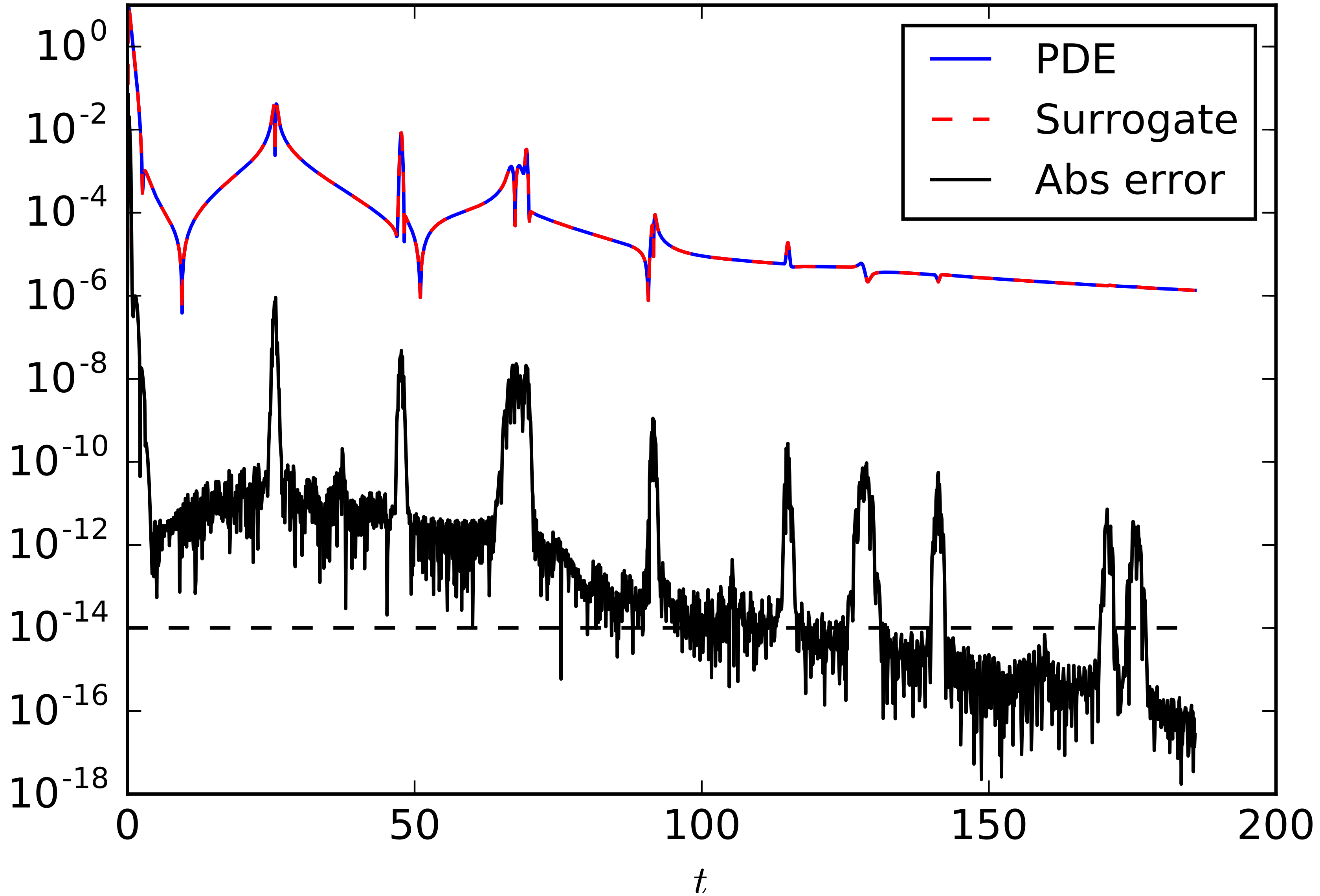

PDE	~380s
Surrogate	~25s
Speed-up*	~15x

**But not quite an apples-to-apples comparison.*

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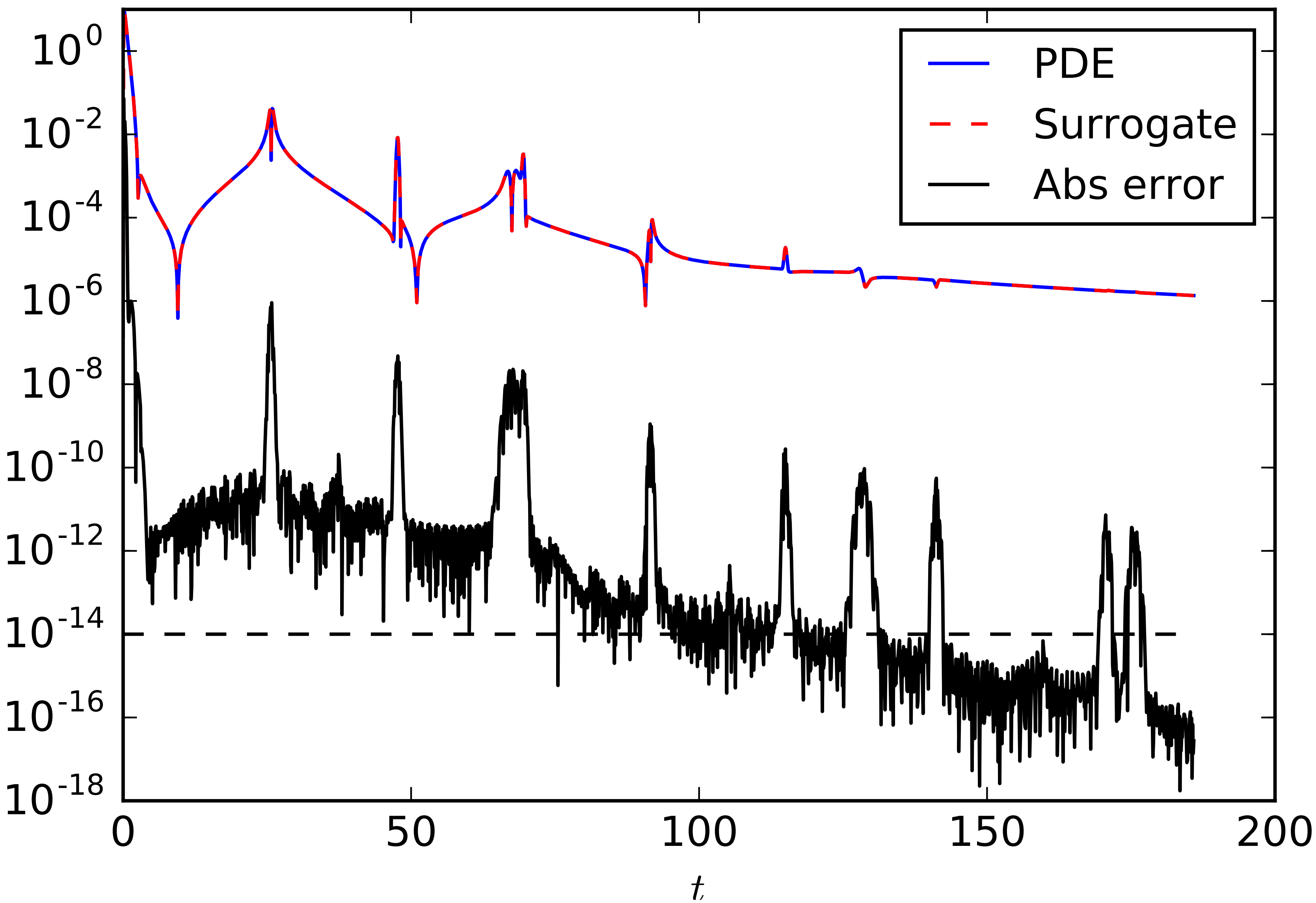

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Physical memory: 30GB reduced to 2GB

Using the surrogate predictions

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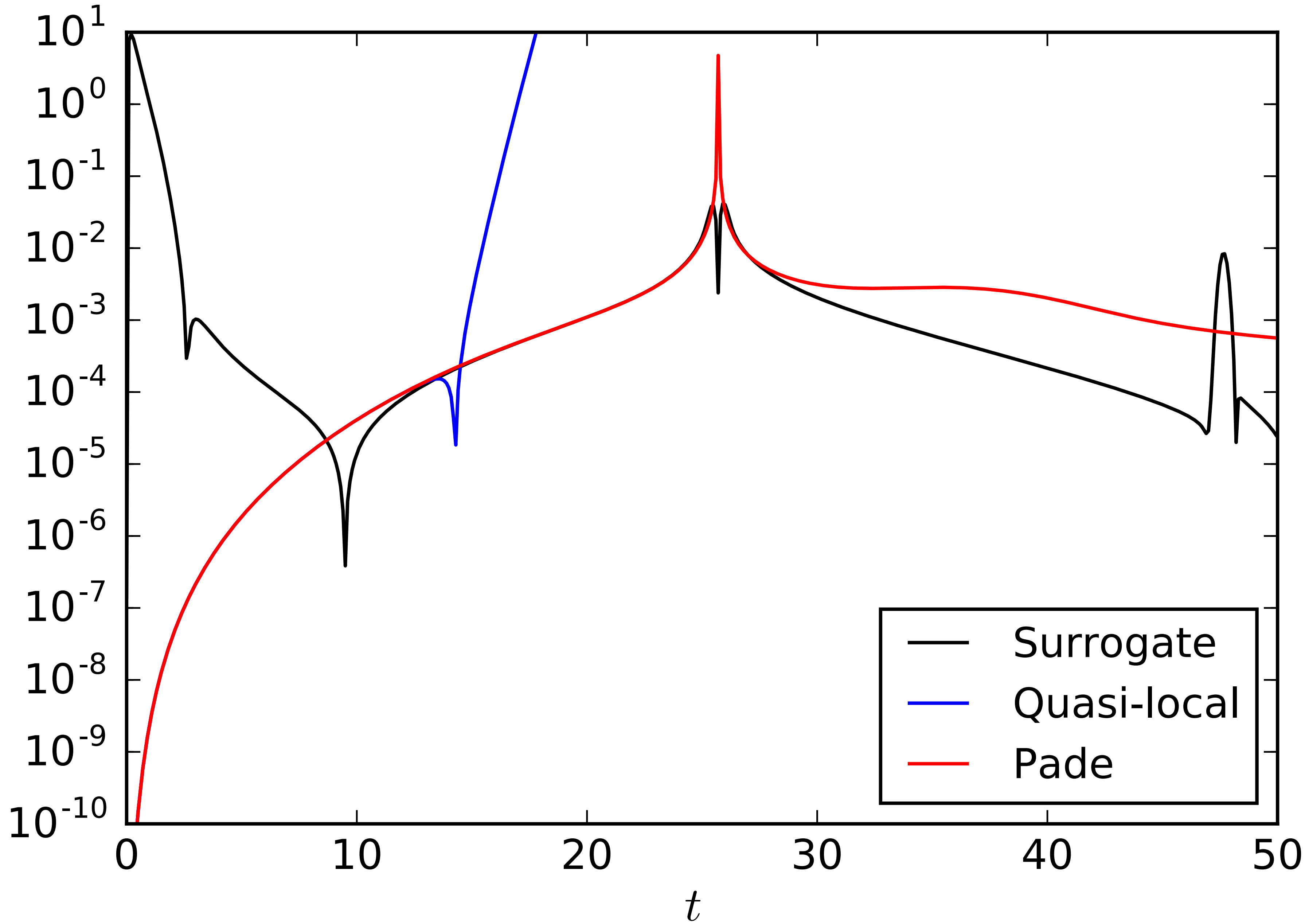

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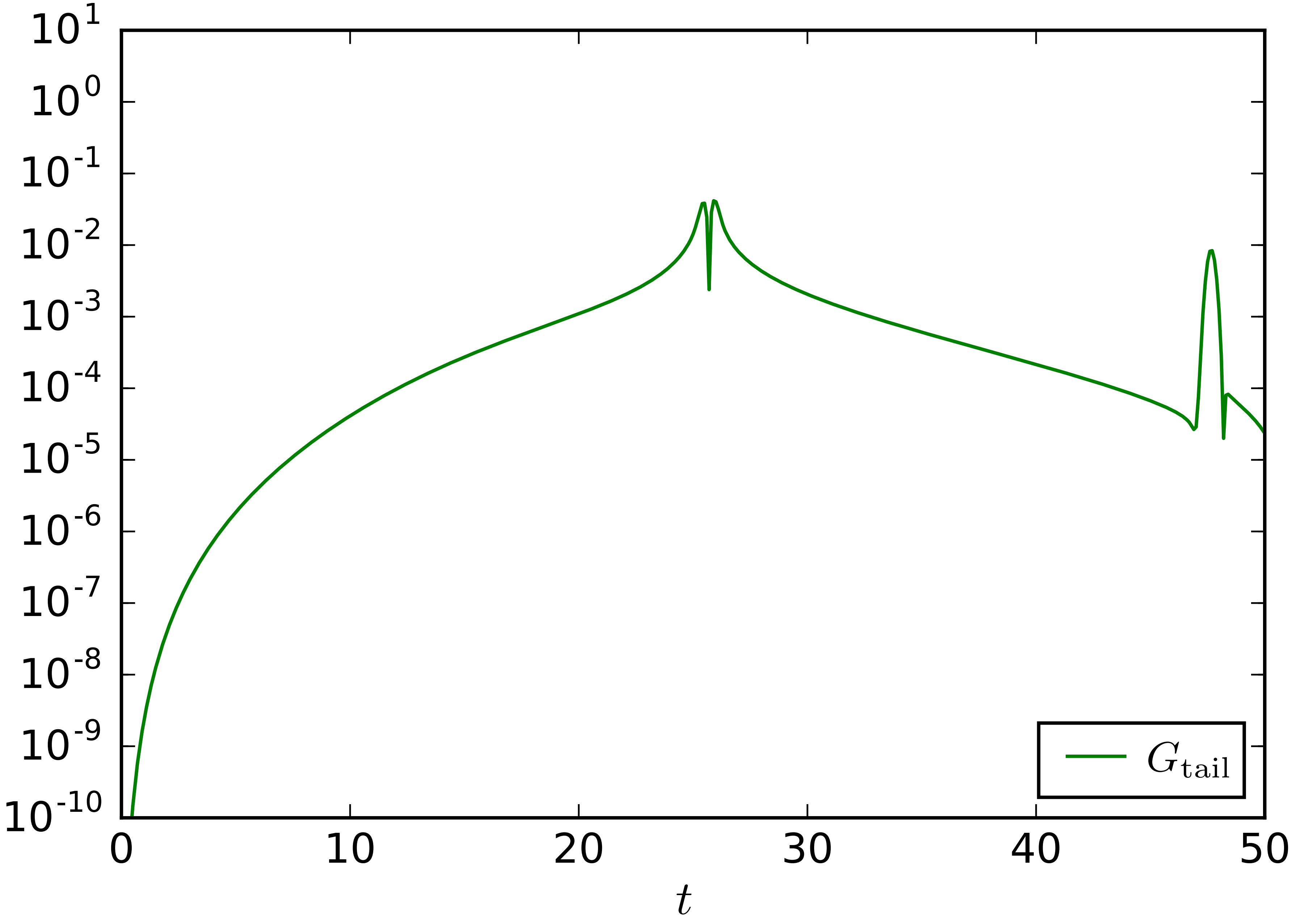

Surrogate self-force evaluation

History-dependent part of first-order scalar self-force in MiSaTaQuWa form *Quinn (00)*

$$F^{\mu}_{\rm hist}(\tau) = q^2 P^{\mu\nu} \lim_{\epsilon \to 0^+} \int_{-\infty}^{\tau-\epsilon} d\tau' \, \nabla_\nu G_{\rm ret}(z^\mu, z^{\mu'})$$

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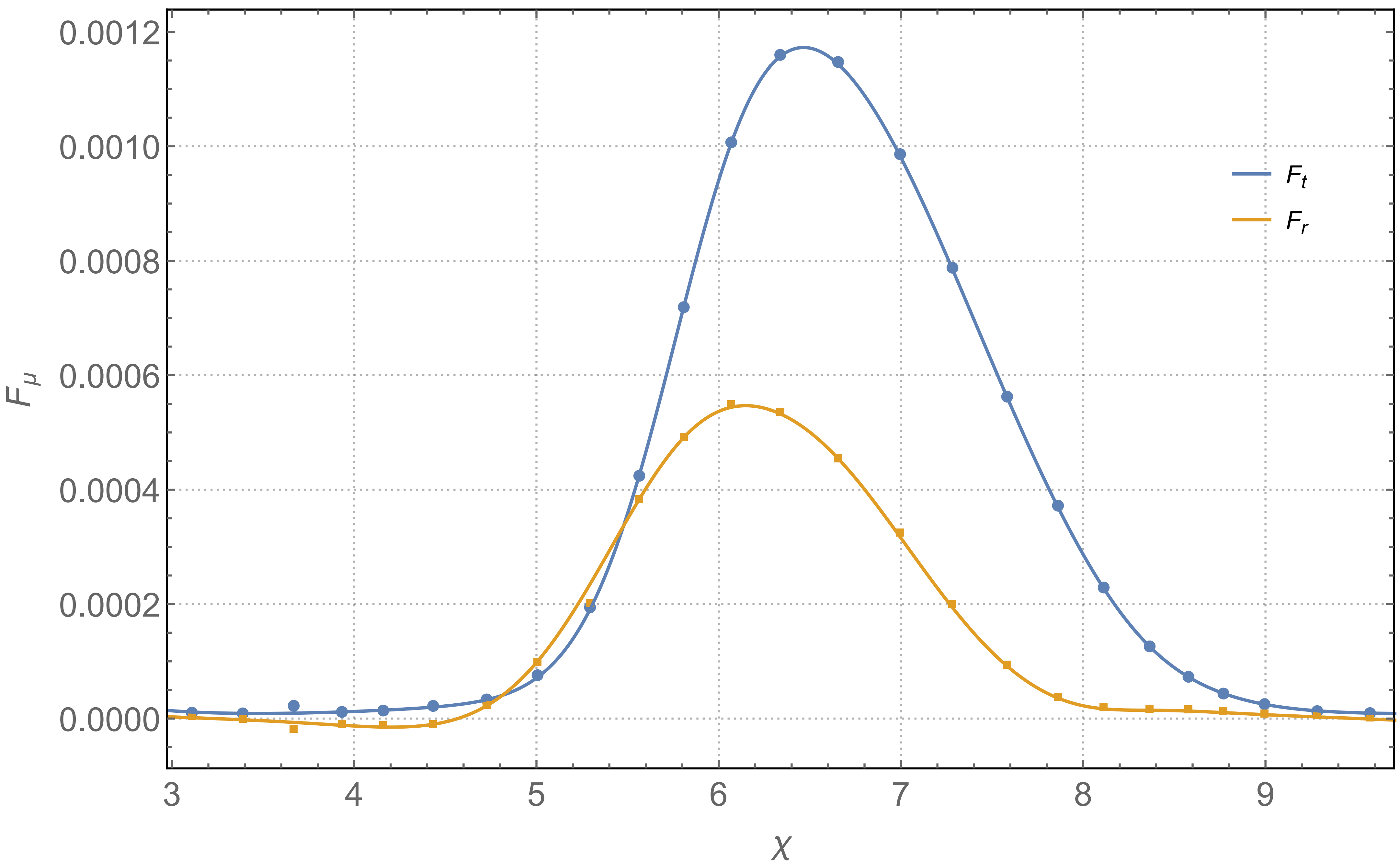

Applications

- Higher-order self-force and radiation *CRG (12a), (12b)*

$$ma^{\mu} \supset P^{\mu\nu}\left(\lim_{\epsilon\to 0^+}\int_{-\infty}^{\tau-\epsilon} d\tau'\,\nabla_{\nu} G_{\rm ret}(z^{\mu}, z^{\mu'})\right)\left(\lim_{\epsilon\to 0^+}\int_{-\infty}^{\tau-\epsilon} d\tau''\, G_{\rm ret}(z^{\mu}, z^{\mu''})\right),$$

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- Self-consistent evolution

$$\frac{d^2 z^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dz^\alpha}{d\tau}\frac{dz^\beta}{d\tau}$$
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- Higher-order, self-consistent evolutions

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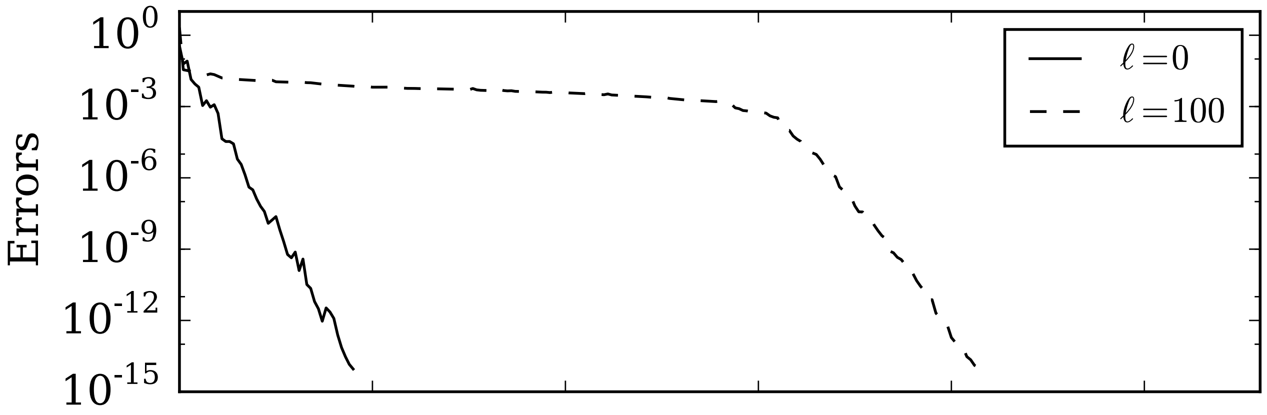
- Many similar applications in gravity plus others (e.g., NS-BH inspirals)

Improving the surrogate building strategy

The plateau in the max projection errors often hints that a different representation of the data may generate a more compact basis



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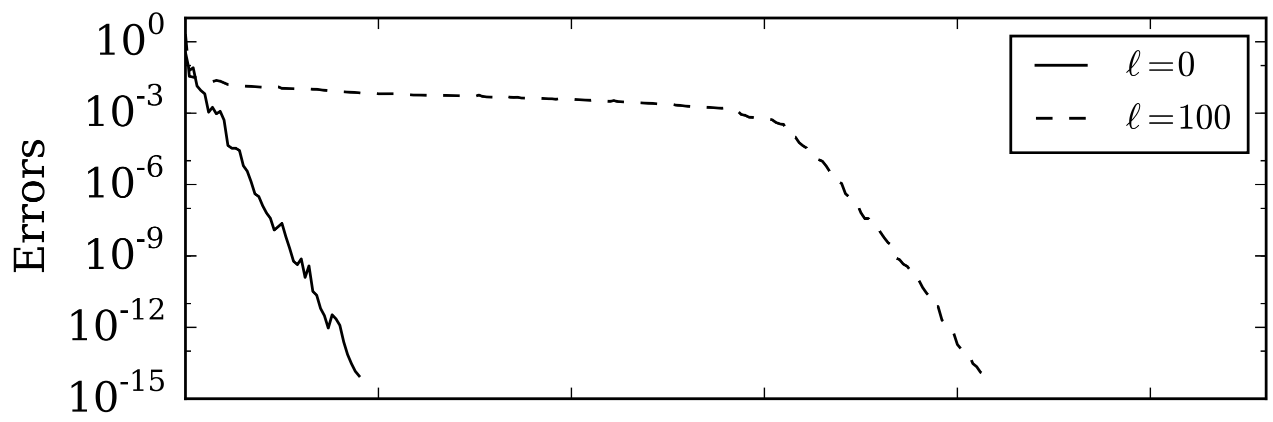

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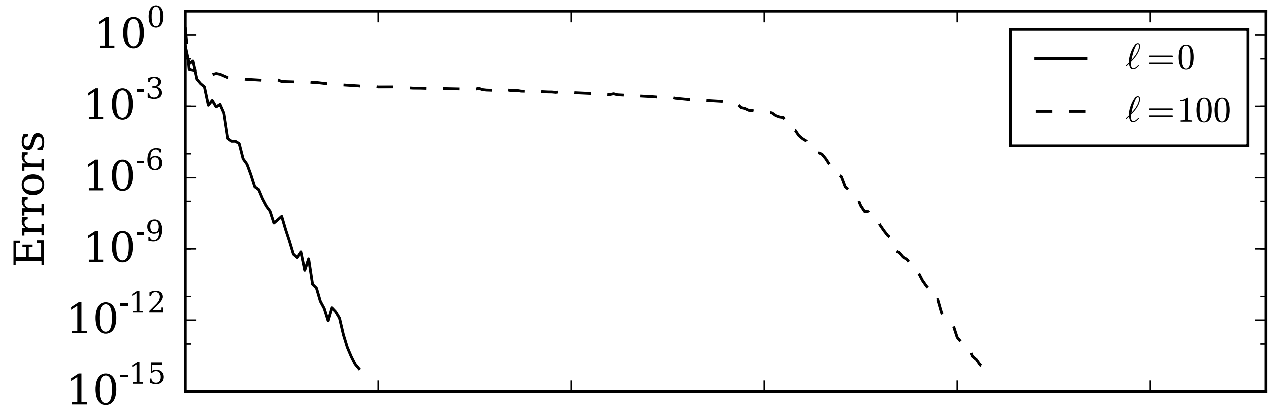

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- Some other way to represent the data?

Different and useful ways to parametrize the data?

- A more “natural” parametrization might be $\lambda = r'_*$ and regard (t, r_*) as the physical dimension

$$G_\ell(t, r_*; r'_*) \approx \sum_{i=1}^{N_\ell} B_i^\ell(t, r_*) G_\ell(T_i, R_{*i}; r'_*)$$

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Maybe try "invasive" approaches that project the wave equation onto the small vector space spanned by the basis

Summary & Outlook

- Green's function methods have many advantages to offer but significant challenges to overcome to be practical
- Reduced-order surrogate modeling offers a promising way to use Green's functions efficiently and accurately for self-force calculations
- For a given worldline, the surrogate is more than 15x faster to evaluate than solving the wave equation, with little loss of accuracy
- Other choices in the surrogate modeling strategy may (should!) improve both the speed and size of the Green's function surrogate
- Extending to Kerr spacetime is straightforward but may involve (much?) larger data sets because of extra parameters and reduced symmetry
- How to compute Green's function for gravitational perturbations?
  - Lorenz gauge has unstable non-radiative modes…
  - Accuracy and speed of metric reconstruction from curvature scalars?