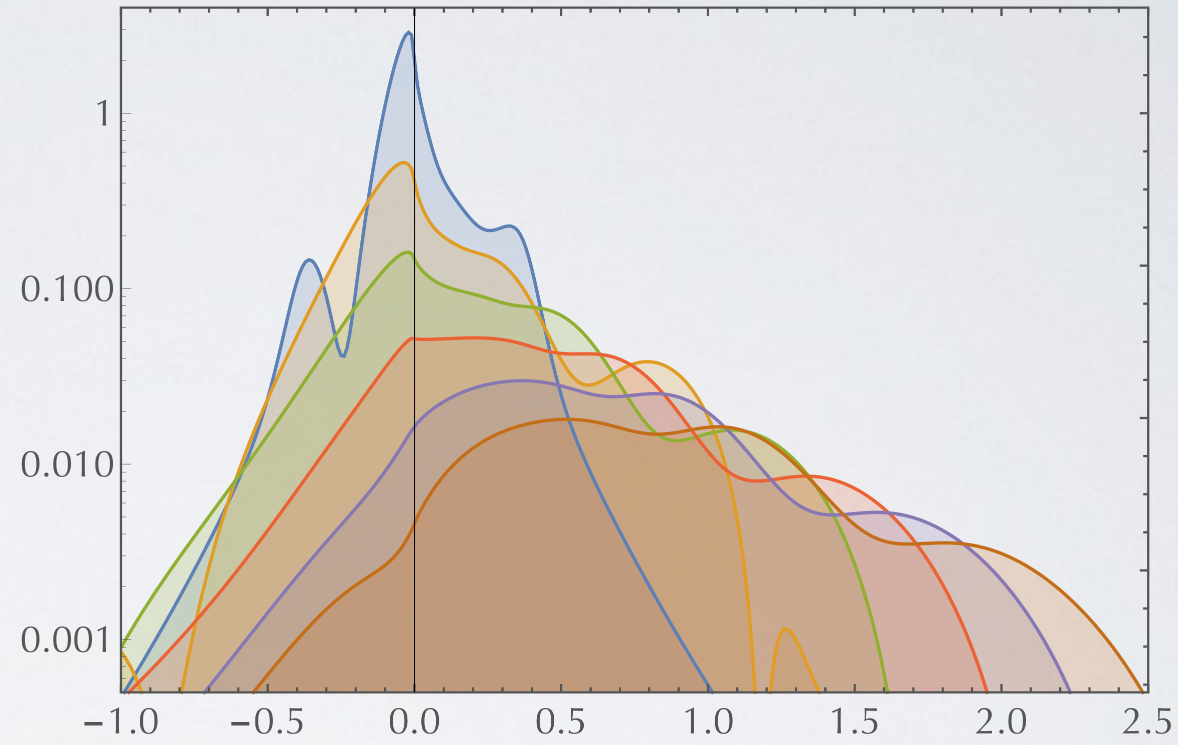
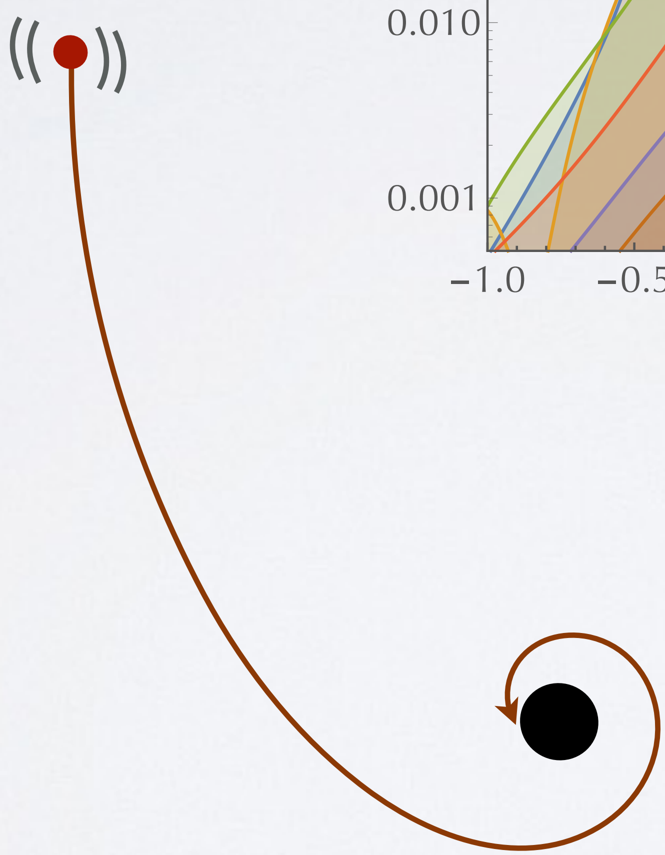
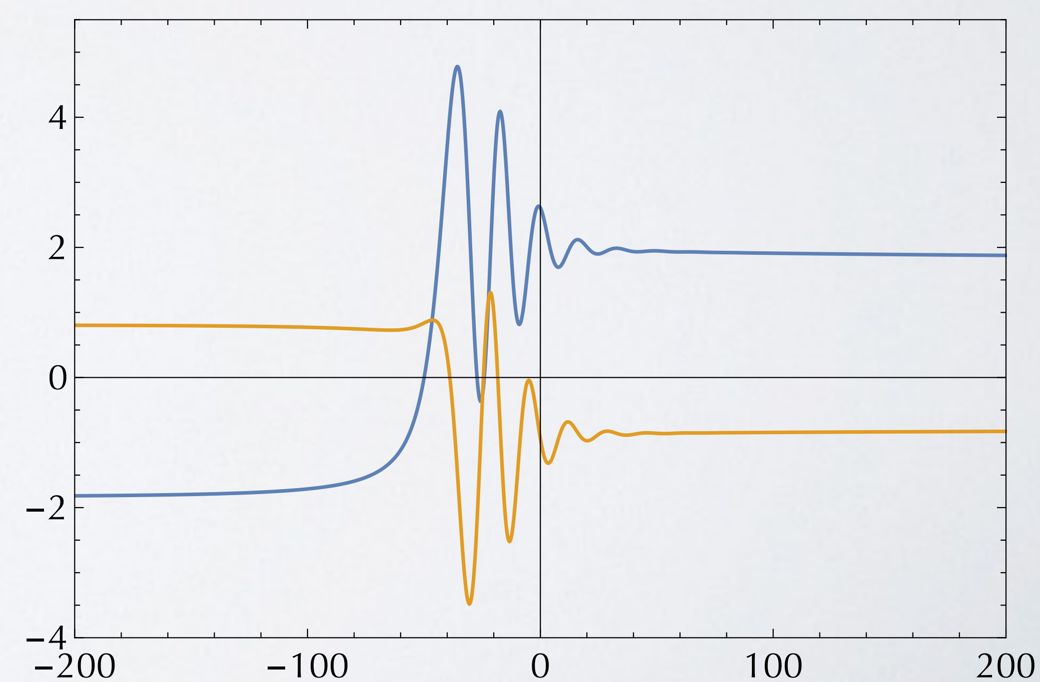
Frequency domain techniques to analyzing unbound motion on Schwarzschild spacetime

Seth Hopper

Vitor Cardoso

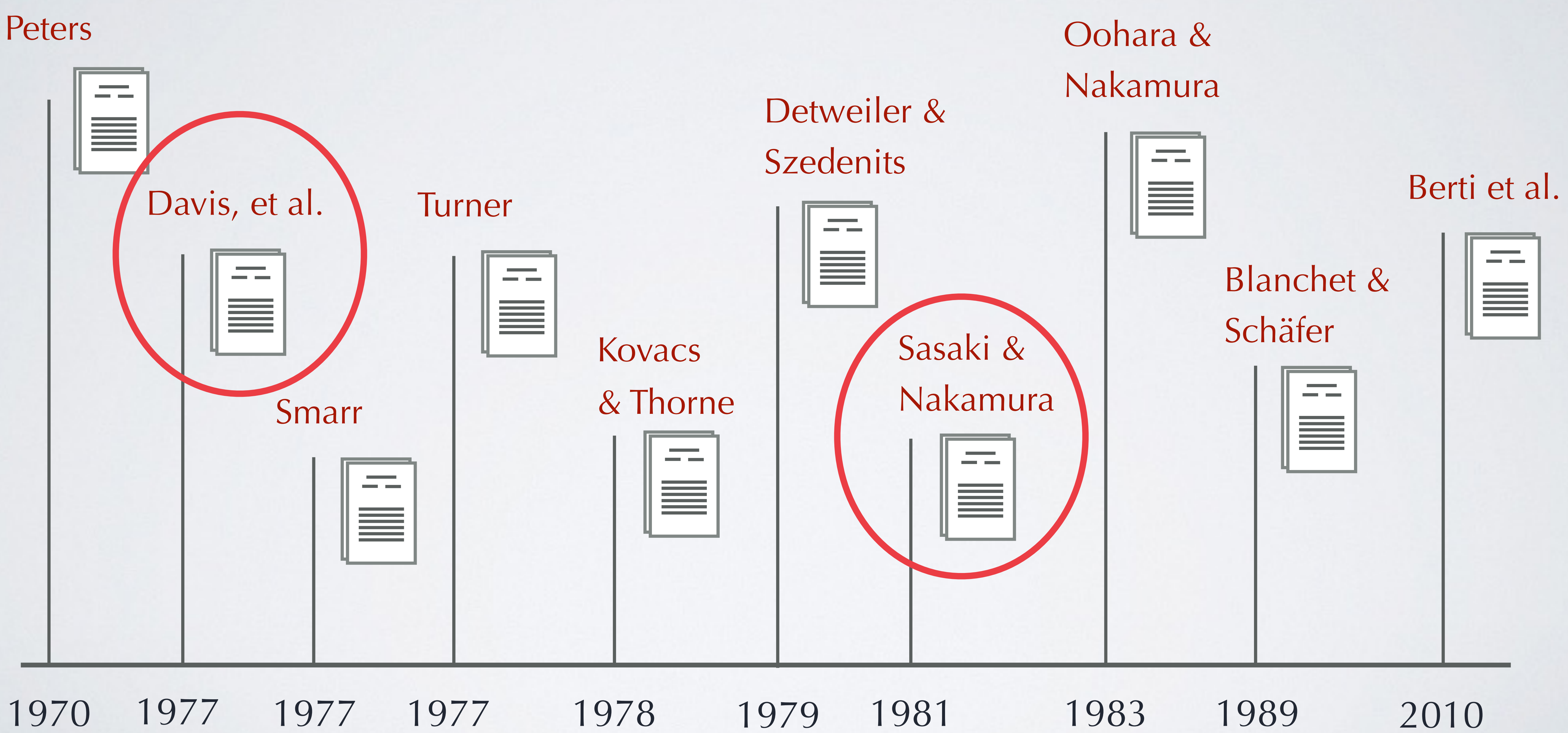






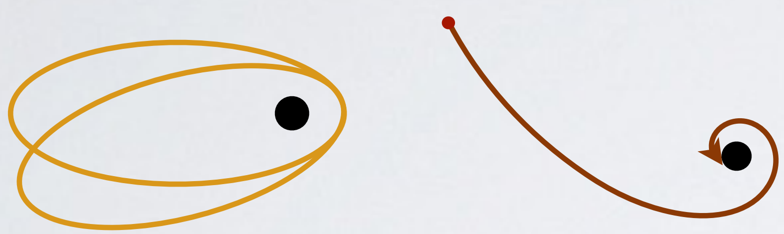
gravitation in técnico

There has been a lot of previous work, but here are a couple highlights

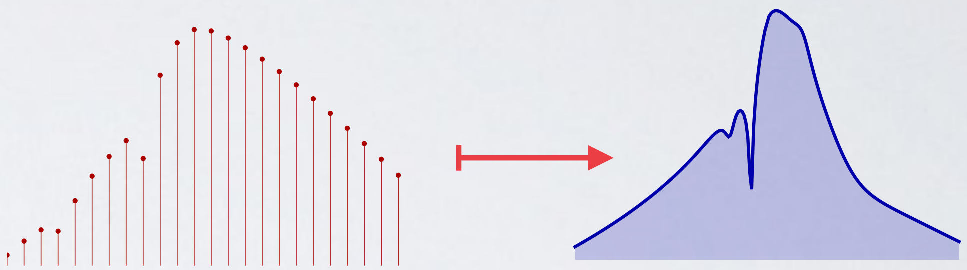


Outline

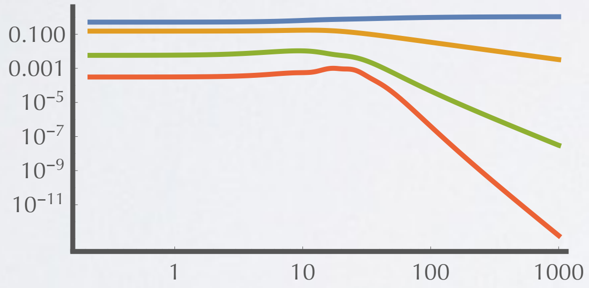
Geodesics, bound and unbound



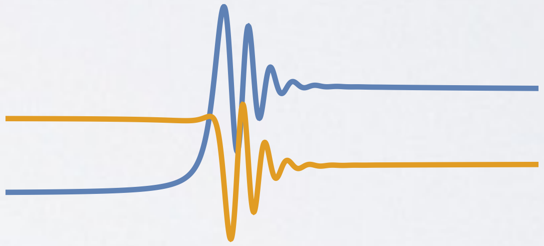
Fourier series to Fourier transform



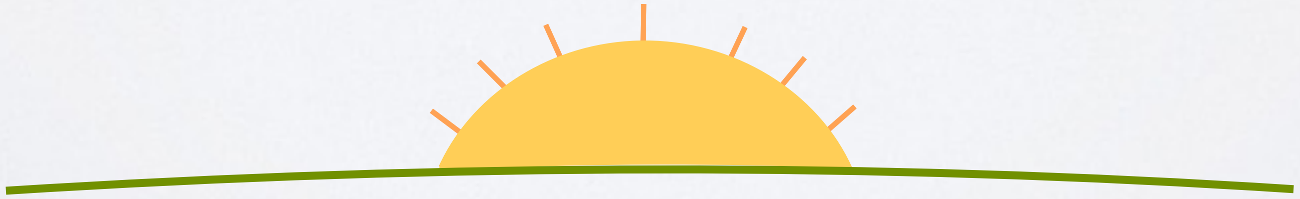
Master functions and source convergence



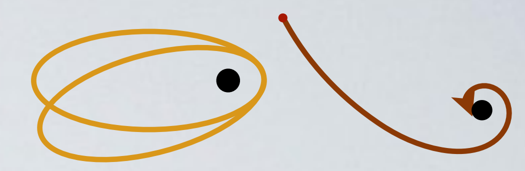
Results, successes and failures

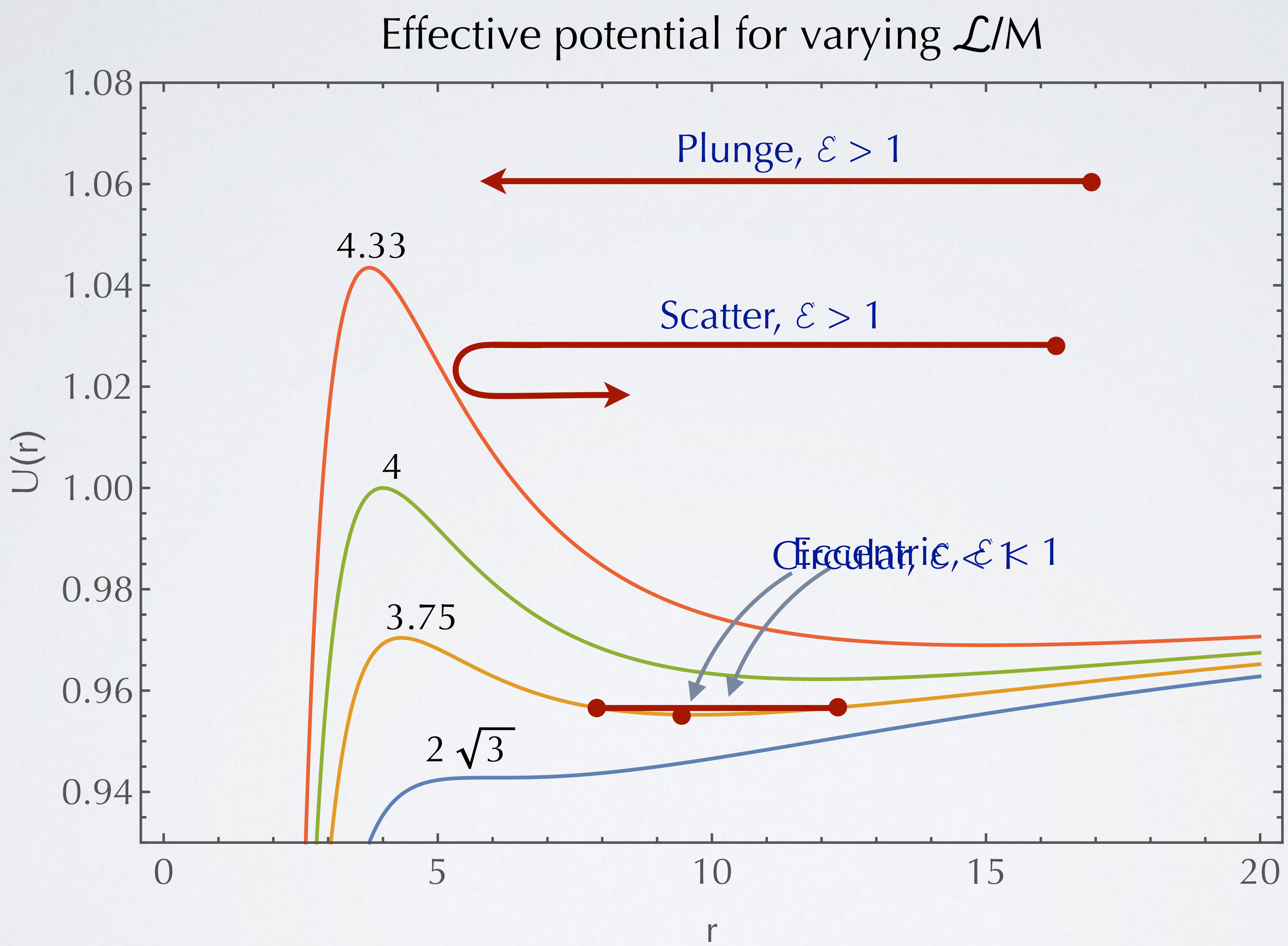


Future prospects and conclusions

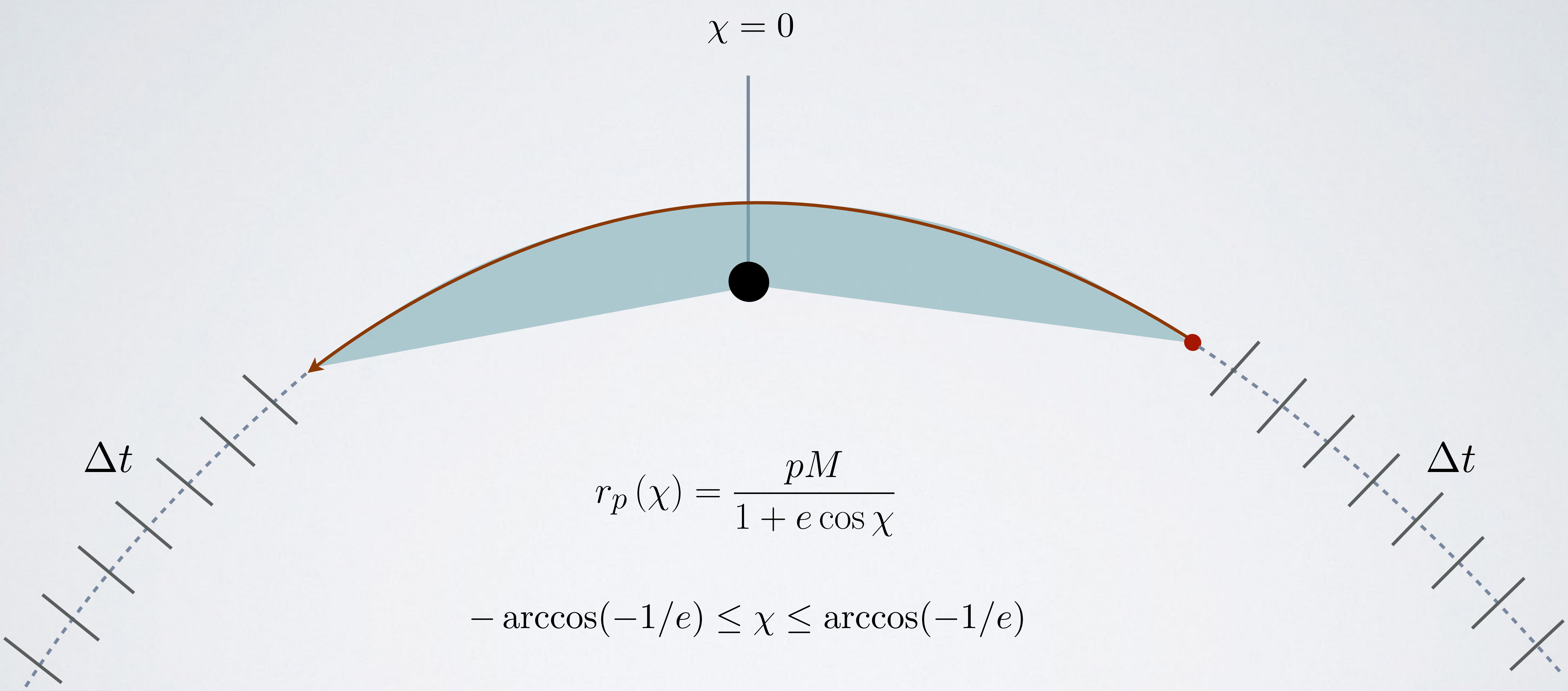


The type of geodesic is determined by the effective potential and the specific energy





Scatters can still be parametrized using the relativistic anomaly

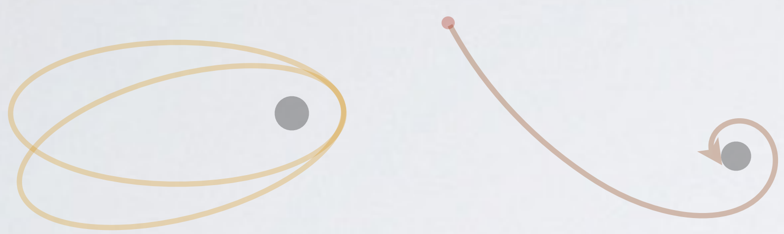


$$r_p(\chi) = \frac{pM}{1 + e\cos\chi}$$

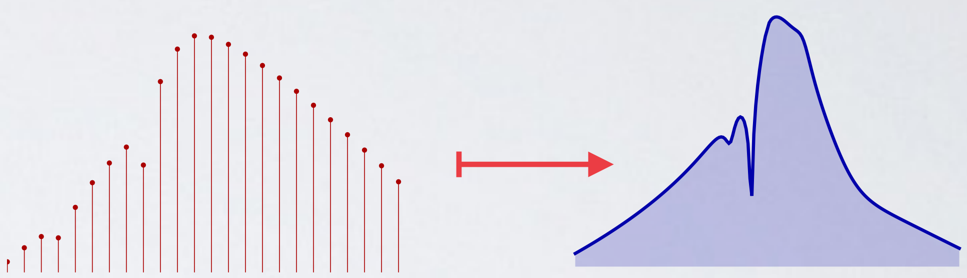
$$-\arccos(-1/e) \leq \chi \leq \arccos(-1/e)$$

Outline

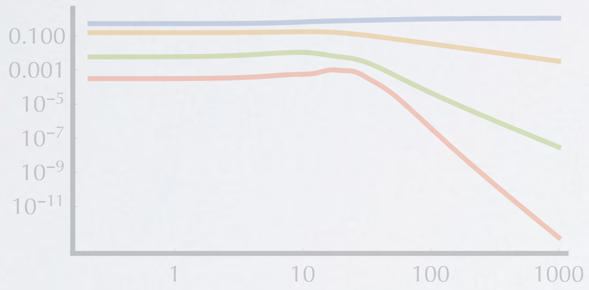
Geodesics, bound and unbound



Fourier series to Fourier transform



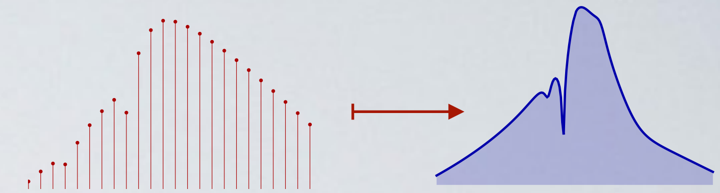
Master functions and source convergence



Results, successes and failures

Future prospects and conclusions

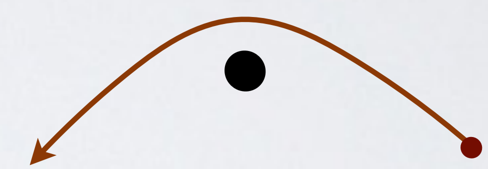
The Fourier series is replaced by a Fourier transform



$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}-V_\ell(r)\right]\Psi_{\ell m}(t,r)=S_{\ell m}(t,r)$$

Bound

Unbound



$$\Psi_{\ell m}(t,r)=\sum_{n=-\infty}^{\infty}X_{\ell mn}(r)\,e^{-i\omega_{mn}t}$$

$$S_{\ell m}(t,r)=\sum_{n=-\infty}^{\infty}Z_{\ell mn}(r)\,e^{-i\omega_{mn}t}$$

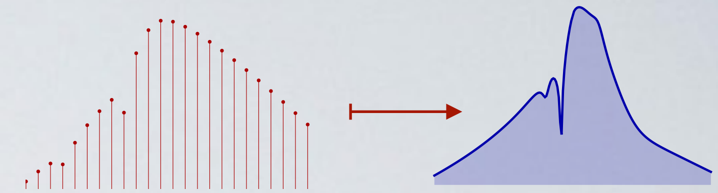
$$\longrightarrow$$

$$\Psi_{\ell m}(t,r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_{\ell m\omega}(r)\,e^{-i\omega t}d\omega$$

$$S_{\ell m}(t,r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}Z_{\ell m\omega}(r)\,e^{-i\omega t}d\omega$$

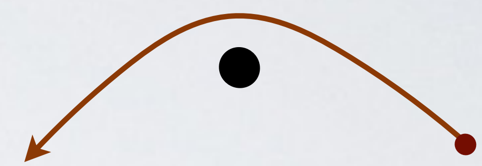
$$\left[\frac{d^2}{dr_*^2}+\omega_{mn}^2-V_\ell(r)\right]X_{\ell mn}(r)=Z_{\ell mn}(r)\quad\longrightarrow\quad\left[\frac{d^2}{dr_*^2}+\omega^2-V_\ell(r)\right]X_{\ell m\omega}(r)=Z_{\ell m\omega}(r).$$

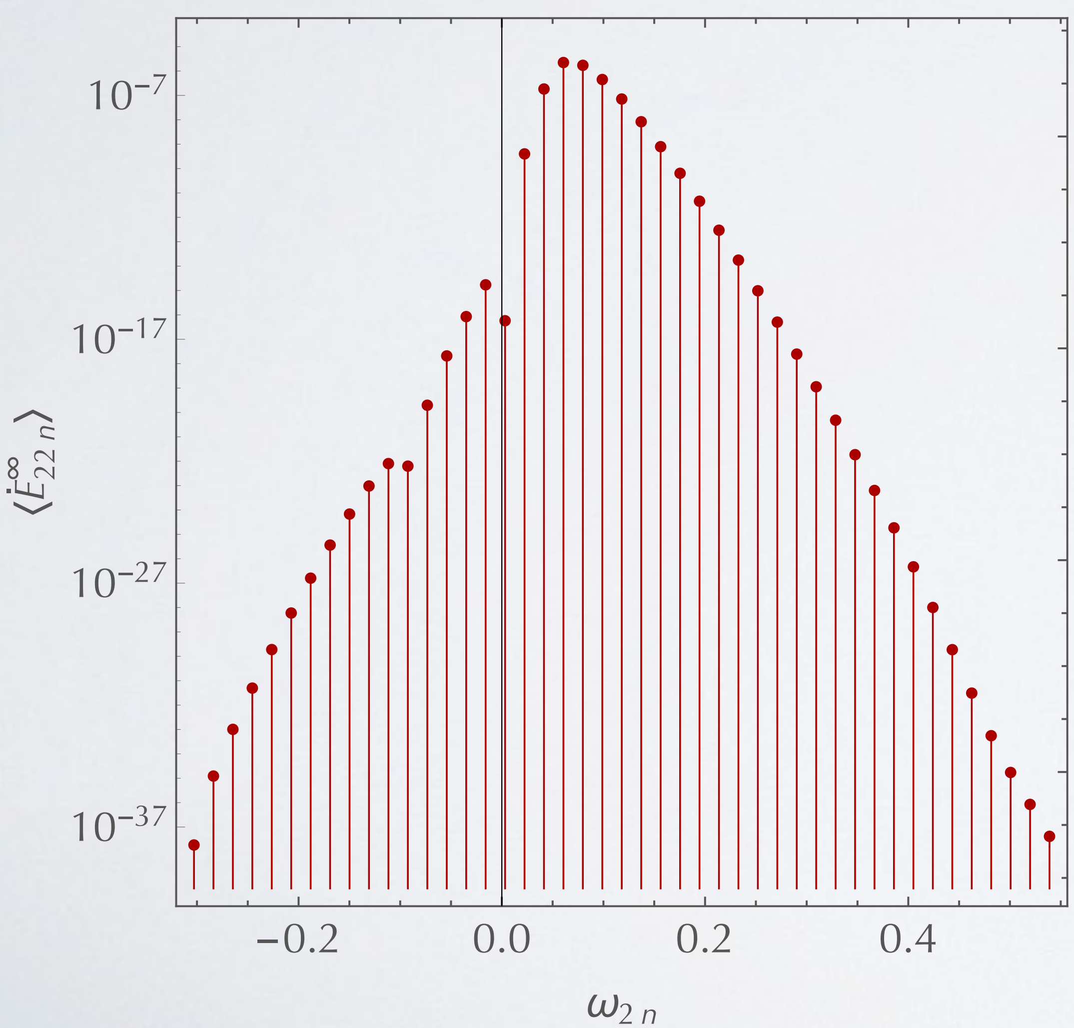
The unbound spectrum is dense

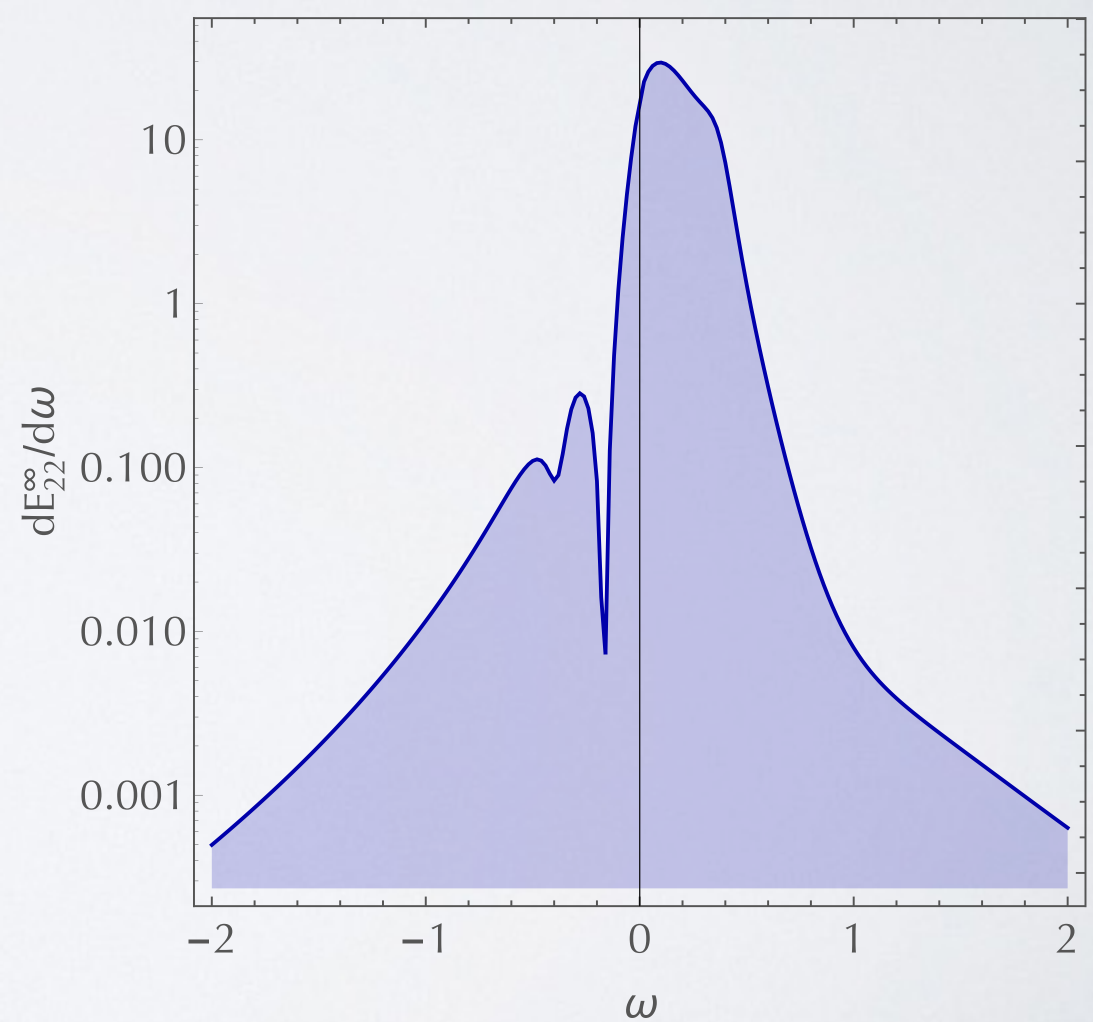


$(p,e)=(10,0.2)$

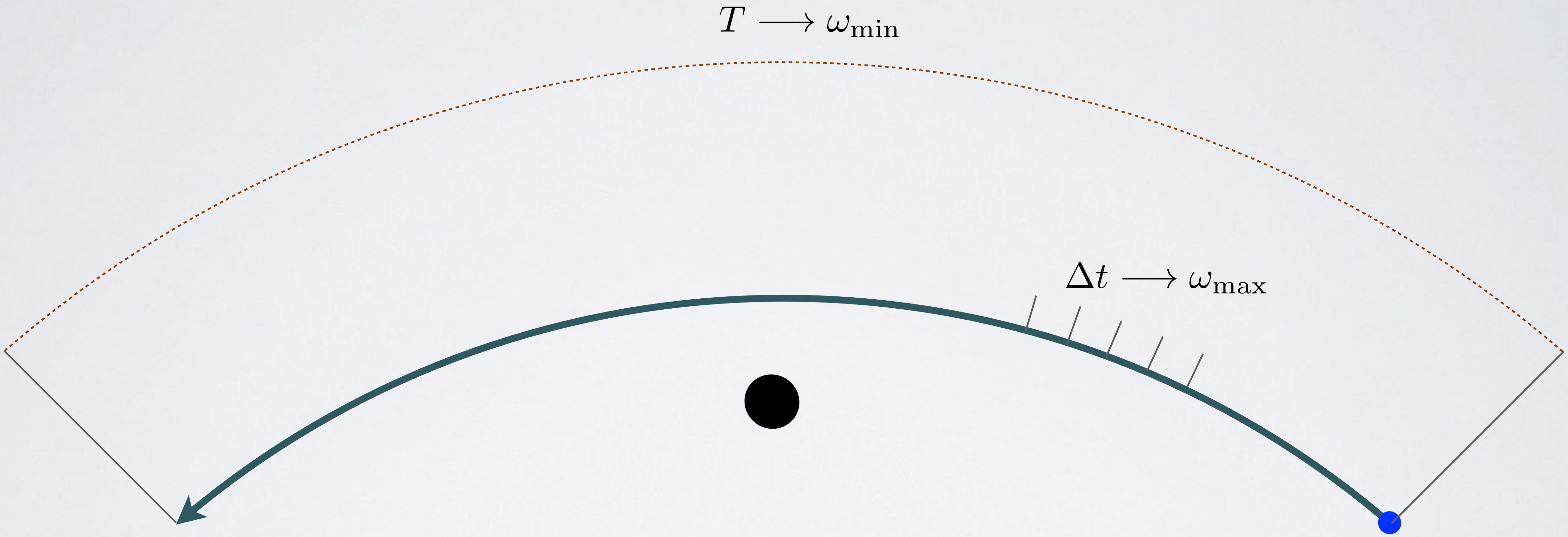
$(\mathcal{E},r_{min})=(1.5,4.3M)$



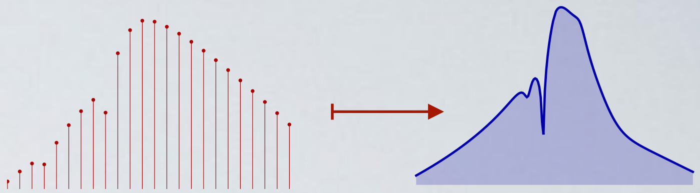




The time domain is dense, too



The normalization integral requires integrating over all time



Extended homogeneous solutions:

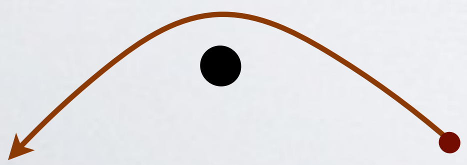
$$\Psi_{\ell m}(t,r) = \Psi^{+}_{\ell m}(t,r)\theta(r-r_p) + \Psi^{-}_{\ell m}(t,r)\theta(r_p - r)$$

Bound

$$\Psi^{\pm}_{\ell m}(t,r) \equiv \boxed{\sum_{n=-\infty}^{\infty}} C^{\pm}_{\ell mn}\hat{X}^{\pm}_{\ell mn}(r)\, e^{-i\omega_{mn}t}$$

$$C^{\pm}_{\ell mn} = \frac{1}{W_{\ell mn}T_r}\boxed{\int_0^{T_r}}\left[\frac{1}{f_p}\hat{X}^{\mp}_{\ell mn}(r_p)G_{\ell m}(t) + \left(\frac{2M}{r_p^2 f_p^2}\hat{X}^{\mp}_{\ell mn}(r_p) - \frac{1}{f_p}\frac{d\hat{X}^{\mp}_{\ell mn}(r_p)}{dr}\right)F_{\ell m}(t)\right]e^{i\omega_{mn}t}\,dt$$

Unbound

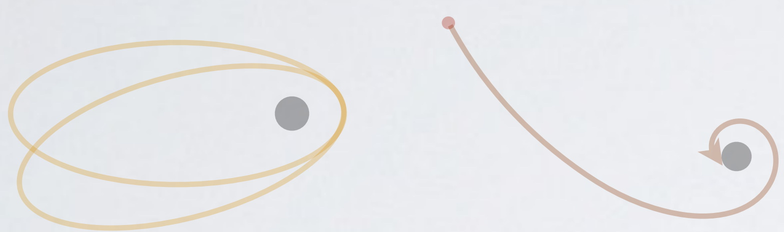


$$\Psi^{\pm}_{\ell m}(t,r) \equiv \frac{1}{2\pi}\boxed{\int_{-\infty}^{\infty}} C^{\pm}_{\ell m\omega}\hat{X}^{\pm}_{\ell m\omega}(r)\, e^{-i\omega t}d\omega$$

$$C^{\pm}_{\ell m\omega} = \frac{1}{W_{\ell m\omega}}\boxed{\int_{-\infty}^{\infty}}\left[\frac{1}{f_p}\hat{X}^{\mp}_{\ell m\omega}(r_p)\boxed{G_{\ell m}(t)} + \left(\frac{2M}{r_p^2 f_p^2}\hat{X}^{\mp}_{\ell m\omega}(r_p) - \frac{1}{f_p}\frac{d\hat{X}^{\mp}_{\ell m\omega}(r_p)}{dr}\right)\boxed{F_{\ell m}(t)}\right]e^{i\omega t}\,dt$$

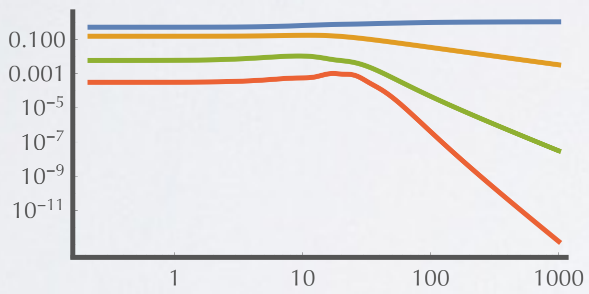
Outline

Geodesics, bound and unbound



Fourier series to Fourier transform

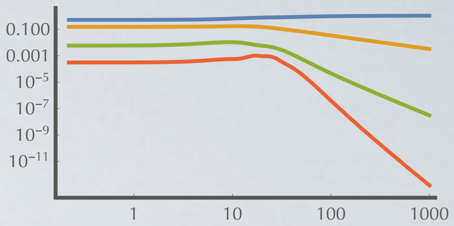
Master functions and source convergence



Results, successes and failures

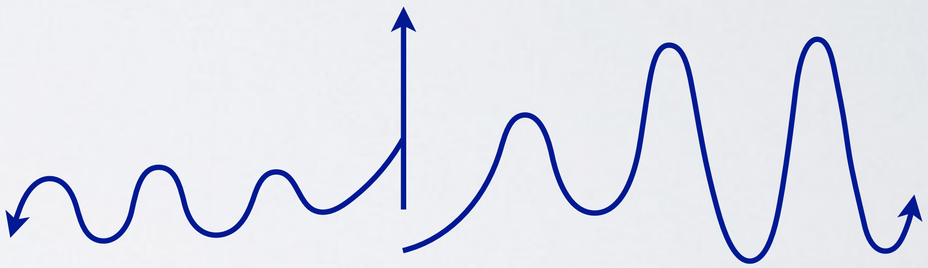
Future prospects and conclusions

With a source present, the Zerilli function requires a delta function to be subtracted

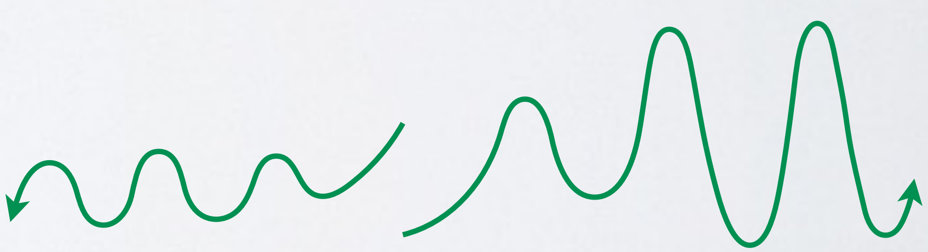


$$\underbrace{\Psi_{\mathrm{ZM}}(t,r) = \frac{r}{\lambda+1}\left[K + \frac{1}{\Lambda}\left(f^2 h_{rr} - r f \partial_r K\right)\right]}_{\text{Defined from metric perturbation}}$$

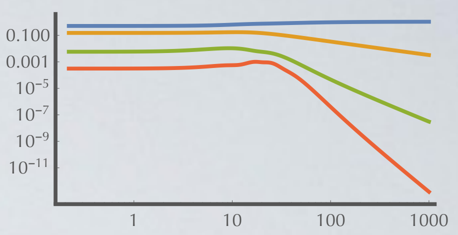
$$\dot{\Psi}_{\mathrm{ZM}}(t,r) = \frac{r}{\lambda+1}\left[\partial_t K + \frac{1}{\Lambda}\left(f^2 \partial_t h_{rr} - r f \partial_t \partial_r K\right)\right]$$



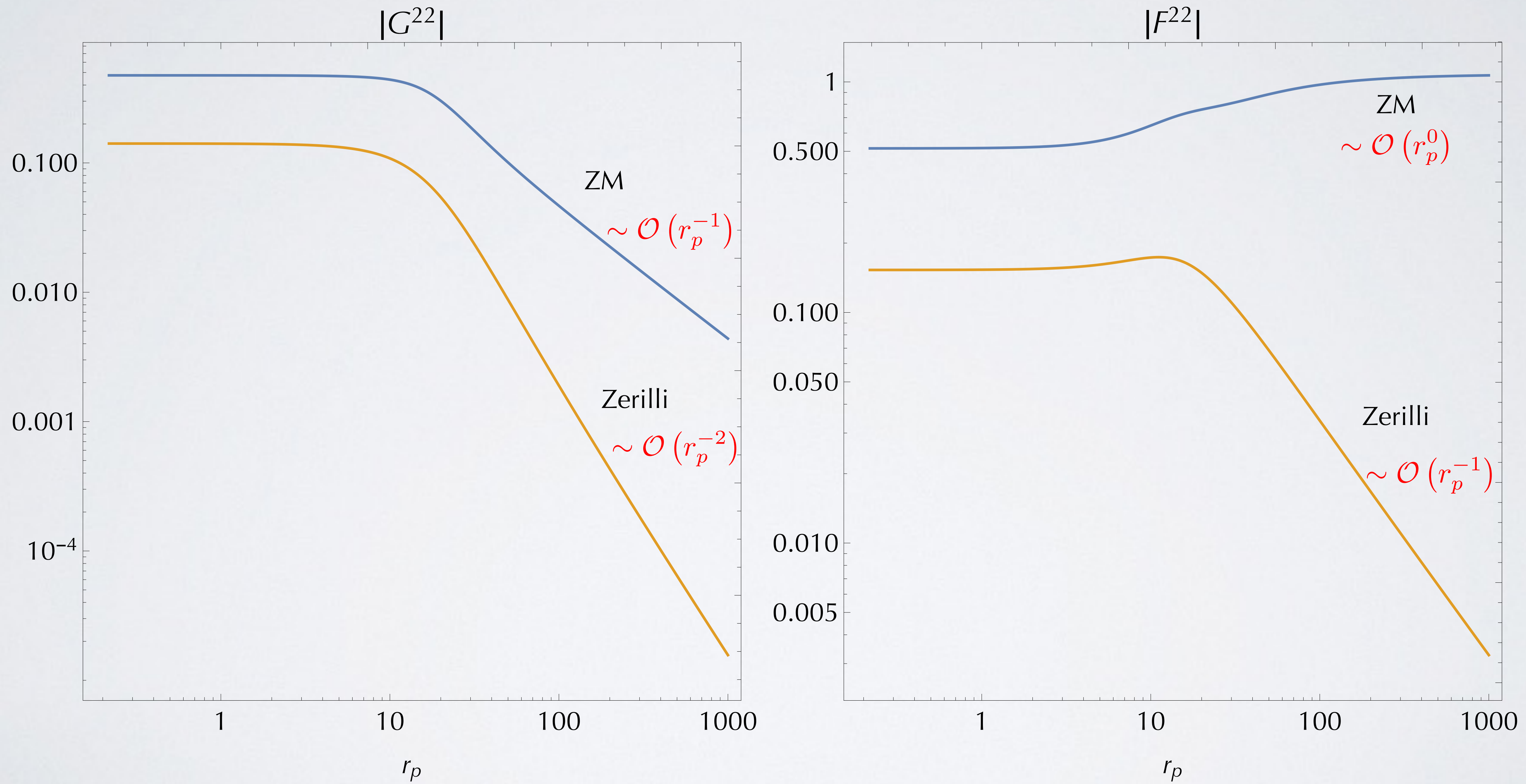
$$\Psi_{\mathrm{Z}}(t,r) \equiv \dot{\Psi}_{\mathrm{ZM}} - \frac{r_p^2 f_p}{\Lambda_p(\lambda+1)} q_{tr}(t)\delta\left(r - r_p\right) = \underbrace{\frac{1}{\Lambda}\left(r\partial_t K - f h_{tr}\right)}_{\text{Metric perturbation}}$$



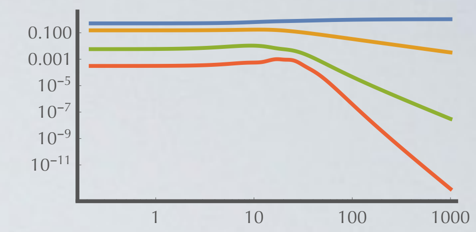
The Zerilli function source converges at large distance, but the ZM source does not



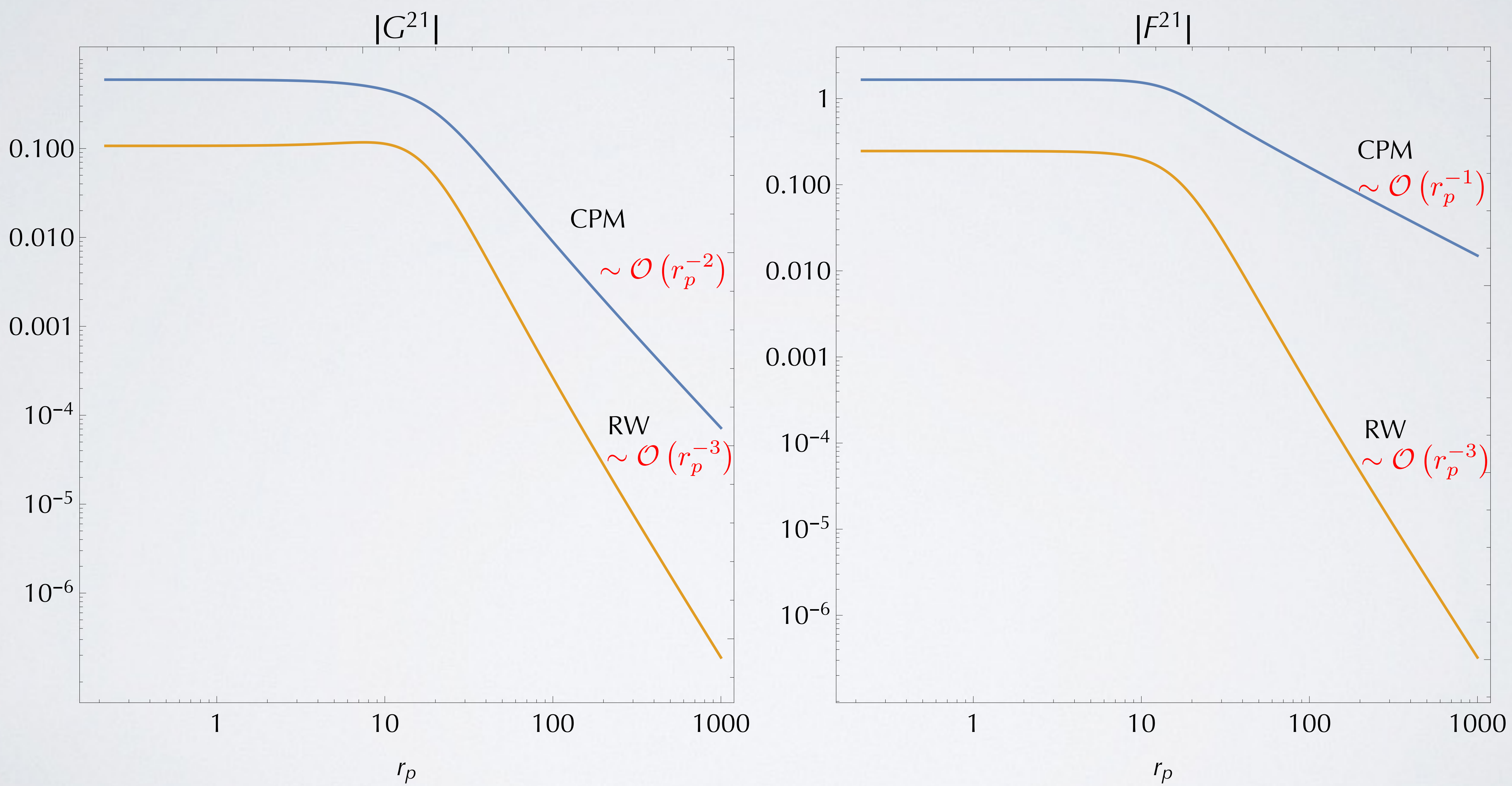
$(\mathcal{E}, r_{\text{min}}) = (1.5, 4.3M)$



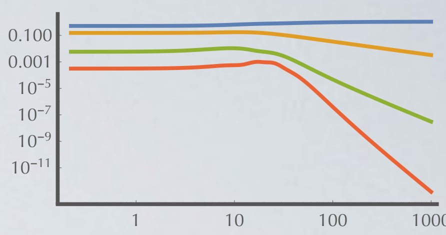
The CPM function source converges at large distance, but the RW source converges faster



$(\mathcal{E}, r_{min}) = (1.5, 4.3M)$

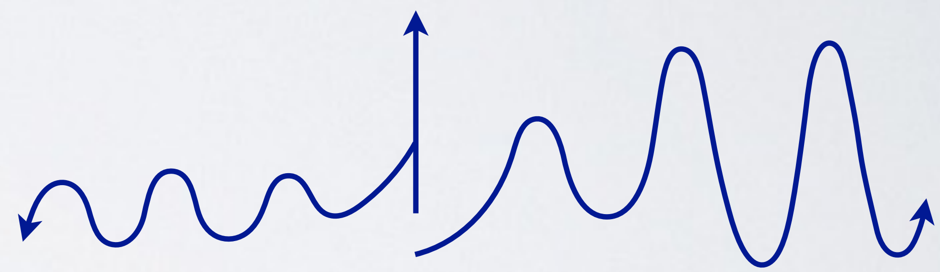


New master functions are related to old ones by taking a time derivative and removing the delta function

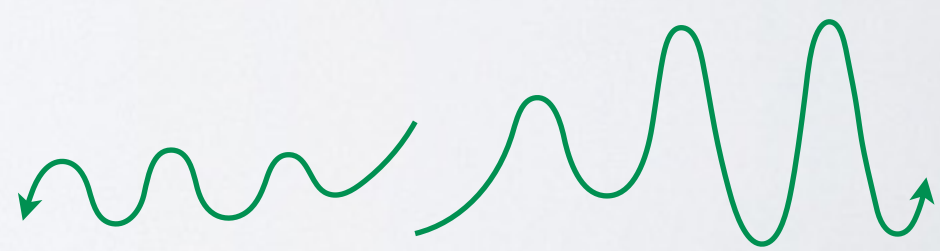


$$\underbrace{\Psi^{(0)}(t,r) = \Psi^{(0),+}(t,r)\theta\left(r-r_p\right) + \Psi^{(0),-}(t,r)\theta\left(r_p-r\right)}_{C^{-1}}$$

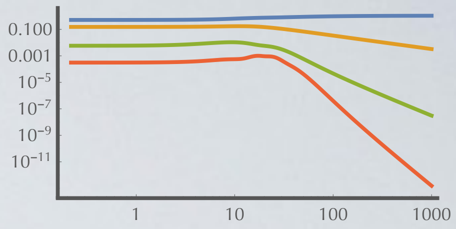
$$\dot{\Psi}^{(0)}(t,r) = \dot{\Psi}^{(0),+}\theta\left(r-r_p\right) + \dot{\Psi}^{(0),-}\theta\left(r_p-r\right) - \dot{r}_p[\![\Psi^{(0)}]\!]_p\delta\left(r-r_p\right)$$



$$\underbrace{\Psi^{(1)}(t,r) \equiv \dot{\Psi}^{(0)} + \dot{r}_p[\![\Psi^{(0)}]\!]_p\delta\left(r-r_p\right)}_{C^{-1}}$$

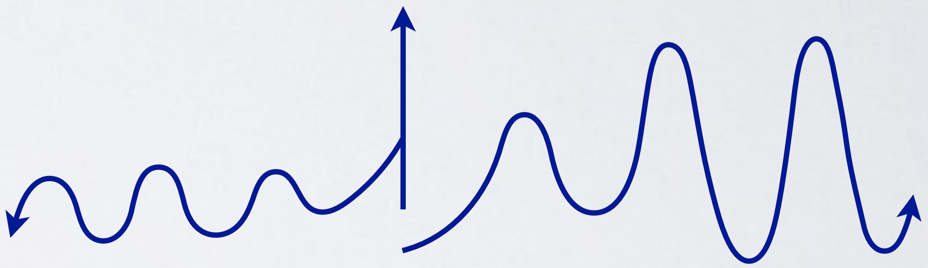


New source terms can be systematically derived from old ones

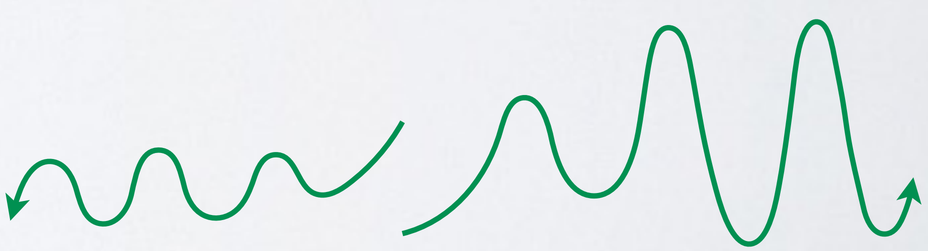


$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V\right] \Psi^{(0)} = G^{(0)} \delta(r - r_p) + F^{(0)} \delta'(r - r_p)$$

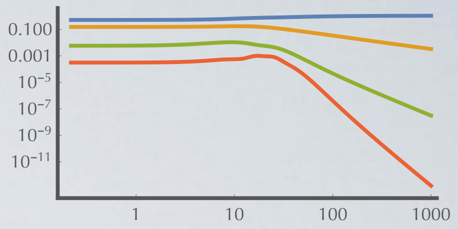
$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V\right] \dot{\Psi}^{(0)} = \dot{G}^{(0)} \delta\left(r - r_p\right) + \left(\dot{F}^{(0)} - \dot{r}_p G^{(0)}\right) \delta'\left(r - r_p\right) - \dot{r}_p F^{(0)} \delta''\left(r - r_p\right)$$



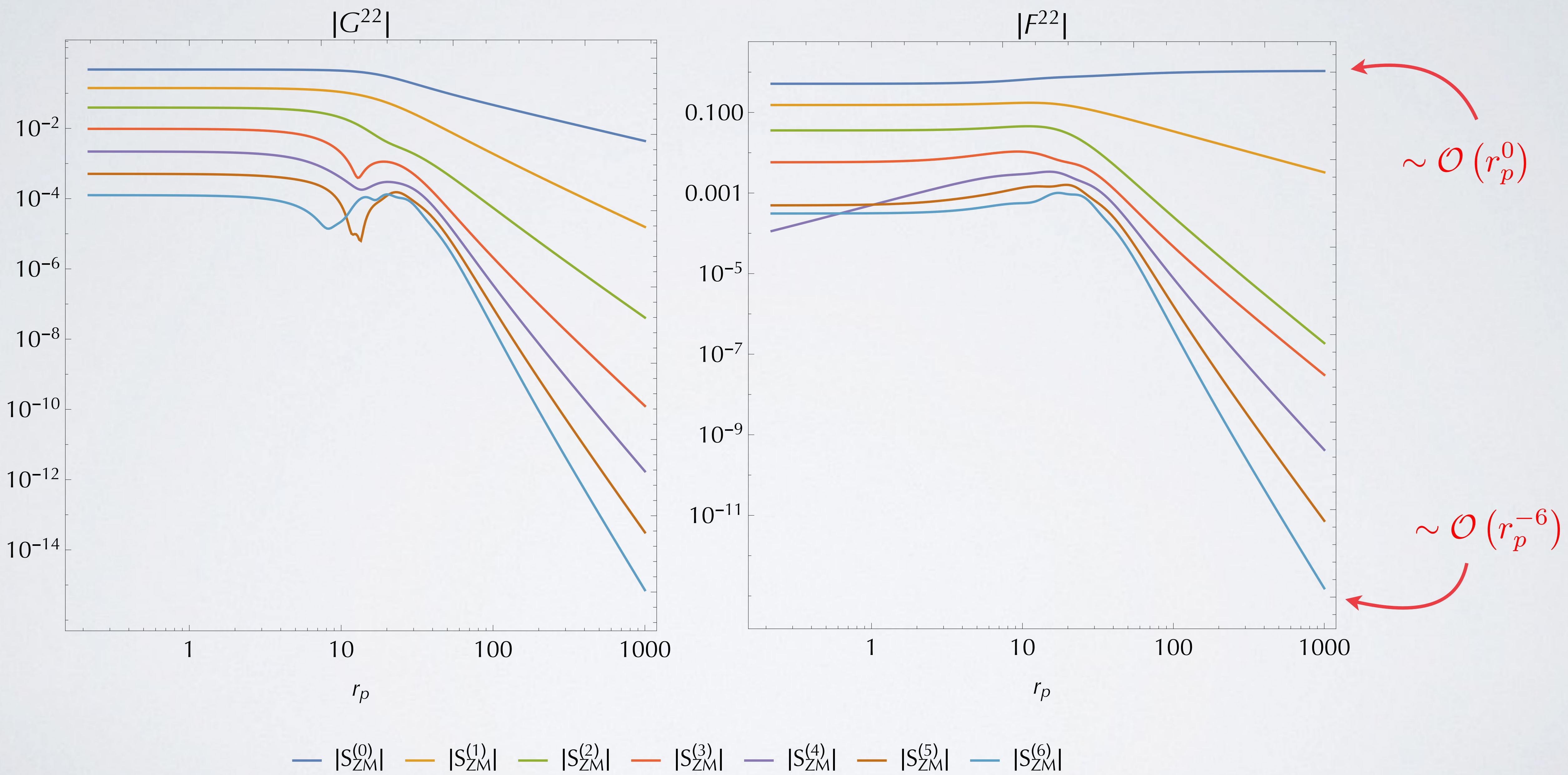
$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V\right] \left[\dot{\Psi}^{(0)} + \dot{r}_p [\![\Psi^{(0)}]\!]_p \delta\left(r - r_p\right)\right] = G^{(1)} \delta(r - r_p) + F^{(1)} \delta'(r - r_p)$$



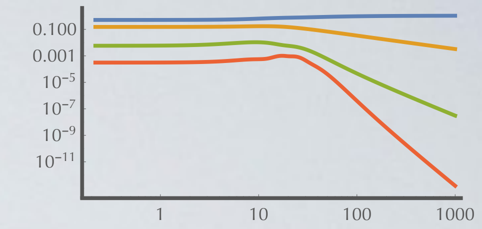
Higher order master function sources always converge faster



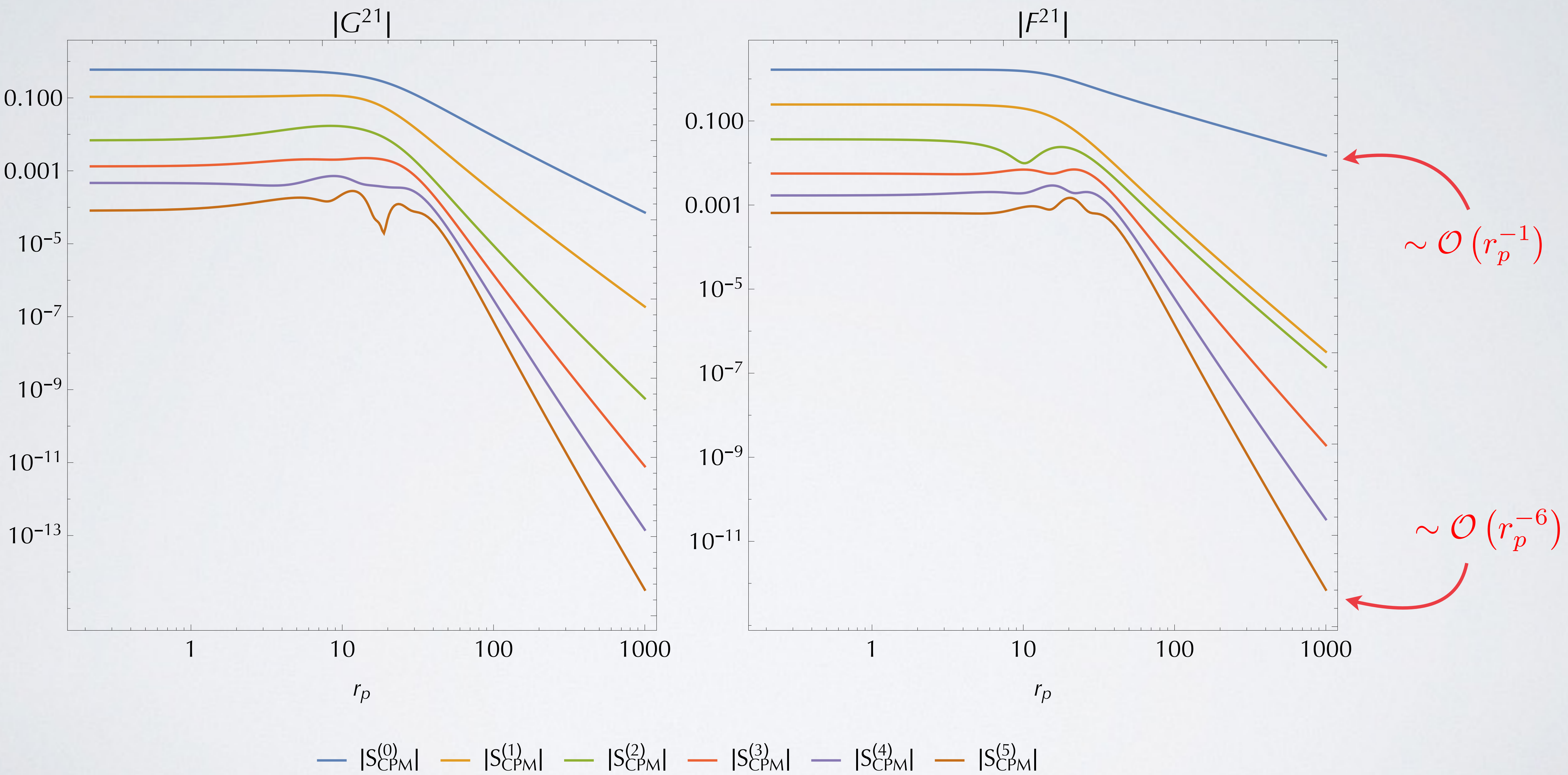
$(\mathcal{E}, r_{\rm min}) = (1.5, 4.3M)$



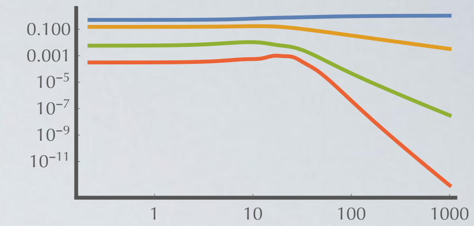
The odd parity has the same behavior

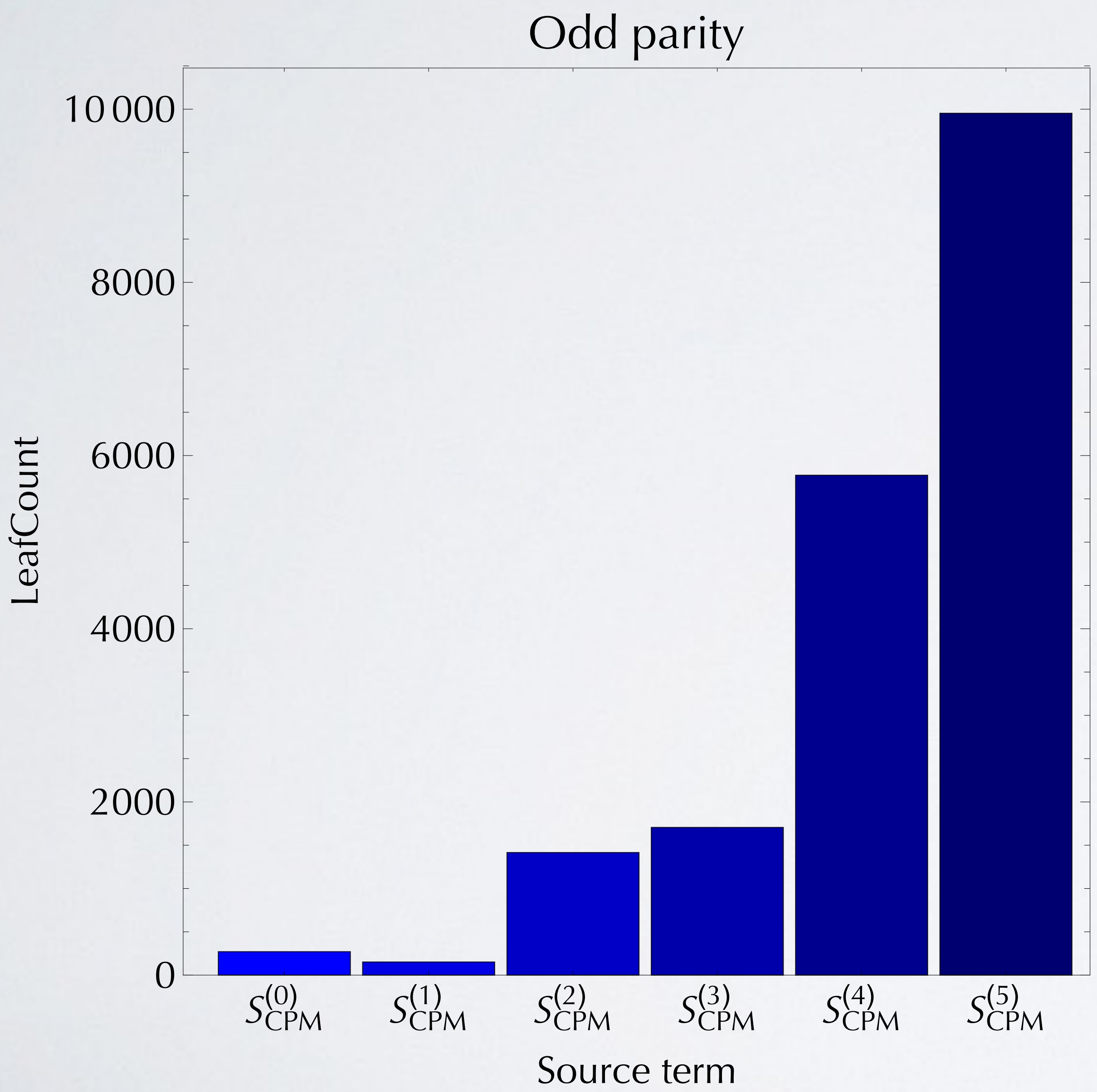


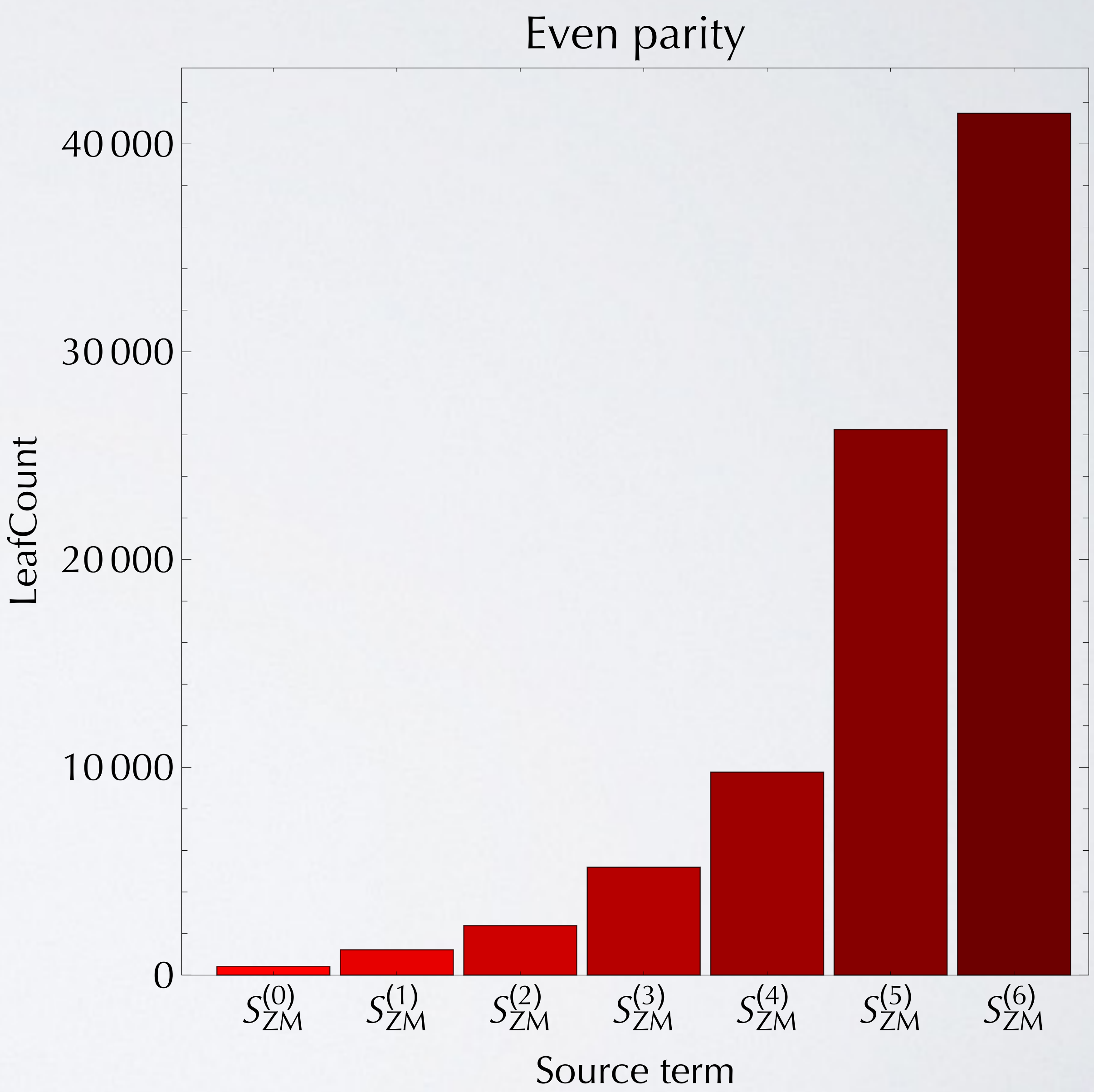
$(\mathcal{E}, r_{\min}) = (1.5, 4.3M)$



Higher-order source terms are much larger, requiring more function evaluations







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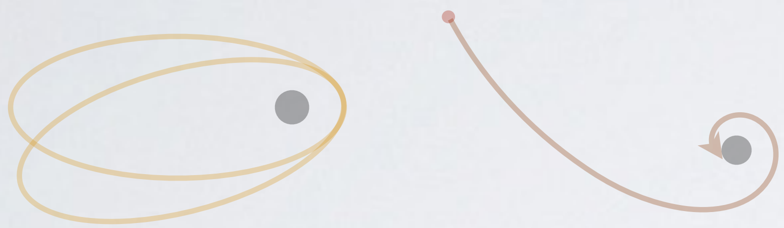
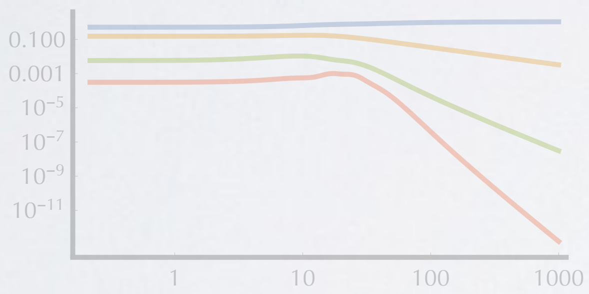
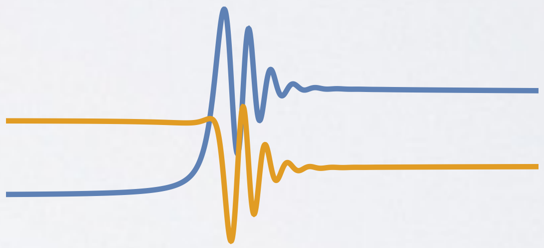


Fourier series to Fourier transform

Master functions and source convergence



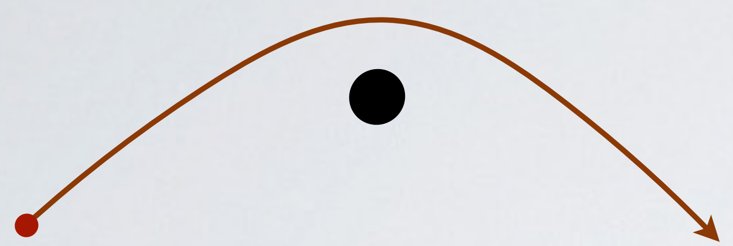
Results, successes and failures

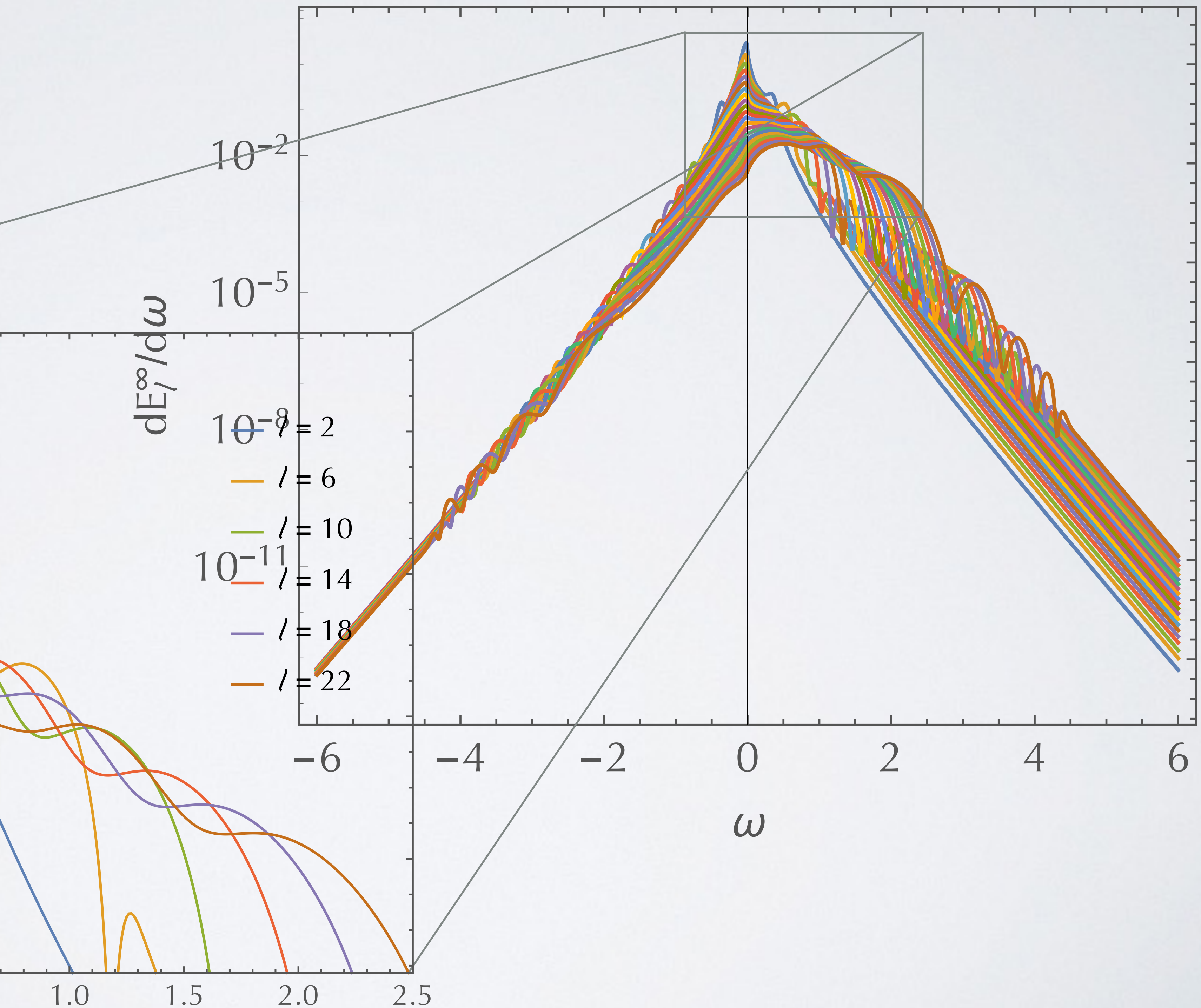


Future prospects and conclusions

Frequency domain allows high Lorentz factor scatters

$(\mathcal{E},b)=(10,20M)$, $v/c = 0.995$

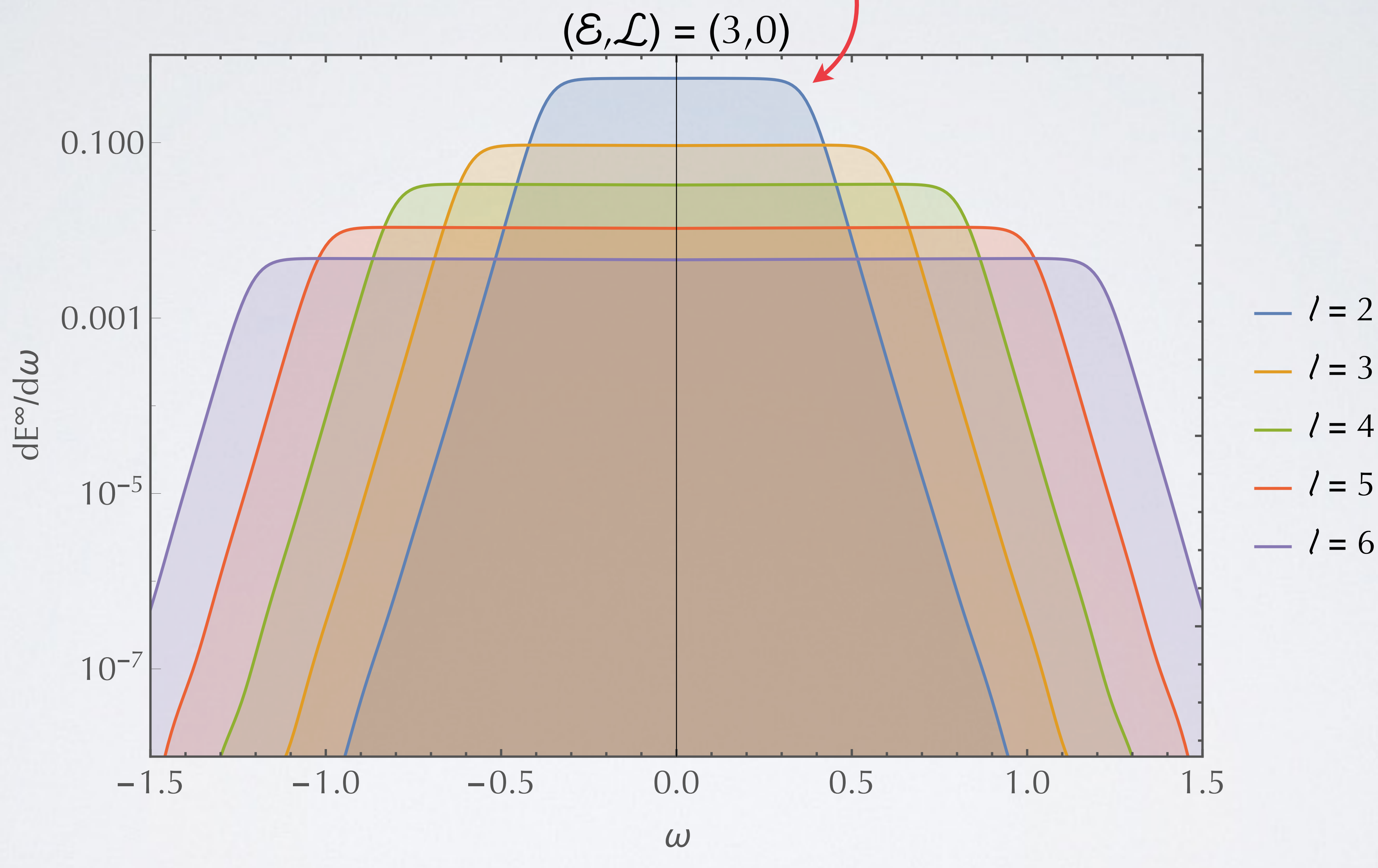




The spectrum at zero frequency is flat

$\Delta E^{\infty}_{\ell m} \sim \omega^2 |C^{+}_{\ell m\omega}|^2$ $\qquad$ $C^{+}_{\ell m\omega} \sim \omega^{-1}$

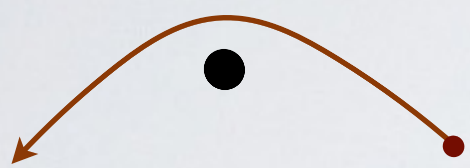
$$\Psi^{\pm}_{\ell m}(t,r) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} C^{\pm}_{\ell m\omega}\hat{X}^{\pm}_{\ell m\omega}(r)\, e^{-i\omega t} d\omega$$

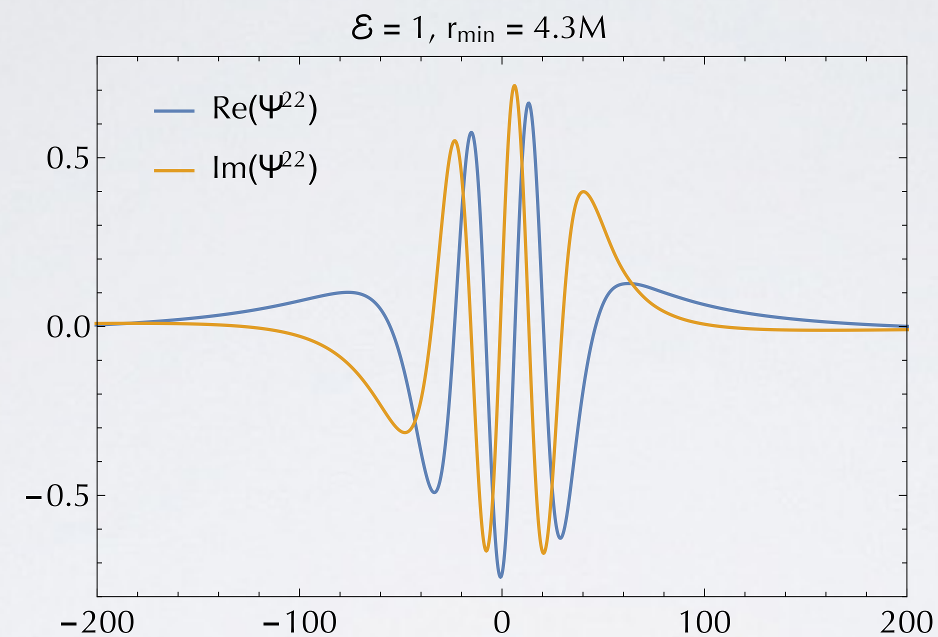


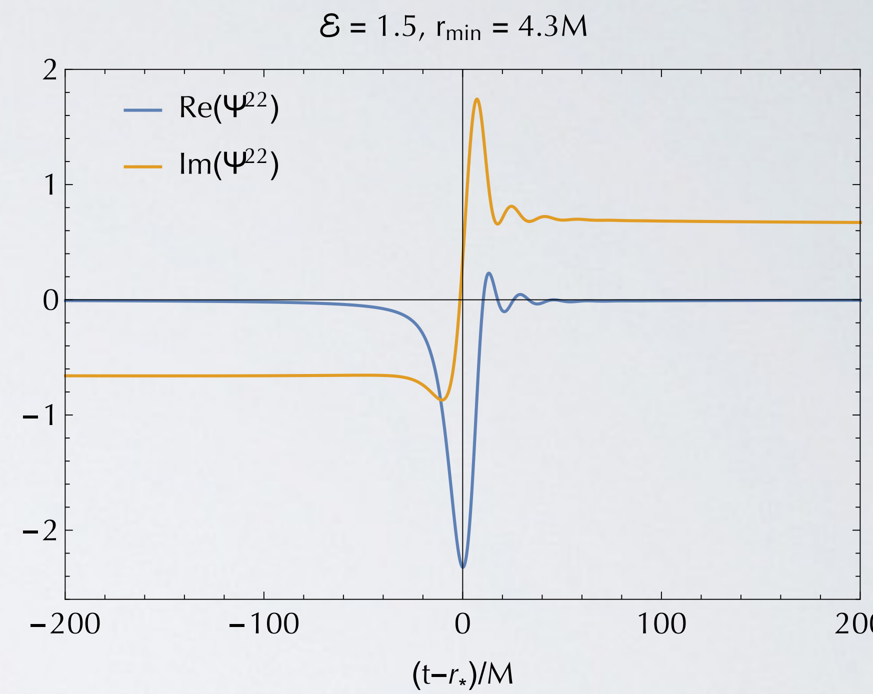
Asymptotically nonzero, constant velocity implies flat ZFL spectrum, Smarr 1977

Here are some waveforms

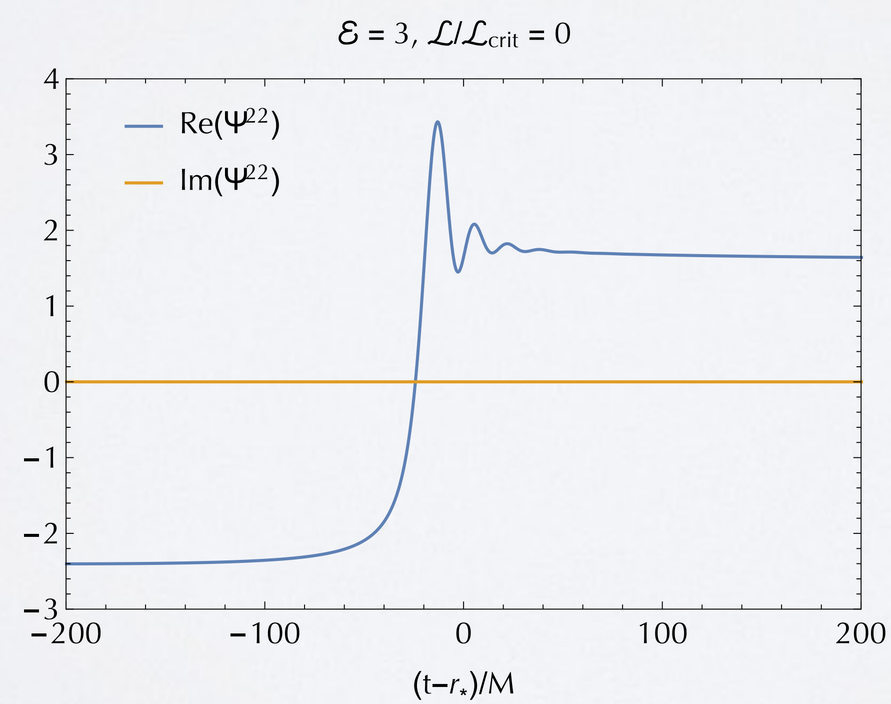
Scatter

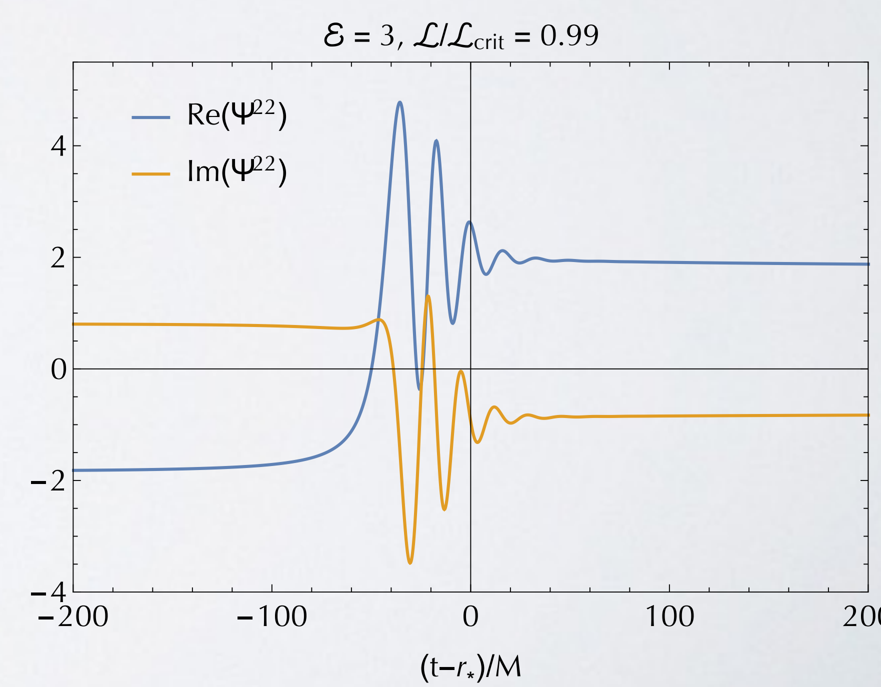




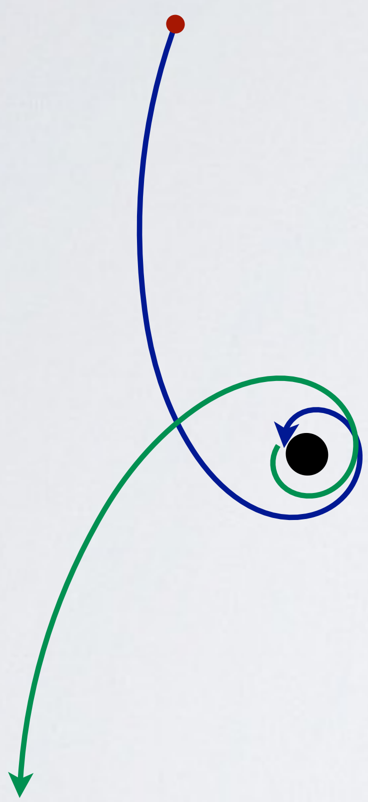


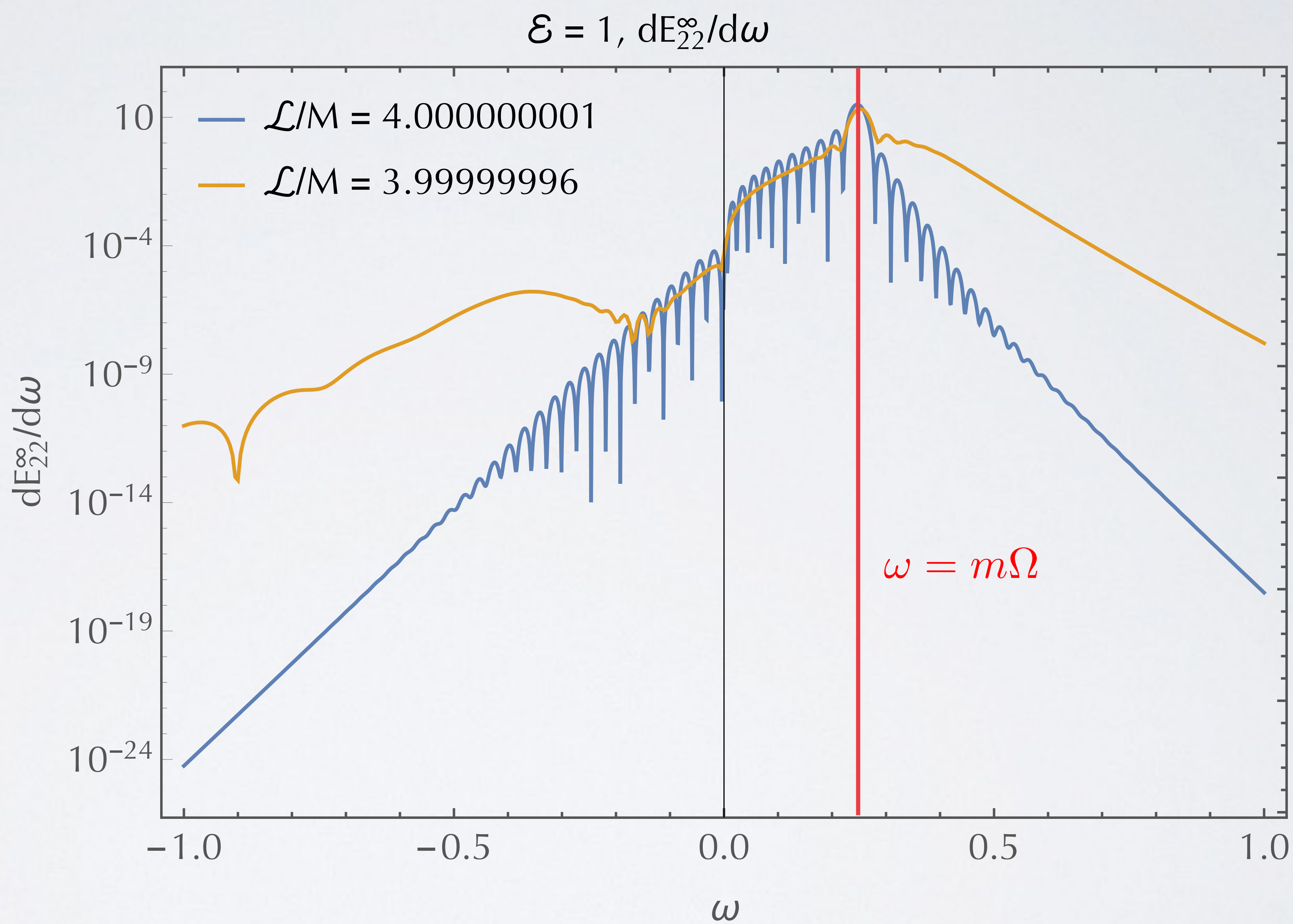
Plunge



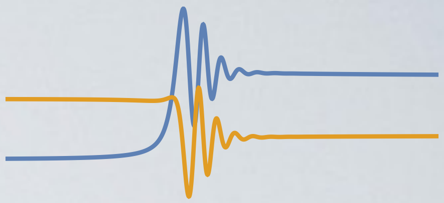


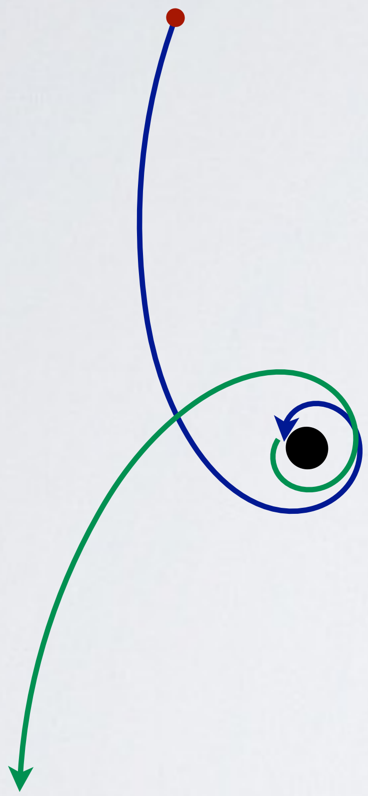
Here are some *almost* marginally bound results

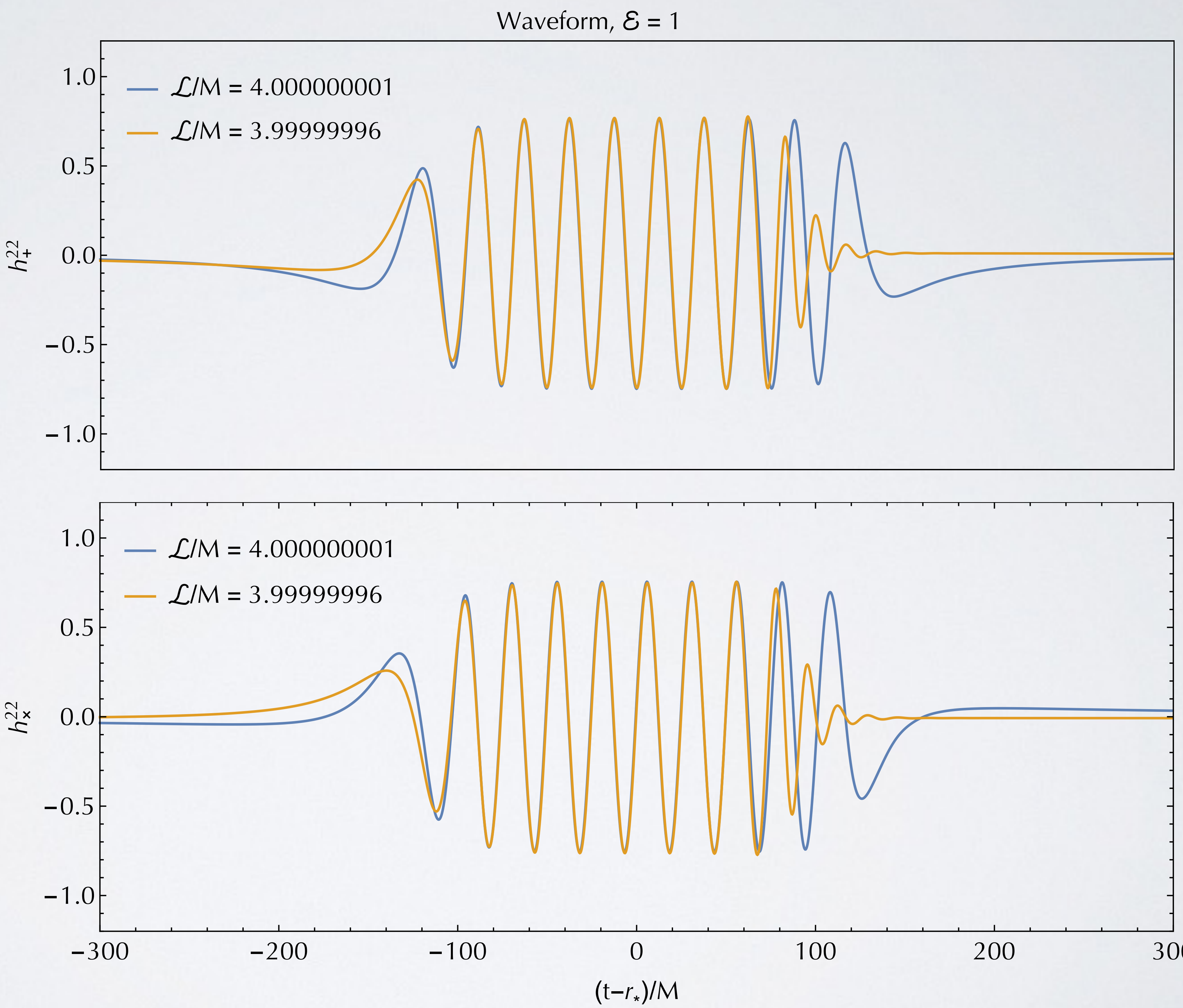




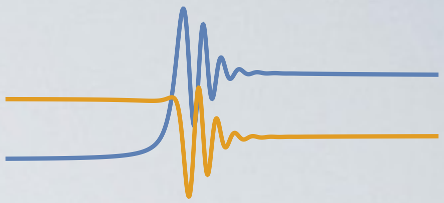
Here are some *almost* marginally bound results



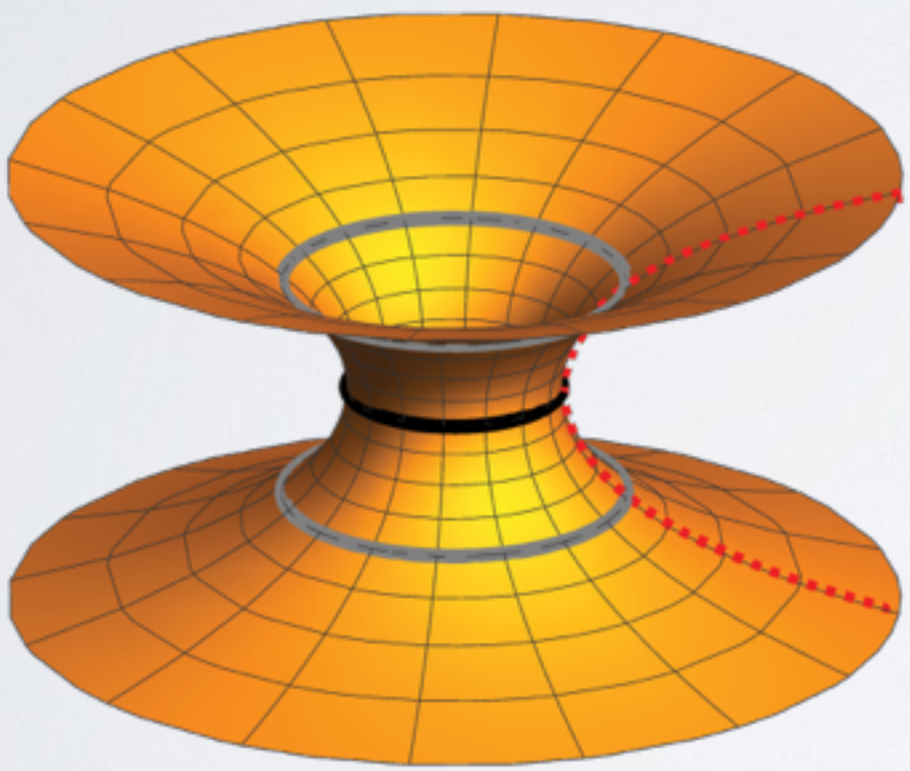


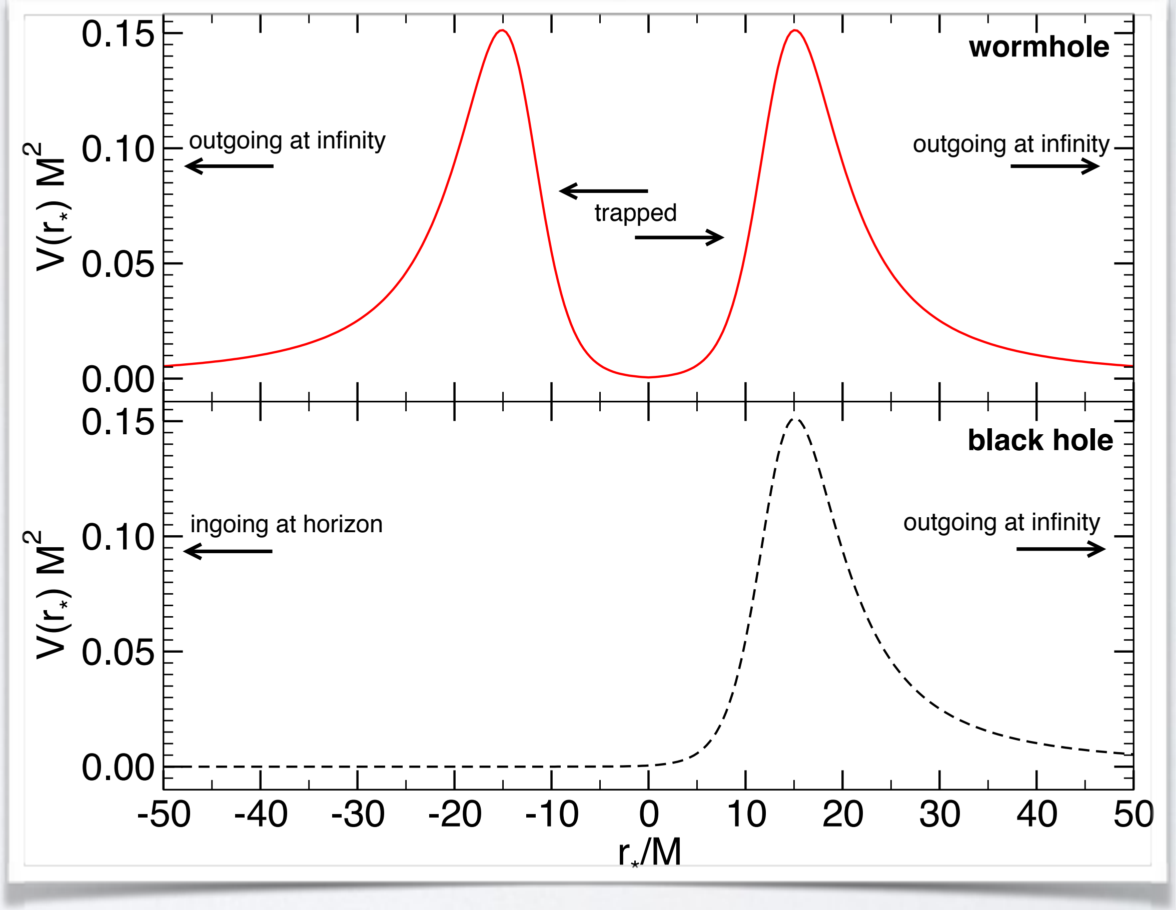


A wormhole can be approximated by "gluing" two Schwarzschild spacetimes together



Wormhole plunge

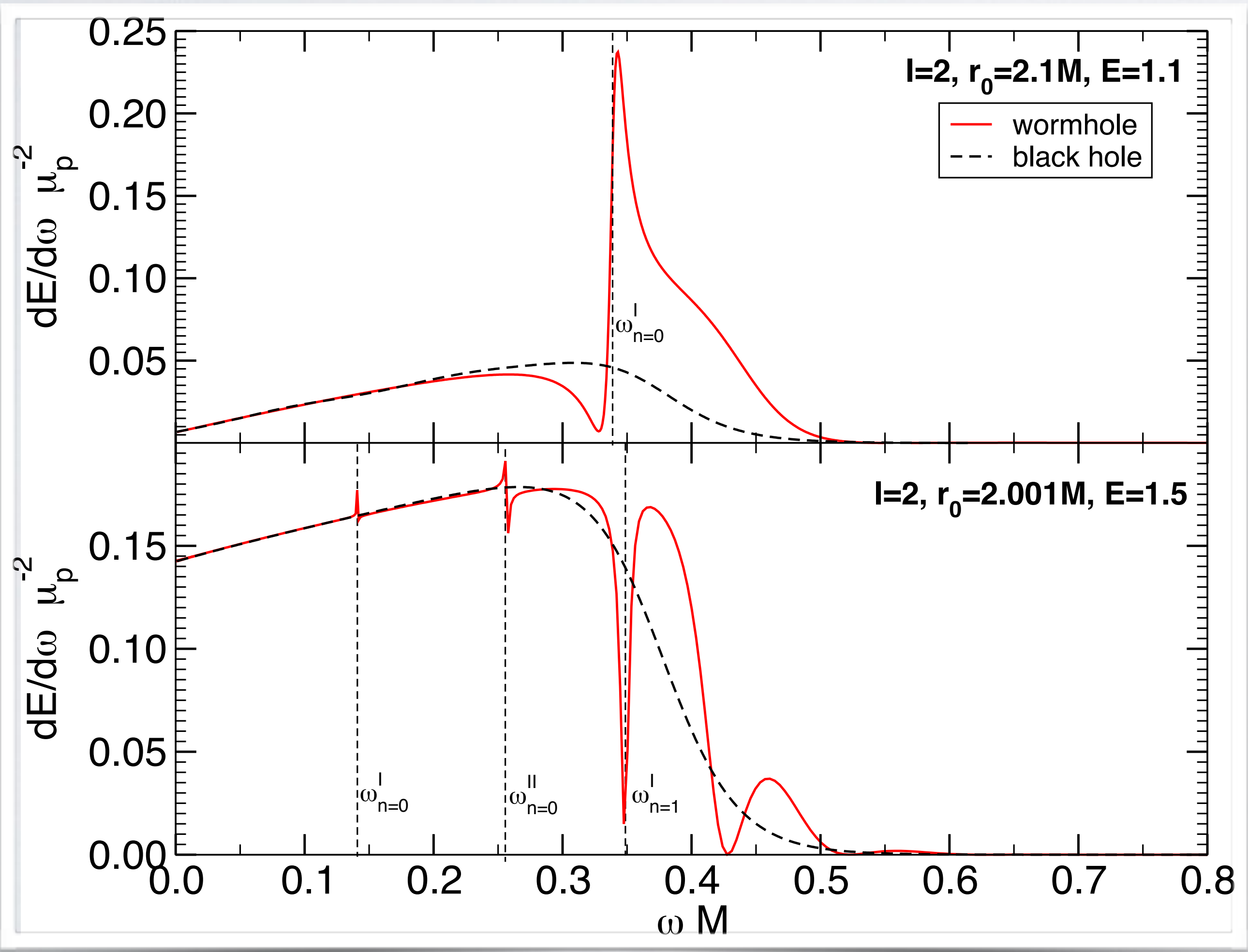


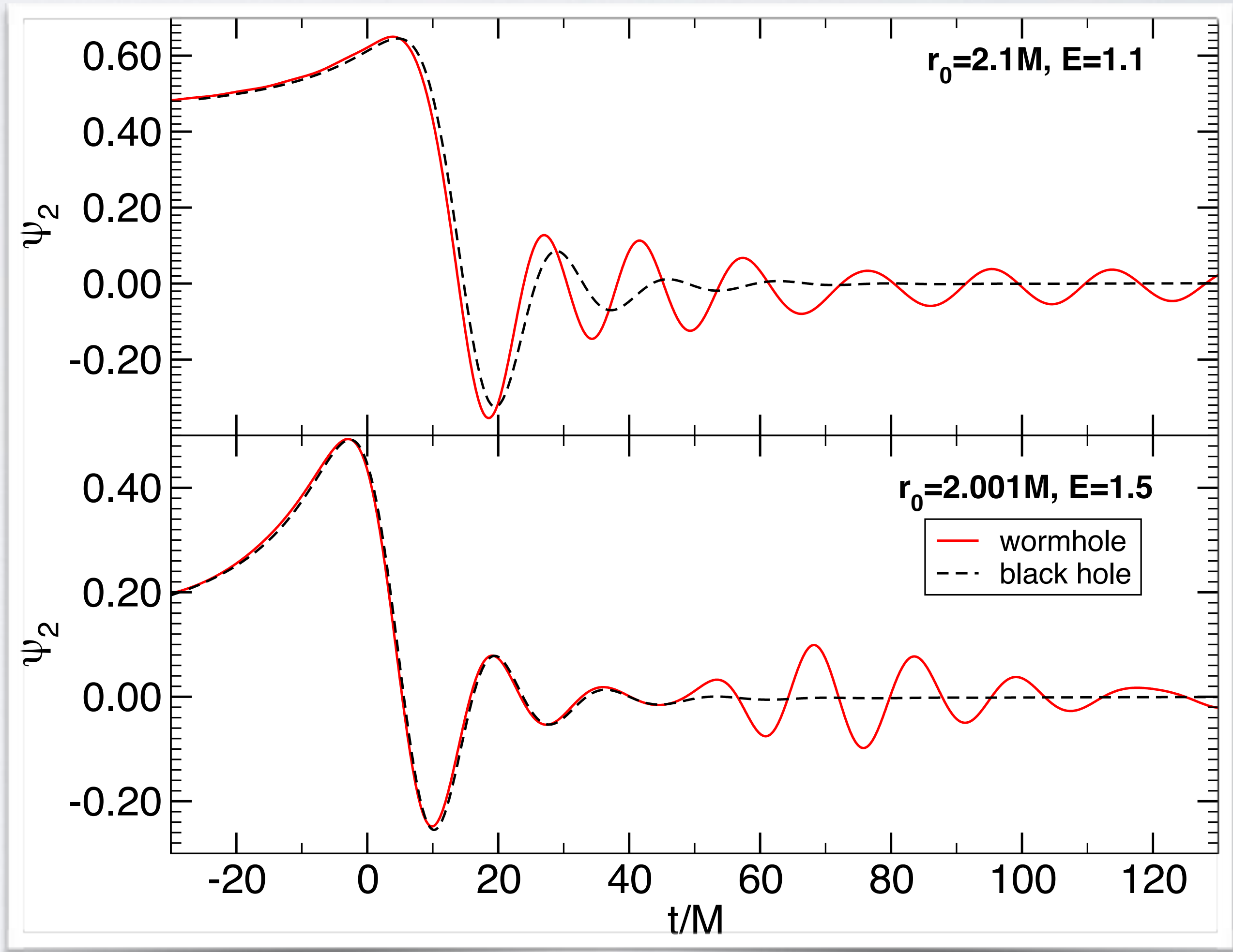


Cardoso et al. PRL 116.171101

A compact object with a light ring will “ring down” even if it has no horizon

Wormhole plunge

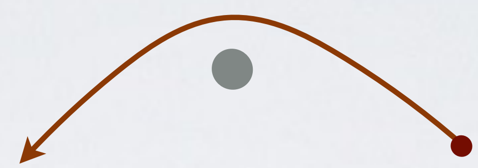


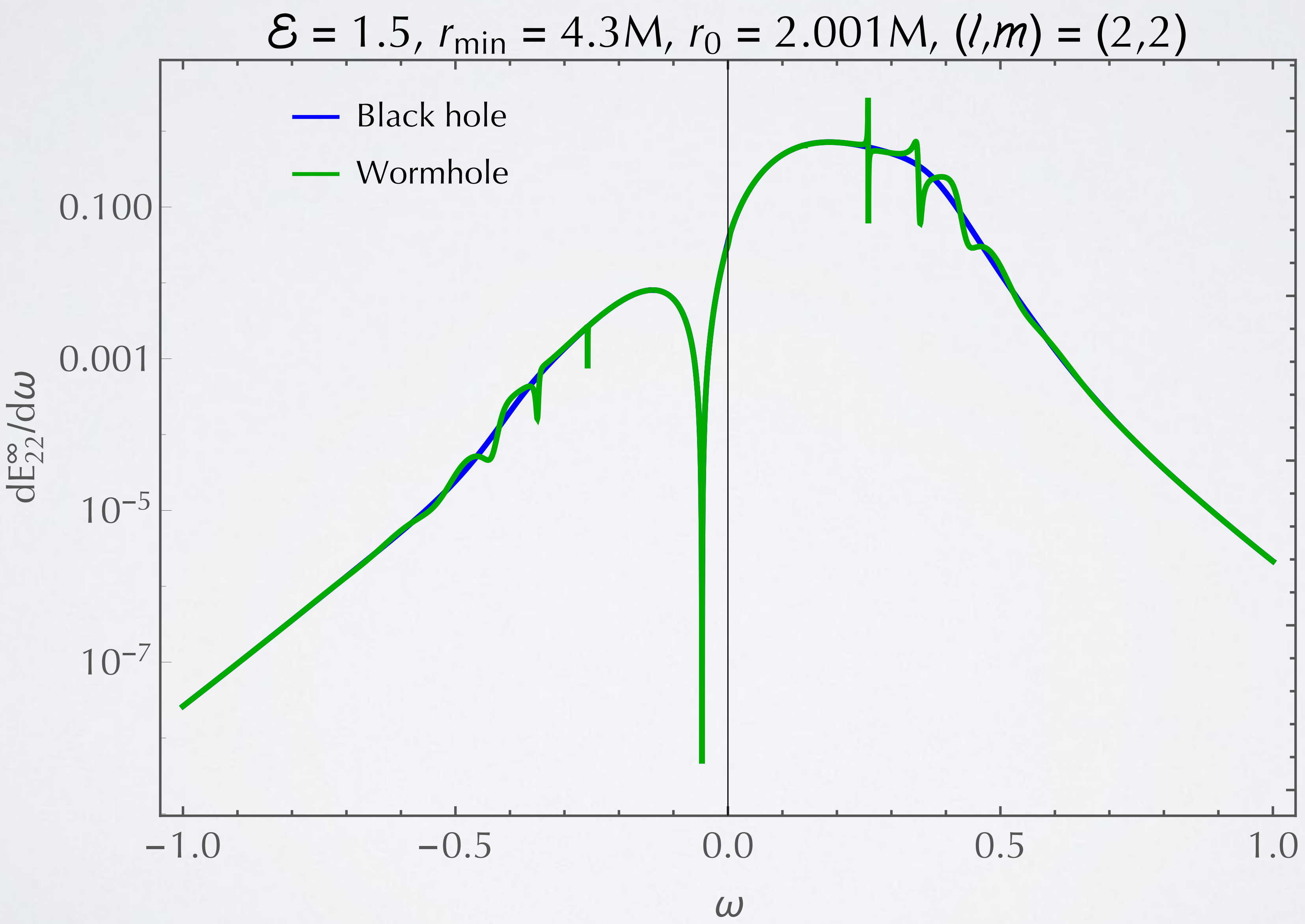


Cardoso et al. PRL 116.171101

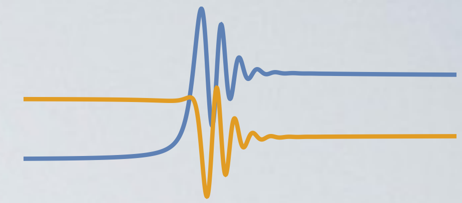
The QNMs of the wormhole are excited by the scatter

Wormhole scatter

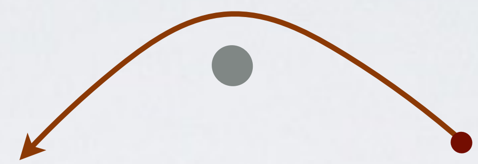




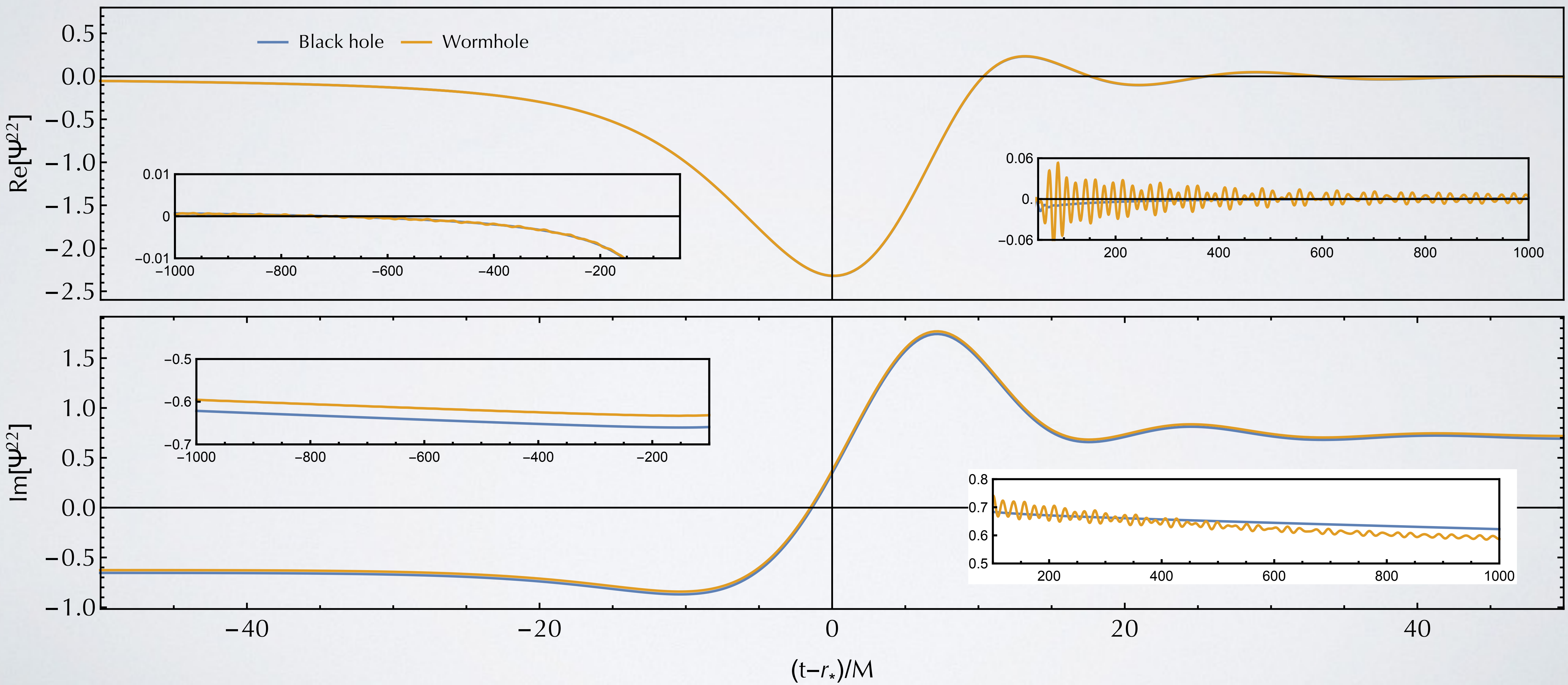
The wormhole rings when being scattered off of, too



Wormhole scatter

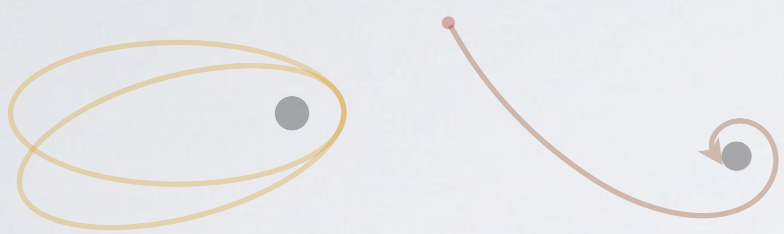


$\mathcal{E} = 1.5$, $r_{min} = 4.3M$, $r_0 = 2.001M$, $(l,m) = (2,2)$

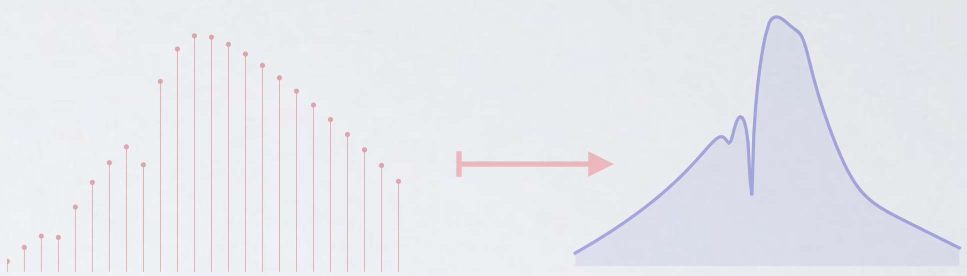


Outline

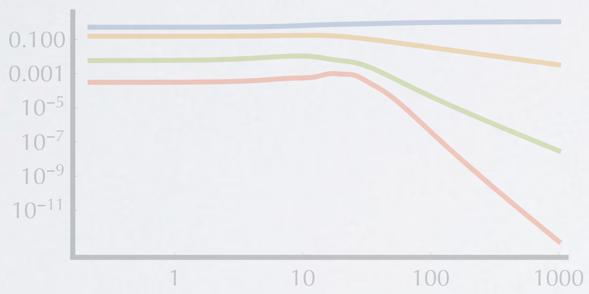
Geodesics, bound and unbound



Fourier series to Fourier transform

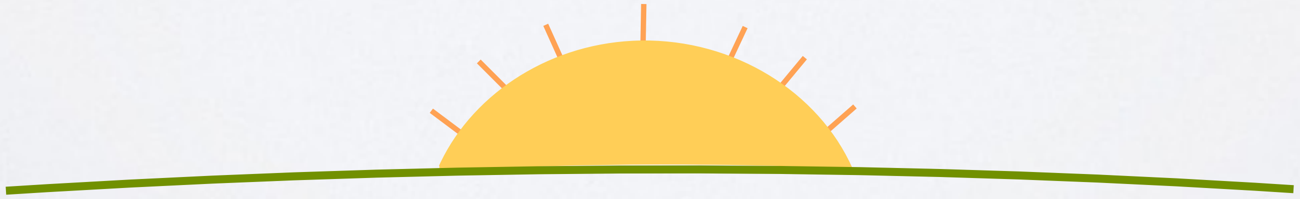


Master functions and source convergence



Results, successes and failures

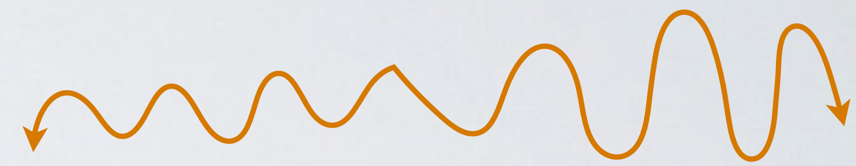
Future prospects and conclusions



Lorenz gauge variables must be differentiated twice to improve source convergence

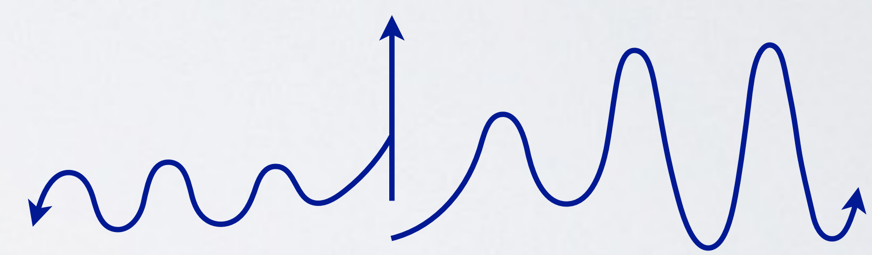
$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}\right) h_i^{(0)}(t,r) + \mathcal{M}_{ij} h_j^{(0)}(t,r) = S_i^{(0)}(t)\,\delta\left(r-r_p\right)$$

$1/r_p^2$



$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}\right) \dot{h}_i^{(0)}(t,r) + \mathcal{M}_{ij} \dot{h}_j^{(0)}(t,r) = \dot{S}_i^{(0)}(t)\,\delta\left(r-r_p\right) - \dot{r}_p S_i^{(0)}(t)\,\delta'\left(r-r_p\right)$$

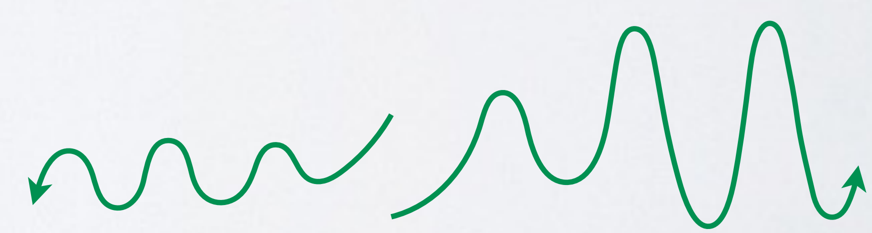
$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}\right) \ddot{h}_i^{(0)} + \mathcal{M}_{ij} \ddot{h}_j^{(0)} = \ddot{S}_i^{(0)}\,\delta\left(r-r_p\right) - 2\dot{r}_p \dot{S}_i^{(0)}\,\delta'\left(r-r_p\right) - \ddot{r}_p S_i^{(0)}\,\delta'\left(r-r_p\right) + \dot{r}_p^2 S_i^{(0)}\,\delta''\left(r-r_p\right)$$



$$h_i^{(2)}(t,r) \equiv \ddot{h}_i^{(0)}(t,r) + \dot{r}_p [\![\dot{h}_i^{(0)}]\!]_p\,\delta\left(r-r_p\right)$$

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}\right) h_i^{(2)}(t,r) + \mathcal{M}_{ij} h_j^{(2)}(t,r) = S_i^{(2)}(t)\,\delta\left(r-r_p\right) + R_i^{(2)}(t)\,\delta'\left(r-r_p\right)$$

$1/r_p^3$



It is not obvious how to generalize this method to the Teukolsky equation

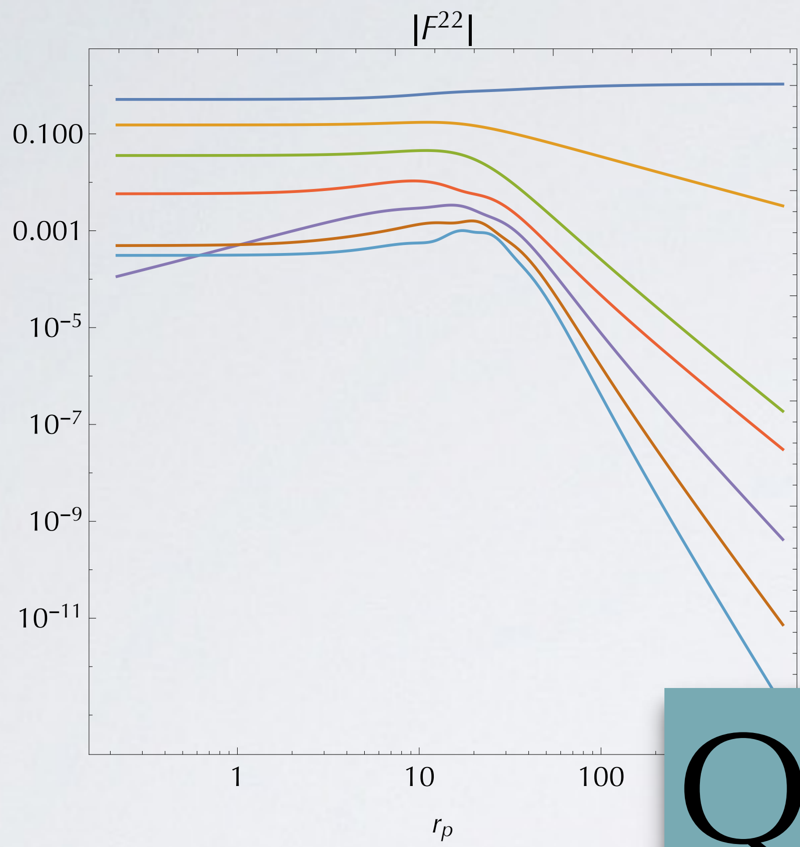
$$\underbrace{\mathcal{W}\left[\psi(t,r,\theta,\varphi)\right] = G(t)\delta^3\left(x^i - x^i_p\right) + F_j(t)\partial_j\delta^3\left(x^i - x^i_p\right) + E_{jk}(t)\partial_k\partial_j\delta^3\left(x^i - x^i_p\right)}_{\text{3+1 Teukolsky equation}} \qquad \{i,j,k\} \in \{r,\theta,\varphi\}$$

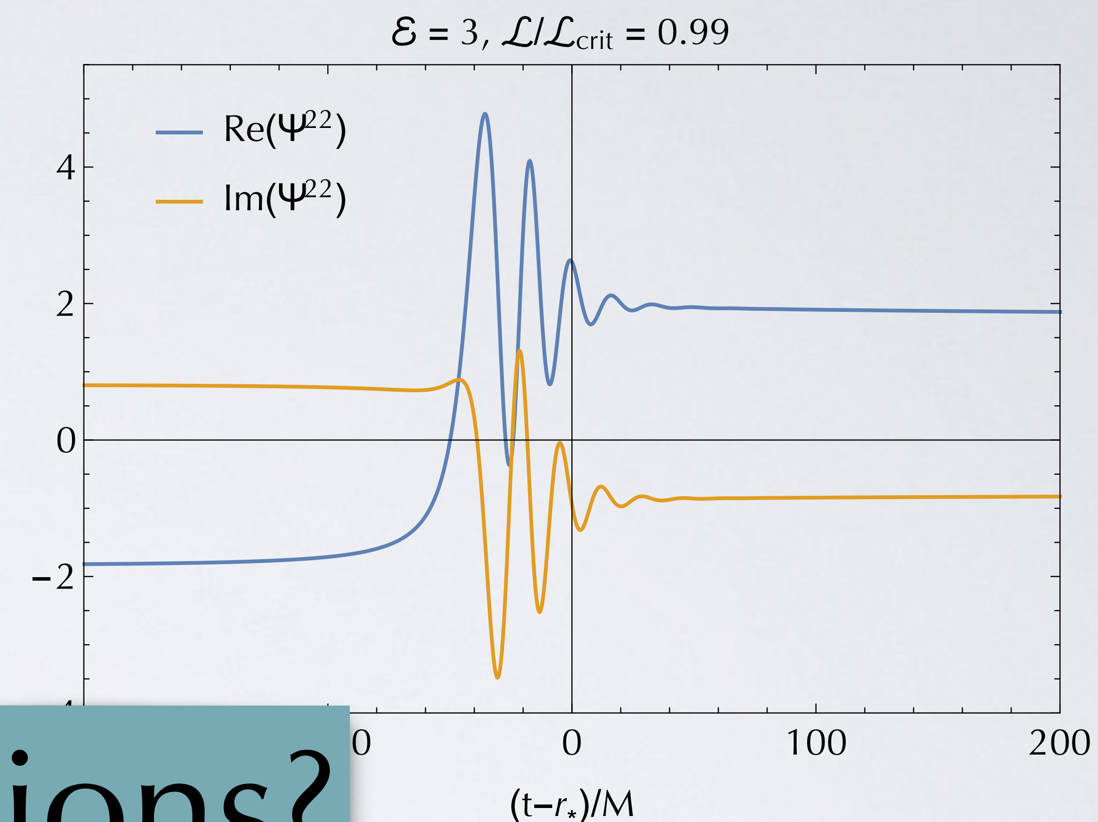
$$\underbrace{\mathcal{W}_m\left[\psi_m(t,r,\theta)\right] = G^m(t)\delta^2\left(x^a - x^a_p\right) + F^m_b(t)\partial_b\delta^2\left(x^a - x^a_p\right) + E^m_{bc}(t)\partial_b\partial_c\delta^2\left(x^a - x^a_p\right)}_{\text{2+1 Teukolsky equation}} \qquad \{a,b,c\} \in \{r,\theta\}$$

$$\underbrace{\cancel{\psi_m(t,r,\theta) = \psi^+_m(t,r,\theta)\theta^2(x^a - x^a_p) + \psi^-_m(t,r,\theta)\theta^2(x^a_p - x^a) + \psi^S_m(t)\delta^2(x^a - x^a_p)}}_{\text{Potential weak form?}}$$

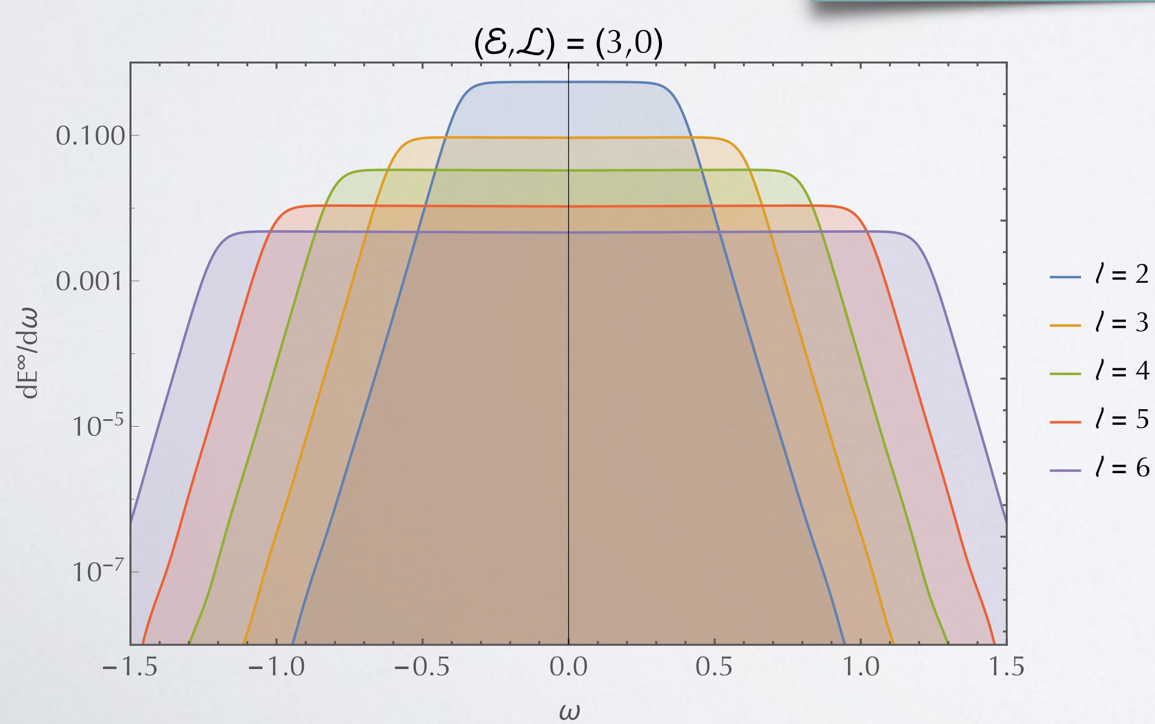
$$\underbrace{\mathcal{W}_{\ell m}\left[\psi_{\ell m}(t,r)\right] = G_{\ell m}(t)\delta\left(r - r_p\right) + F_{\ell m}(t)\delta'\left(r - r_p\right) + E_{\ell m}(t)\delta''\left(r - r_p\right)}_{\text{1+1 Teukolsky equation, with } l \text{ mode coupling?}}$$

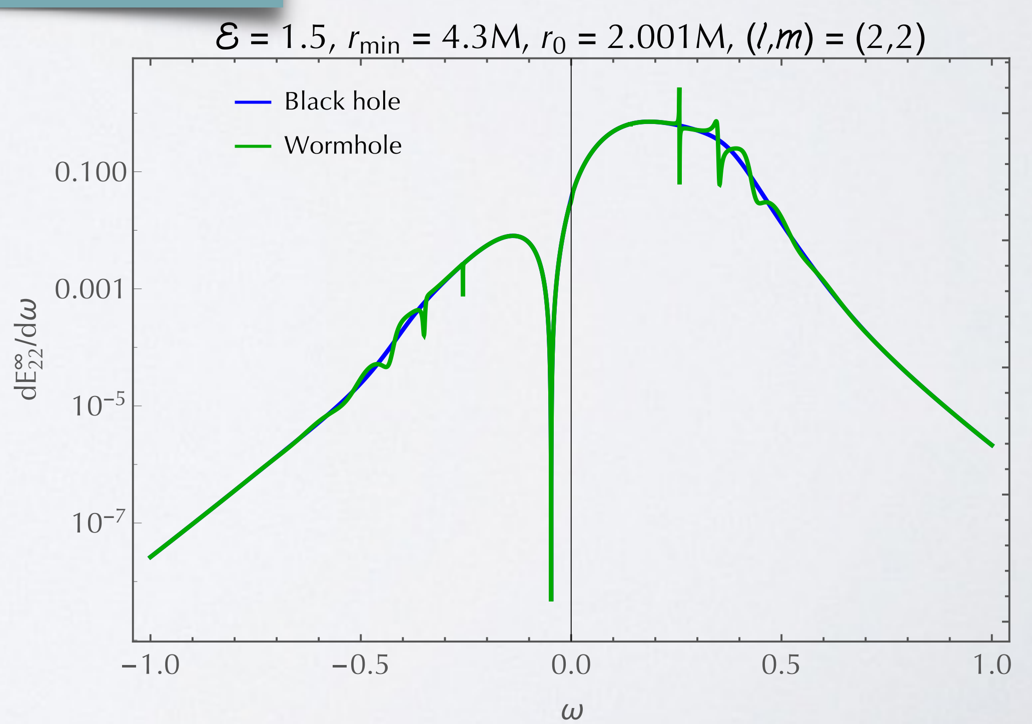
These are the main points



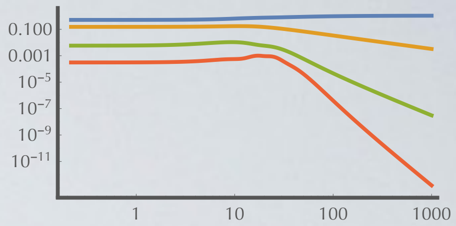


Questions?

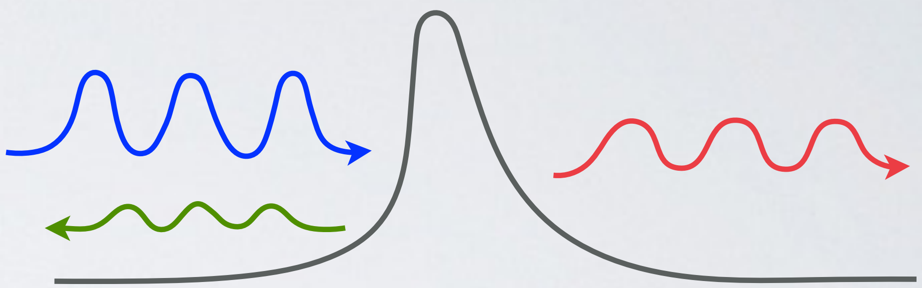




The Zerilli function is the time derivative of the ZM function



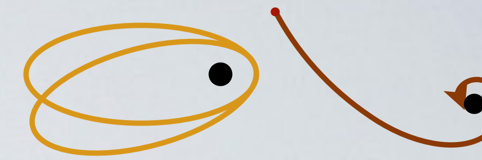
$$\underbrace{\Psi_{\rm ZM}(t,r) = \frac{r}{\lambda+1}\left[K + \frac{1}{\Lambda}\left(f^2 h_{rr} - r f \partial_r K\right)\right]}_{\text{Defined from metric perturbation}}$$

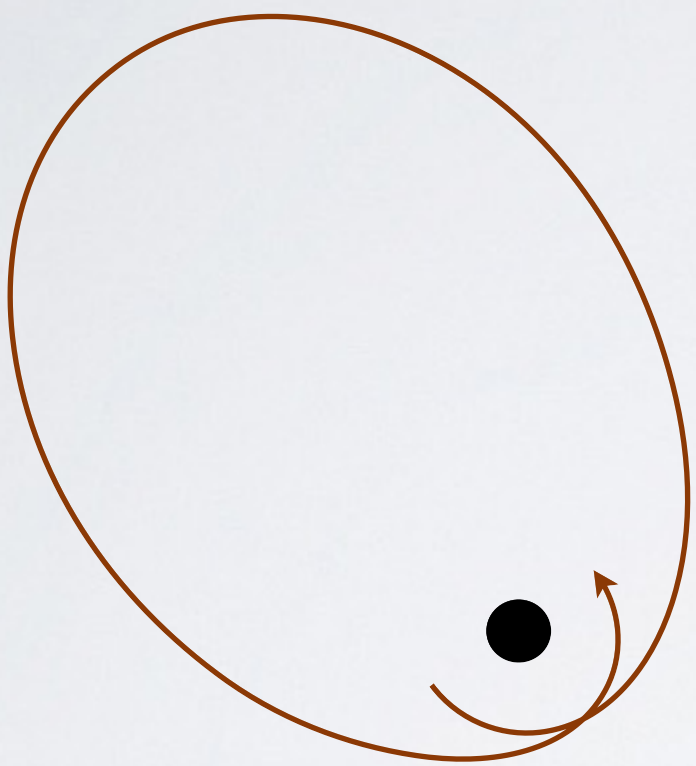


$$\overbrace{-\partial_t\partial_r K - \frac{r-3M}{r^2 f}\partial_t K + \frac{f}{r}\partial_t h_{rr} + \frac{\lambda+1}{r^2} h_{tr} = 0}^{\text{Homogeneous field equation}}$$

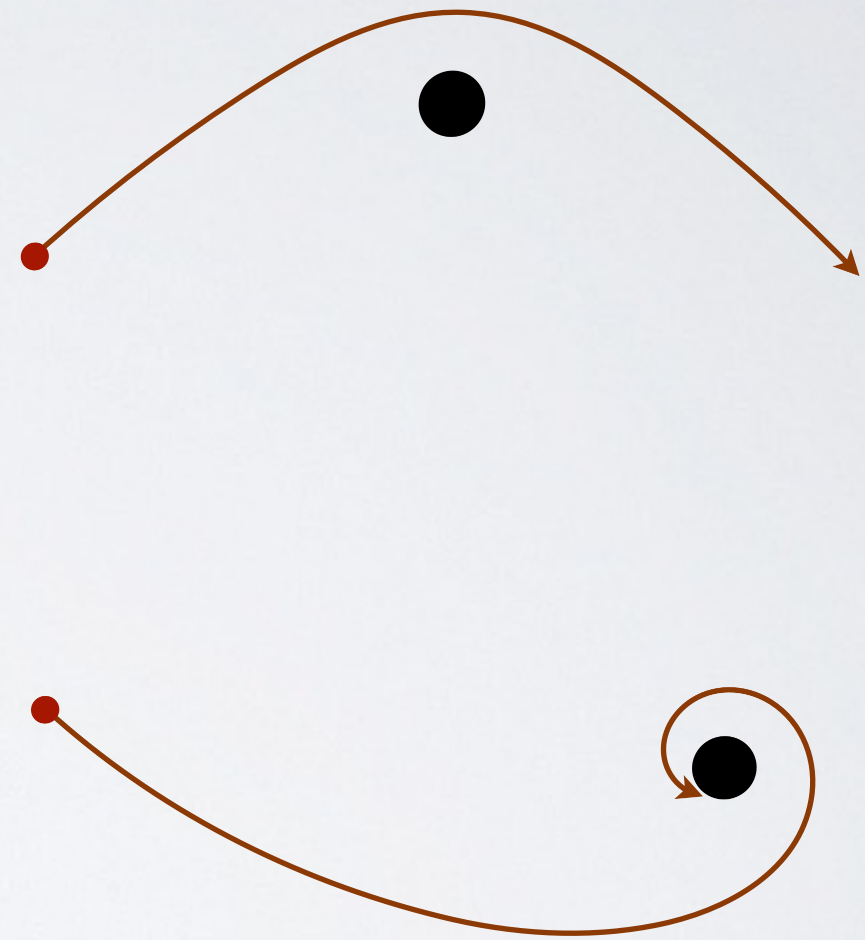
$$\Psi_{\rm Z}(t,r) \equiv \dot{\Psi}_{\rm ZM} = \frac{1}{\Lambda}\left(r\partial_t K - f h_{tr}\right)$$

Eccentric orbits on Schwarzschild have two frequencies, but unbound motion does not





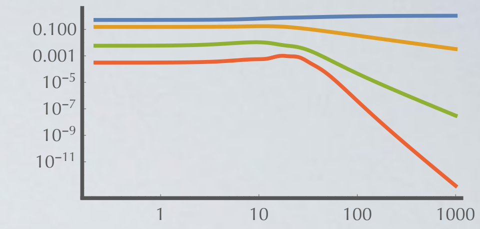
$$\longmapsto$$



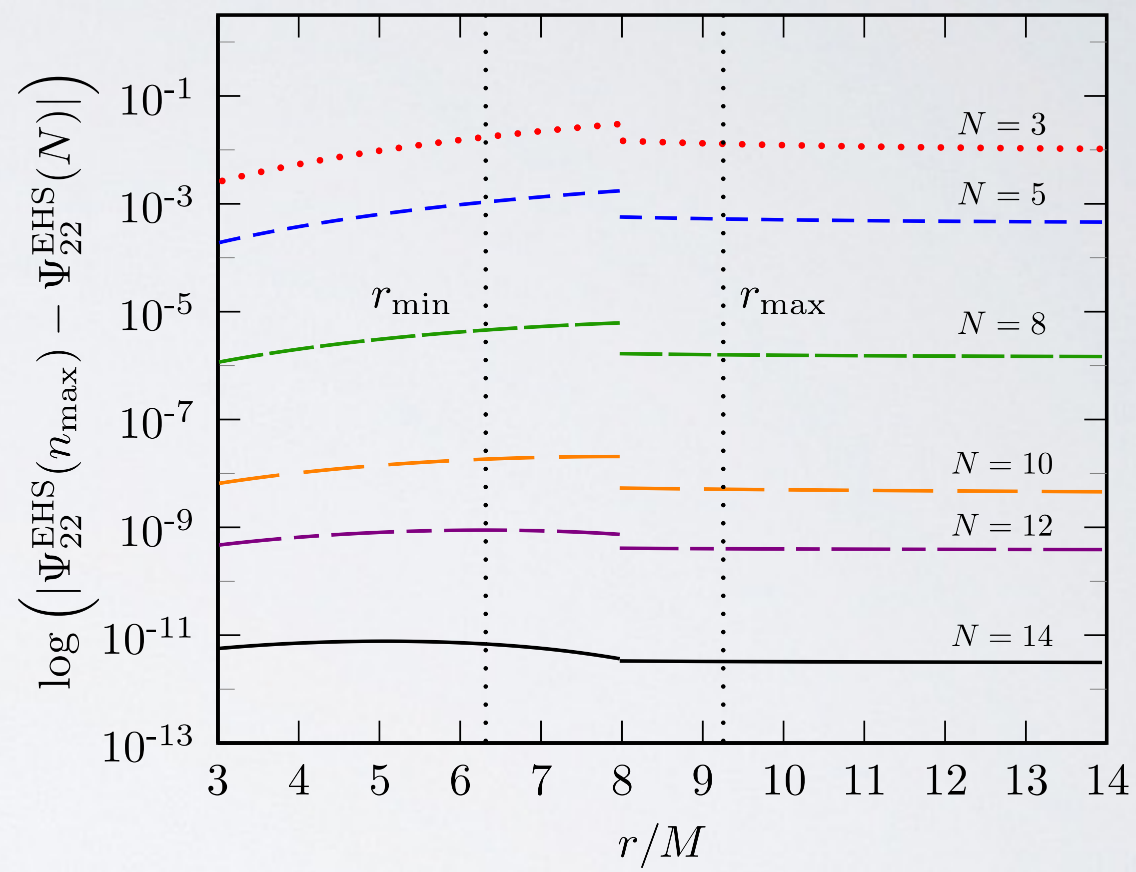
$$r_{\text{min}} \to r_{\text{min}}$$

$$0 \leq t \leq T_r$$

The time domain ZM can be reconstructed from the frequency domain Zerilli modes



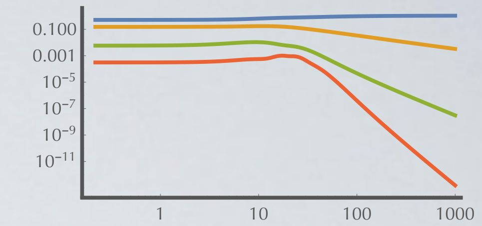
$$\underbrace{\Psi^{\mathrm{ZM},\pm}_{\ell m}(t,r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C^{\mathrm{ZM},\pm}_{\ell m\omega}\hat{X}^{\pm}_{\ell m\omega}(r)e^{-i\omega t}d\omega}_{\text{Time domain EHS}}$$



$$\Psi_{\mathrm{Z}}(t,r) \equiv \dot{\Psi}_{\mathrm{ZM}} - \frac{r_p^2 f_p}{\Lambda_p(\lambda+1)} q_{tr}(t)\delta(r-r_p)$$

$$\underbrace{\Psi^{\mathrm{ZM},\pm}_{\ell m}(t,r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{C^{\mathrm{Z},\pm}_{\ell m\omega}\hat{X}^{\pm}_{\ell m\omega}(r)}{-i\omega}e^{-i\omega t}d\omega}_{\textit{Also}\text{ time domain EHS}}$$

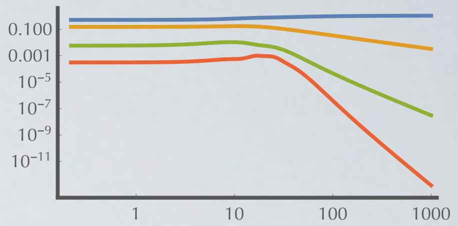
New source terms can be derived from old ones, and they always converge faster

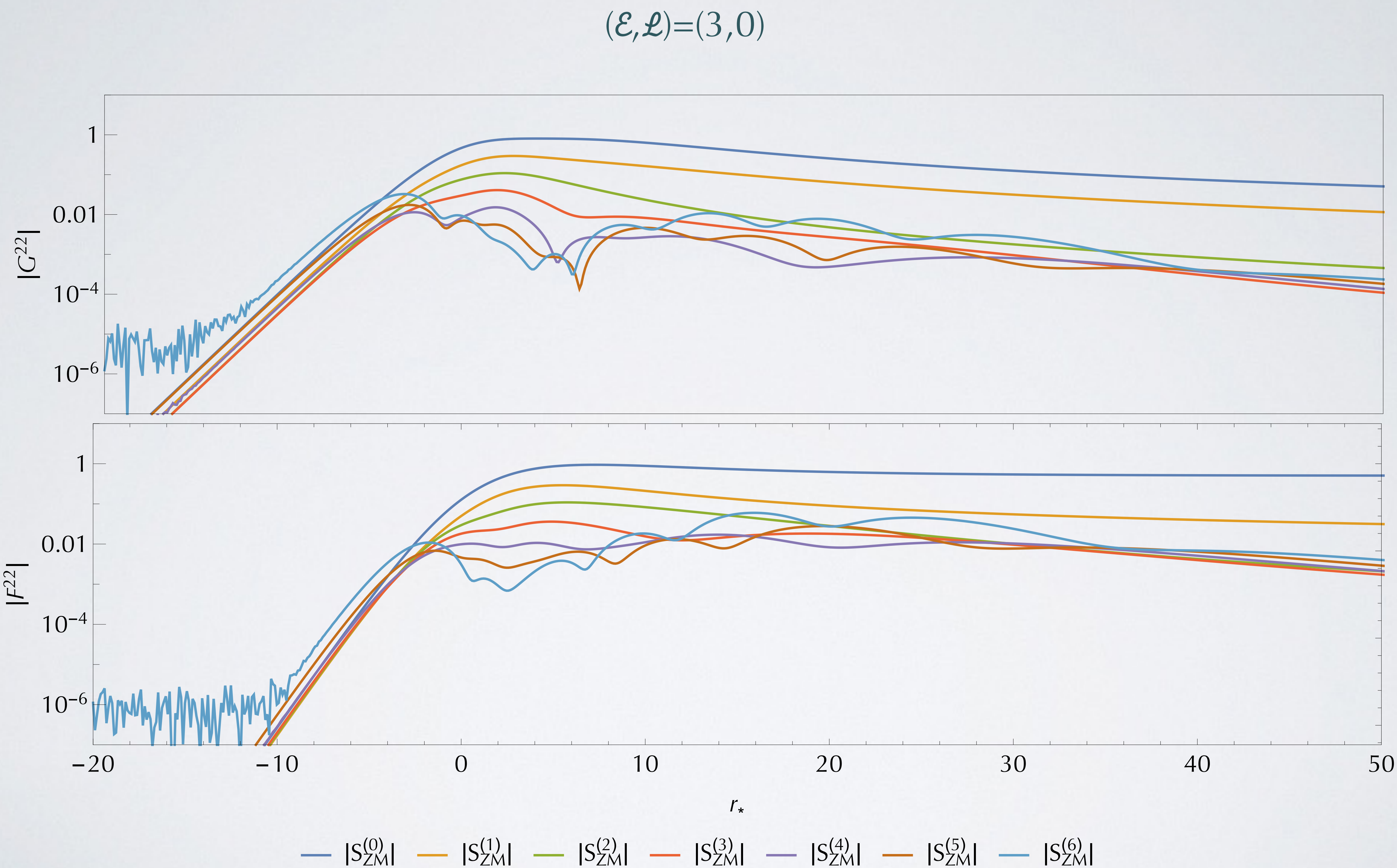


$$
\begin{aligned}
G^{(1)} = {} & \dot{G}^{(0)} + \frac{\mathcal{E}^2 \ddot{F}^{(0)} \dot{r}_p}{f_p^2 U_p^2} - \frac{2\dot{F}^{(0)}}{r_p^7 U_p^4}\Big[10\mathcal{L}^4 M^2 - 7\mathcal{L}^4 M r_p + \left(\mathcal{L}^4 + 16\mathcal{L}^2 M^2\right) r_p^2 + 2\mathcal{L}^2 M \left(4\mathcal{E}^2 - 5\right) r_p^3 \\
& + \left(6M^2 + \mathcal{L}^2\left(1 - 2\mathcal{E}^2\right)\right) r_p^4 + M\left(4\mathcal{E}^2 - 3\right) r_p^5\Big] + \frac{\dot{r}_p \mathcal{E}^2 V_p}{f_p^2 U_p^2} F^{(0)} \\
& - \frac{F^{(0)} \dot{r}_p}{r_p^{11} U_p^6}\Big[20\mathcal{L}^6 M^3 - 30\mathcal{L}^6 M^2 r_p + 4\left(17\mathcal{L}^4 M^3 + 3\mathcal{L}^6 M\right) r_p^2 - \mathcal{L}^4\left(\mathcal{L}^2 + 2M^2\left(10\mathcal{E}^2 + 51\right)\right) r_p^3 \\
& + \left(60\mathcal{L}^2 M^3 + 2\mathcal{L}^4 M\left(4\mathcal{E}^2 + 21\right)\right) r_p^4 - 2\mathcal{L}^2\left(M^2\left(45 - 12\mathcal{E}^2\right) + \mathcal{L}^2\left(\mathcal{E}^2 + 2\right)\right) r_p^5 \\
& + 12M\left(M^2 - 3\mathcal{L}^2\left(\mathcal{E}^2 - 1\right)\right) r_p^6 + 3\left(M^2\left(4\mathcal{E}^2 - 6\right) + \mathcal{L}^2\left(2\mathcal{E}^2 - 1\right)\right) r_p^7 + 6M\left(1 - 2\mathcal{E}^2\right) r_p^8\Big]
\end{aligned}
$$

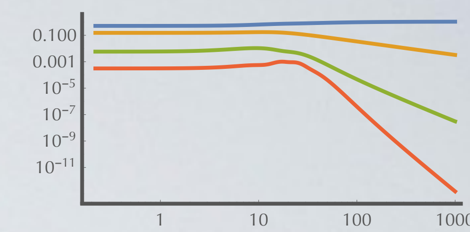
$$
\begin{aligned}
F^{(1)} = {} & \dot{F}^{(0)} - \dot{r}_p G^{(0)} + \frac{\dot{F}^{(0)}}{U_p^2 r_p^3}\Big[4\mathcal{L}^2 M - 2\mathcal{L}^2 r_p + 4M r_p^2 + \left(2\mathcal{E}^2 - 2\right) r_p^3\Big] \\
& + \frac{F^{(0)} \dot{r}_p}{r_p^7 U_p^4}\Big[-10\mathcal{L}^4 M^2 + 7\mathcal{L}^4 M r_p - \left(\mathcal{L}^4 + 16\mathcal{L}^2 M^2\right) r_p^2 \\
& + 10\mathcal{L}^2 M\left(1 - 2\mathcal{E}^2\right) r_p^3 - \left(6M^2 + \mathcal{L}^2\left(1 - 4\mathcal{E}^2\right)\right) r_p^4 + 3M\left(1 - 4\mathcal{E}^2\right) r_p^5\Big]
\end{aligned}
$$

Time derivatives introduce more structure near the black hole





The Jhingan-Tanaka function is an even-parity master function that uses the RW potential



$$\underbrace{\mathcal{T}^{\pm} \equiv \left[\lambda(\lambda+1) + \frac{9M^2}{r^2}\frac{f}{\Lambda}\right] \pm 3Mf\partial_r}_{\text{Chandrasekhar transformation}}$$

$$\mathcal{T}^{-}\Psi^{\rm ZM}_{\ell m}(t,r) = \underbrace{\Psi^{\rm JT}_{\ell m}(t,r) \equiv r\lambda K^{\ell m} - r^2 f\partial_r K^{\ell m} + rf^2 h^{\ell m}_{rr}}_{\text{Jhingan-Tanaka function}}$$

$$\left[-\partial_t^2 + \partial_{r_*}^2 + \underbrace{\frac{f}{r^2}\left(2(\lambda+1) - \frac{6M}{r}\right)}_{\text{Regge-Wheeler potential}}\right]\Psi^{\rm JT}_{\ell m}(t,r) = S^{\rm JT}_{\ell m}(t,r)$$

$$\begin{aligned} S^{\rm JT}(t,r) = {} & \frac{2f^2(r + r\lambda - 3M)}{r}Q_r + f^2(2M + r\lambda - r)Q_{rr} + (2M + r\lambda - r)Q_{tt} + rf^2 Q^{\flat} \\ & - \frac{f(-18M^2 + 6Mr + r^2\lambda(\lambda+1))}{r^3}Q^{\sharp} + r^2 f f_p^2 \partial_r Q_{rr} + \frac{r^2 f^3}{f_p^2}\partial_r Q_{tt} + \frac{3Mf^2}{r}\partial_r Q^{\sharp} \end{aligned}$$