Hamiltonian formulation of self-forced motion in Kerr: conservative dynamics

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Based on works with Alex. Le Tiec and Kyoto team

(Ryuichi Fujita, Hiroyuki Nakano, Nori Sago, Takahiro Tanaka and Kei Yamada)

Motivating Question

Given a **gauge-dependent** self force (GSF), how can one use it to learn **"observables"** that describe post-geodesic corrections to the orbital dynamics in Kerr?

"Direct integration of the MiSaTaQuWa equation: Possible in principle, but, ***not efficient in practice.****"*

Yasushi Mino (**2004**)

Radiation-reaction formula in Kerr

The four **first integrals *P*** of Kerr geodesic are

$$E = -u^\alpha \xi^{(t)}_\alpha \qquad L_z = u^\alpha \xi^{(\phi)}_\alpha \qquad \mu^2 = -g^{(0)}_{\alpha\beta} u^\alpha u^\beta \qquad Q = K_{\alpha\beta} u^\alpha u^\beta$$

Carter ('68)

The dissipative part of the first-order GSF is related with their time-averaged change rates.

$$\left\langle \frac{\mathrm{d}P_\alpha}{\mathrm{d}\tau} \right\rangle = \left\langle \frac{\partial P_\alpha}{\partial u_\beta} f^\beta [h^{\mathrm{diss.}}] \right\rangle$$

✔ This formula is **gauge-invariant**.

✔ Far easy to implement as its r.h.s. are related to gravitational-wave fluxes at infinity and the horizon

Mino, arXiv:0302075

Drasco, Hughes and Flanagan, arXiv:0505075

Sago, Tanaka, Hikida and Nakano, arXiv:0506092

Cute ideas! But...

The radiation reaction formula does not capture
the conservative effect of the first-order GSF in Kerr.

- ✔ Detweiler's redshift variable
- ✔ ISCO shift
- ✔ Orbital precession
- ✔ Tidal invariants ...

Where are they?

Mino's approach in modern view

Take the 4-dim affine-parametrized geodesic Hamiltonian

$$H^{(0)}(x,\ u) = \frac{1}{2} g^{\mu\nu}_{(0)} u_\mu u_\nu$$

After a canonical transformation from ***(x, u)*** *to* ***(X, P)***, Hamilton's equations read

$$\frac{\mathrm{d}P_\alpha}{\mathrm{d}\tau} = 0 \qquad \frac{\mathrm{d}X^\alpha}{\mathrm{d}\tau} = \nu^\alpha$$

where

$$\nu^\alpha = (1,\ 0,\ 0,\ 0)$$

Schmidt, arXiv:0202090

✔ The radiation reaction formula is relevant to the first equation.

✔ The conservative GSF is relevant to the second equation.

c.f. Mino, arXiv:0506003

So, our answer is to…

Develop the 4-dim Hamiltonian formulation of the MiSaTaQuWa equation, and use it.

Today: focus on only the **conservative dynamics** ("turning off" dissipation) and its applications.

- ✔ ISCO formula with inclination.
- ✔ "A first law" of binary mechanics on generic orbits.

Perturbed Hamiltonian in Kerr

Given a **fixed perturbed metric**, the perturbed motion is described by the 4-dim affine-parametrized Hamiltonian

$$H[x,\, u\,;\gamma] := \frac{1}{2} g^{\mu\nu}_{(0)}(x) u_\mu u_\nu - \underbrace{\frac{1}{2} h^{\mu\nu}_{(R)}[x\,;\gamma] u_\mu u_\nu}_{\text{Interaction Hamiltonian}} + \mathcal{O}(\mu^2)$$

where

$$H[\gamma]|_\gamma = -\frac{1}{2}$$

The fixed source orbit:
an "osculating" Kerr geodesic

Detweiler and Whiting, arXiv:0202086

Pound, arXiv:1506.06245

4-dim action-angle variables

We use **the action-angle variables** for bound geodesics in Kerr as the reference canonical coordinates.

✔ Angle variables $w^{\alpha} := w^{\alpha}(x, u)$

Measure the phase of motion in the temporal/spatial directions.

✔ Action variables $J_{\alpha} := J_{\alpha}(x, u)$

The one-to-one function of the first integrals of motion ***P***

Hinderer and Flanagan, arXiv:0805.3337

Perturbed Hamilton's equations

$$\dot{J}_\alpha = -\left(\frac{\partial H^{(1)}[\gamma]}{\partial w^\alpha}\right)_J \qquad \dot{w}^\alpha = \omega^\alpha_{(0)}(J) + \left(\frac{\partial H^{(1)}[\gamma]}{\partial J_\alpha}\right)_w$$

Derivative w.r.t. the proper time in the perturbed metric

$H^{(1)}[w, J, \gamma]$: the interaction Hamiltonian

Since $\langle \dot{J}_\alpha \rangle = 0$, the orbital average

$$\omega^\alpha := \left\langle \omega^\alpha_{(0)}(J) \right\rangle + \left\langle \frac{\partial H^{(1)}[\gamma]}{\partial J_\alpha} \right\rangle \qquad z := (\omega^t)^{-1}$$

implies the 4-dim frequencies of the perturbed orbit.

(except the resonant orbits.)

Gauge and motion

Perturbed motion is gauge-dependent. Pound, arXiv:1506.02894

For an infinitesimal gauge transformation $\hat{\delta}_\xi x^\mu = \xi^\mu + \mathcal{O}(\eta^2)$

$$\hat{\delta}_\xi J_\alpha = -\frac{\partial \Xi}{\partial w^\alpha} \qquad \hat{\delta}_\xi w^\alpha = \frac{\partial \Xi}{\partial J_\alpha} \qquad \hat{\delta}_\xi H^{(1)}[\gamma]\Big|_\gamma = \dot{\Xi}$$

where
$\Xi := u_\mu \xi^\mu$

c.f. Vine and Flanagan, arXiv:1503.04727

In our formulation, only the perturbed frequencies and the averaged change rate of ***J*** are gauge-invariant.

$$\hat{\delta}_\xi \omega^\alpha = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}\tau \left(\hat{\delta}_\xi \dot{w}^\alpha\right) = \lim_{T\to\infty} \frac{\hat{\delta}_\xi w^\alpha(T) - \hat{\delta}_\xi w^\alpha(-T)}{2T} = 0$$

Gauge (canonical) transformation

Using the Green function and Fourier series expansion,

$$H^{(1)}[\gamma] \approx \int_{\gamma_{(0)}} d\tau' u^\mu u^\nu \, G^{R,\mathrm{sym}}_{\mu\nu\alpha'\beta'}(x, x')\, u^{\alpha'} u^{\beta'}$$

$$\approx \sum_{\mathbf{n},\mathbf{n}'} \mathcal{G}_{\mathbf{n},\mathbf{n}'}(J\,;J^{(\gamma)})\, e^{-i\varpi_{\mathbf{n}'}(J^{(\gamma)}) w^t + i(\mathbf{n}\cdot\mathbf{w})}$$

Symmetric under $J_\alpha \leftrightarrow J^{(\gamma)}_\alpha$

All the periodically varying terms can be always eliminated by a gauge transformation.

Due to the symmetry, the averaged term depends only on ***J*.**

$$\mathcal{H}_{\mathrm{int}}(J)|_\gamma := \left\langle H^{(1)}[\gamma] \right\rangle = \langle h_{uu}[\gamma] \rangle$$

Local particle Hamiltonian

For some gauge-fixed ***J***, one can always replace the source-dependent Hamiltonian to the "averaged" local Hamiltonian.

$$\mathcal{H}(J) := H^{(0)}(J) + \frac{1}{2}\mathcal{H}_{\rm int}(J)$$

Avoid the double count of "source" and "field" contributions.

The Hamilton's equations

$$\dot{J}_\alpha = -\frac{\partial \mathcal{H}(J)}{\partial w^\alpha} = 0 \qquad \omega^\alpha = \dot{w}^\alpha = \frac{\partial \mathcal{H}(J)}{\partial J_\alpha}$$

describe gauge-invariant parts of the perturbed frequencies.

Innermost Stable Spherical Orbit

Using action-angle form, the Innermost Stable Spherical Orbit ("circular inclined orbit"): **ISSO** is defined by

$$\omega^r_{\mathrm{ISSO}} = \left(\frac{\partial \mathcal{H}}{\partial J_r}\right)\Big|_{J_r=0} = 0$$

"Constant-radius" spherical orbit

Wilkins ('72)

Changing variables $J_\alpha \to \Omega^\alpha := \omega^\alpha/\omega^t$, the ISSO is given by

$$\det \frac{\partial^2 \tilde{z}(\Omega)}{\partial \Omega^a \partial \Omega^b}\Big|_{\mathrm{ISSO}} = 0$$

$$a = (\theta, \phi)$$

where

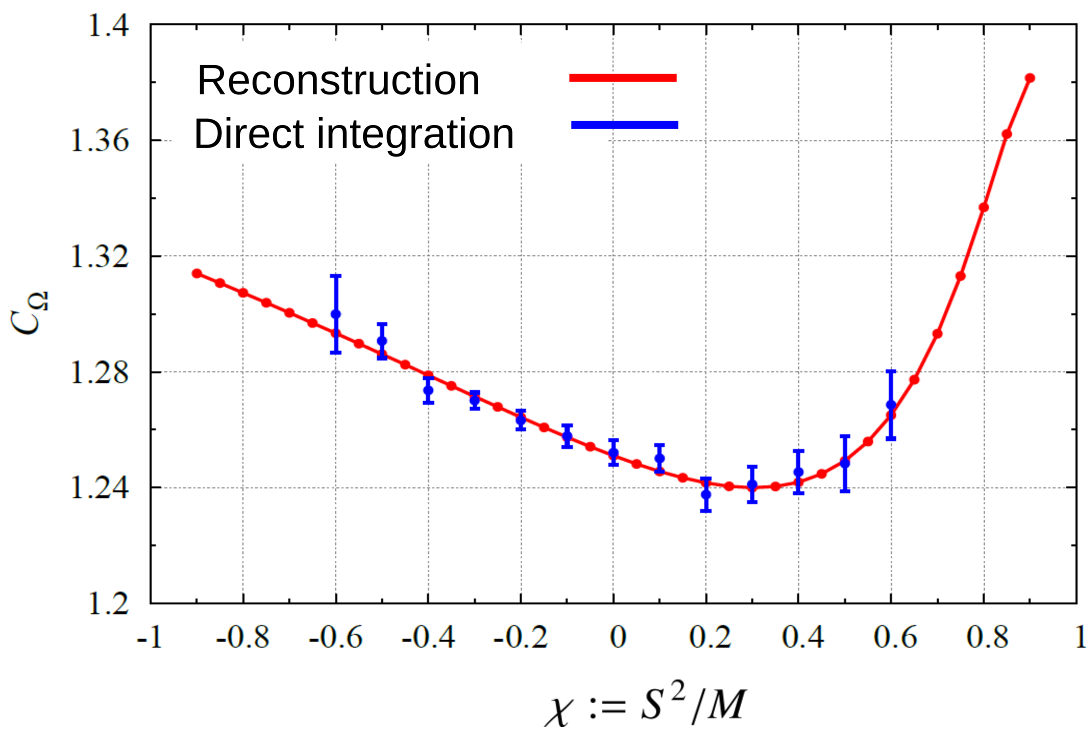
$$\tilde{z} := z^{(0)} + \frac{\eta}{2} z^{(1)} + \mathcal{O}(\eta^2)$$

~ the redshift variable

ISCO Frequency shift vs Spin

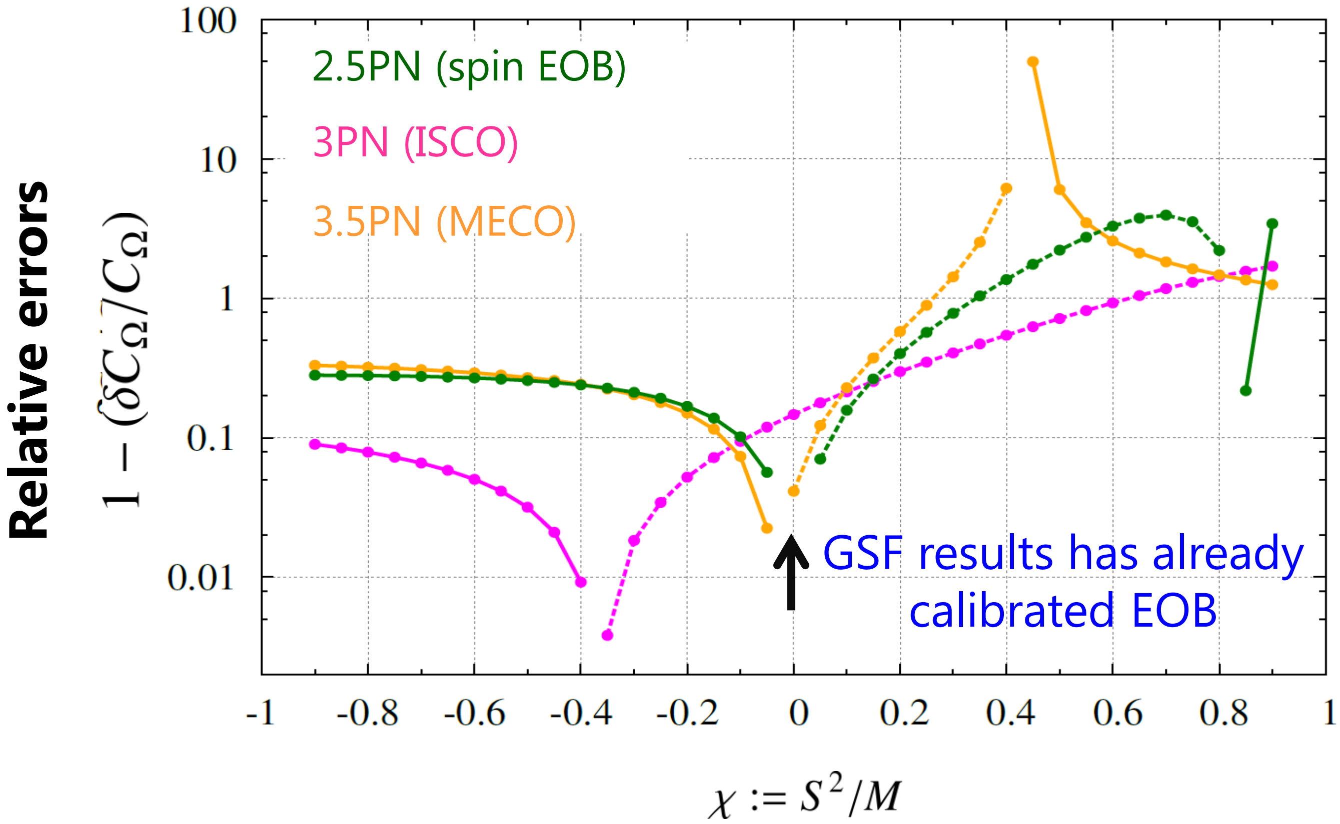
Parametrization: $(M+\mu)\Omega_{\rm ISCO} = M\Omega^{(0)}_{\rm ISCO}(\chi)\left(1+\eta\, C_\Omega(\chi)+O(\eta^2)\right)$

Equatorial limit of the ISSO condition: $(\partial^2\tilde{z}/\partial\Omega^{\phi\,2})|_{\rm ISCO} = 0$



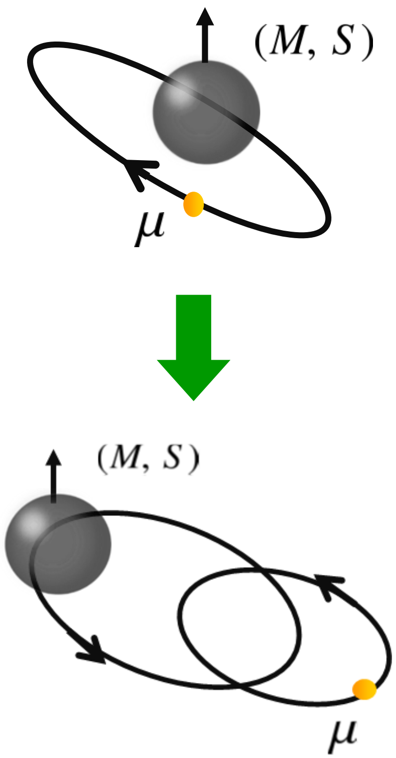
S.I. + Barack, Dolan, Le Tiec, Nakano, Shah, Tanaka and Warburton, arXiv:1404.6133

Benchmark results



Given an analytic model of spinning binaries, one can judge whether it is good to describe the strong-field dynamics.

“A” first law of binary mechanics



The GSF dynamics in Kerr background is the large mass-ratio limit of binaries.

Strictly speaking, our Hamiltonian is a function of $\boldsymbol{\mu}$**,** $\boldsymbol{M}$ and $\boldsymbol{S}$, too.

$$\mathcal{H} = \mathcal{H}(J;\, \mu,\, M,\, S)$$

Particle Hamiltonian first law

Variation of Hamiltonian ***δH*** and Hamilton's equation read

$$\begin{cases} \delta(M + \tilde{E}) - \Omega^i \, \delta\tilde{J}_i = z \, \delta\mu + z_{\rm BH} \, \delta M + \Omega_{\rm BH} \, \delta S \; + \mathcal{O}(\mu^3) \\ (M + \tilde{E}) - 2\Omega^i \, \tilde{J}_i = z \, \mu + z_{\rm BH} \, M + 2\Omega_{\rm BH} \, S \end{cases}$$

c.f. Le Tiec, arXiv:1311.3836

where

$$z_{\rm BH} := 1 + \frac{\tilde{z}}{\mu^2} \left(\frac{\partial \mathcal{H}}{\partial M} \right)_{(J,\mu,S)}$$

"Redshift variable" of Kerr

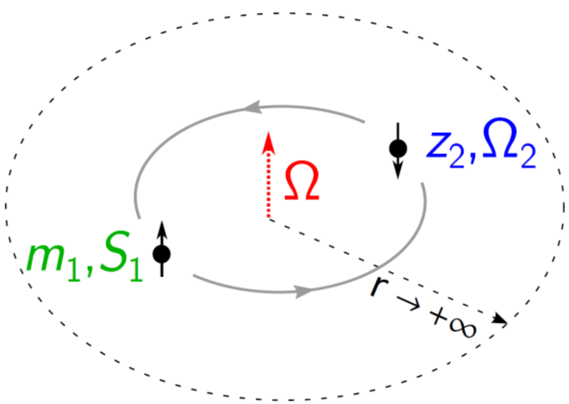
$$\tilde{J}_\alpha := \mu J_\alpha \left(1 - \frac{1}{2} \mathcal{H}_{\rm int} \right) \qquad \tilde{E} := -\tilde{J}_t$$

These relations are valid for generic strong-field orbits in Kerr, but are established only "along the orbit" as we use local GSF.

Two-body Hamiltonian first law

In the post-Newtonian (PN) theory, the first laws of binary BH mechanics are established, using the 2-body Hamiltonian.

$$(M_{\mathrm{ADM}},\, L_{\mathrm{ADM}})$$



$$H_{\mathrm{ADM}}(\mathbf{r},\, \mathbf{p}\,; m_a,\, \mathbf{S}_a)$$

$$\delta M_{\mathrm{ADM}} - \Omega \delta L_{\mathrm{ADM}} = \sum_a (z_a \delta m_a + \Omega_a \delta S_a)$$

$$M_{\mathrm{ADM}} - 2\Omega L_{\mathrm{ADM}} = \sum_a (z_a m_a + 2\Omega_a S_a)$$

M **and** ***L*** **are the global ADM quantities**

Le Tiec, Blanchet and Whiting, arXiv:1111.5378

Blanchet, Buonanno and Le Tiec, arXiv:1211.1060

Le Tiec, arXiv:1506.05648

Synergy with the PN theory

Both first laws are **formally identical if**

✔ the effective metric is that of Detweiler and Whiting.

✔ objects' physical parameters are identical in each theory.

Conjecture:

Given fixed frequencies, our gauge choice for local J implies

$$(M + \tilde{E},\, \tilde{J}_\phi) \leftrightarrow (M_{\rm Bondi},\, L_{\rm Bondi})$$

for generic perturbed orbits in Kerr.

Supporting evidences: Le Tiec, Barausse and Buonanno, arXiv:1111.5609

Gralla and Le Tiec, arXiv:1210.8444

Summary

Hamiltonian formulation is very efficient
for devising observables due to GSF in Kerr.

Things we've learned:

Local observables: the orbital-averaged frequencies and the redshift variable.

Global observables: likely the total energy and the angular momentum of the binary system.

The choice of gauge would be important.

Future directions

Formalism:

Including the dissipative GSF to model inspirals

Proof the conjectures on the particle Hamiltonian first law

What is the action for our Hamiltonian?

Applications:

Implement the ISSO formula.

(Re)derive other observables; Periastron advance?

Synergy with PN/NR/EOB via the first law; what can we do?

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(Merci beaucoup.)