Entropy in classical mechanics, general relativity, and the gravitational two-body problem

Marius Oltean[1,2,3,4],

Luca Bonetti[1,2], Alessandro D.A.M. Spallicci[1,2] and Carlos F. Sopuerta[4]

[1]*Observatoire des Sciences de l'Univers en région Centre, Université d'Orléans, France*

[2]*Laboratoire de Physique et Chimie de l'Environnement et de l'Espace, CNRS, France*

[3]*Departament de Física, Facultat de Ciències, Universitat Autònoma de Barcelona, Spain*

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UNIVERSITE D'ORLEANS

CAMPUS FRANCE

UAB Universitat Autònoma de Barcelona

CSIC Consejo Superior de Investigaciones Científicas

IEEC

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30 June 2016 — Meudon, France

Motivation

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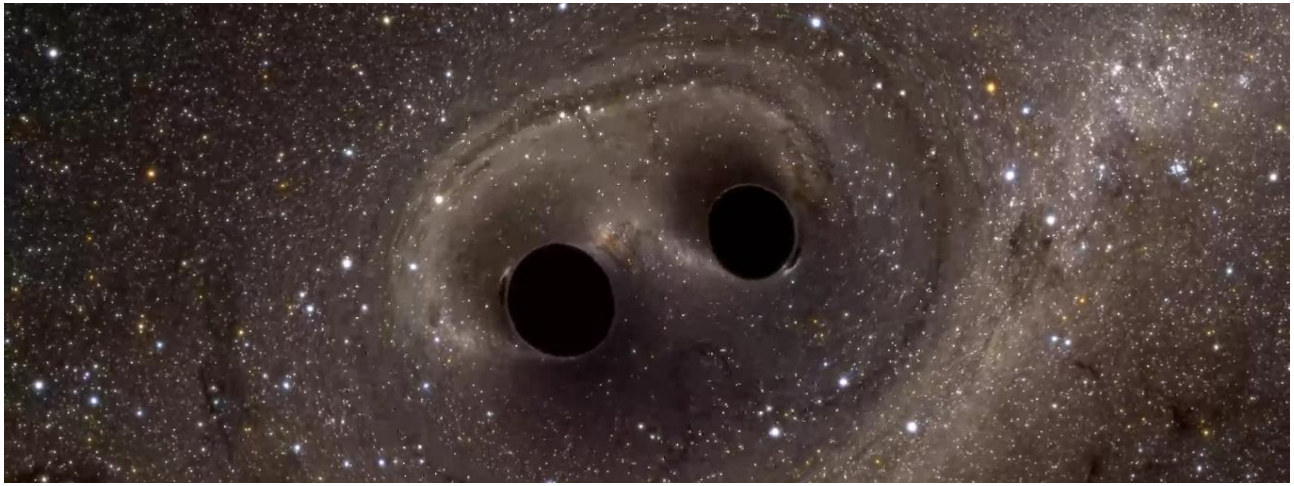

[simulation of GW150914, *Simulating eXtreme Spacetimes* project]

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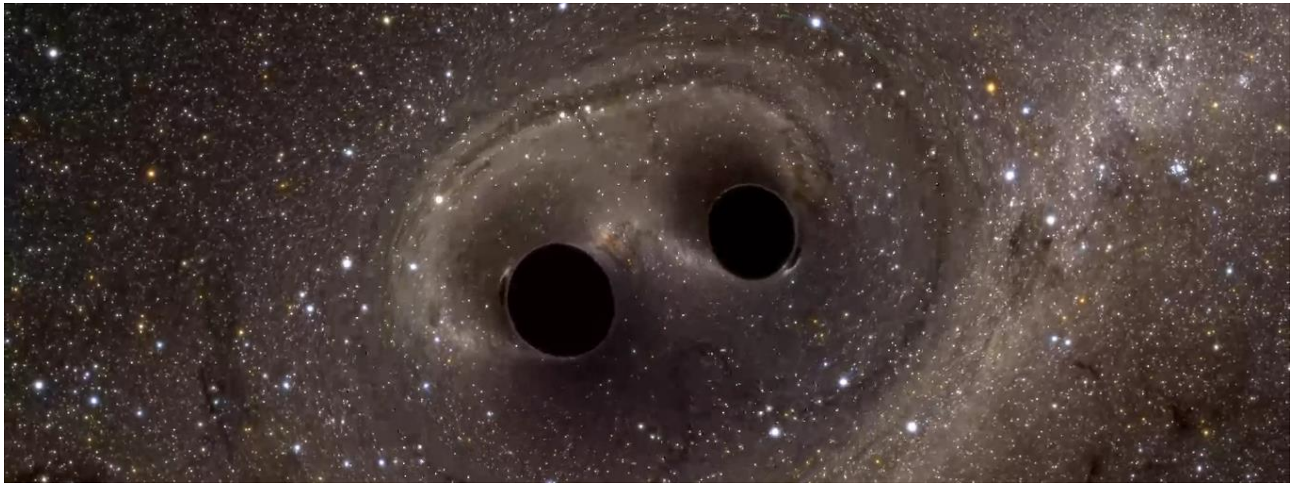

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- In the GR two-body problem, we expect entropy increase from GW emission.

- But what is "entropy" in GR? How do we compute it in general?
And how/why should it obey the second law of thermodynamics?

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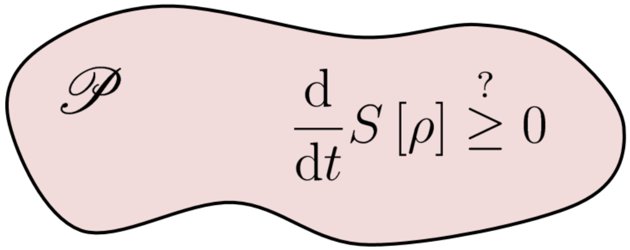

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*Does there exist a **functional** $S[\rho]$ **of a probability density** ρ **on** $\mathscr{P}$ which monotonically increases in time?*

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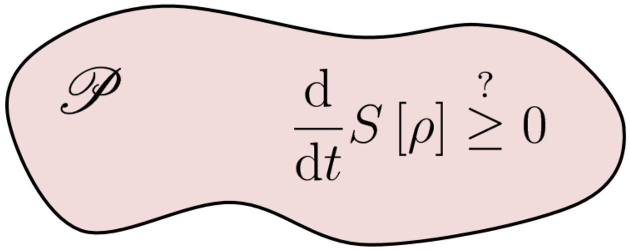

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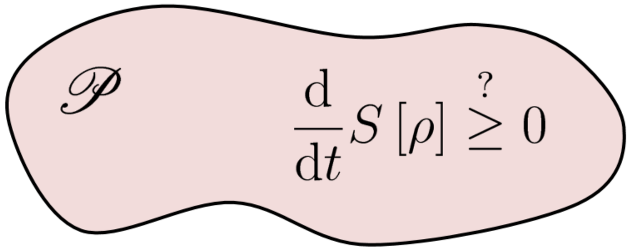

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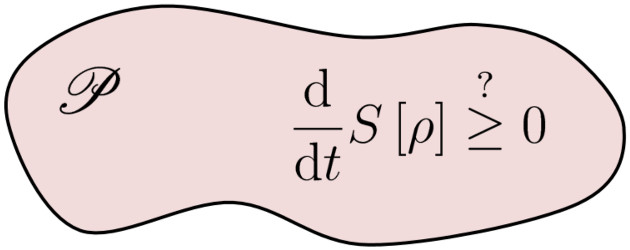

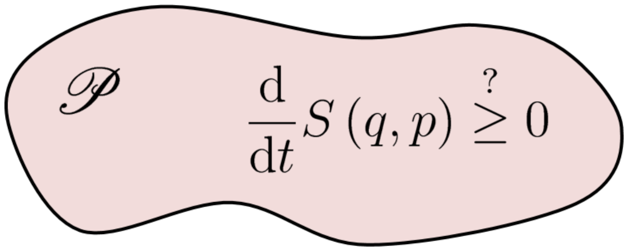

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- Well-known arguments:
  - Loschmidt reversibility [Sitzungsber. Kais. Akad. Wiss. Wien, Math. Naturwiss. Classe **73**, 128 (1876)];
  - Poincaré recurrence theorem [Acta mathematica **13**, 1 (1890)].

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- We will look at some less well-known arguments:
  1. ***Perturbative* approach**: the idea for a proof was sketched by Poincaré [C. R. Acad. Sci. (Paris) **108**, 550 (1889)].
  2. ***Topological* approach**: proof by Olsen [Found. Phys. Lett. **6(4)**, 327 (1993)].

Perturbative approach

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Taylor expand:

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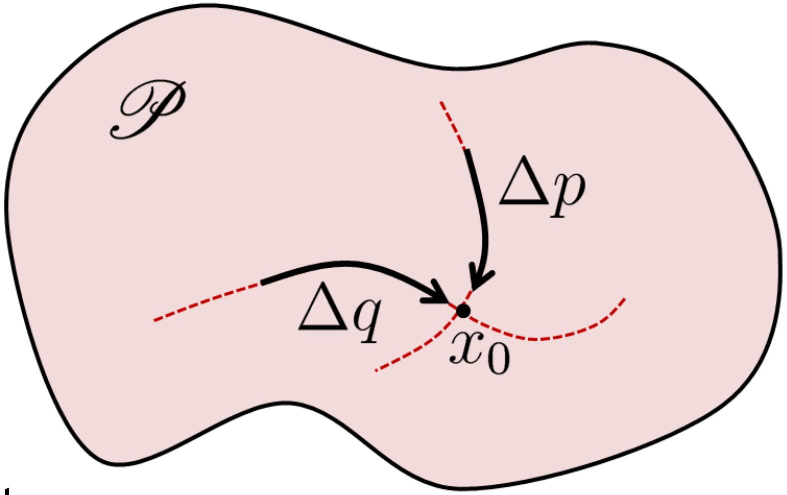
about a hypothetical equilibrium point x_0;
get a contradiction with $\dot{S} > 0$ away from it.



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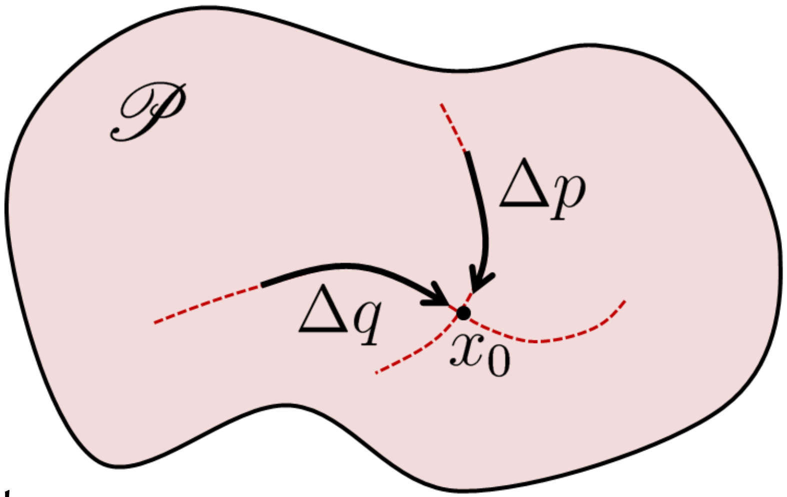

- The full proof (in our forthcoming paper) uses some assumptions on S and H including (among obvious ones): $\mathbf{Hess}(\dot{S}) \succ 0$ and $\sum_{i=1}^{N} (\partial^2 H/\partial q_i \partial q_j) \geq 0$ at x_0.

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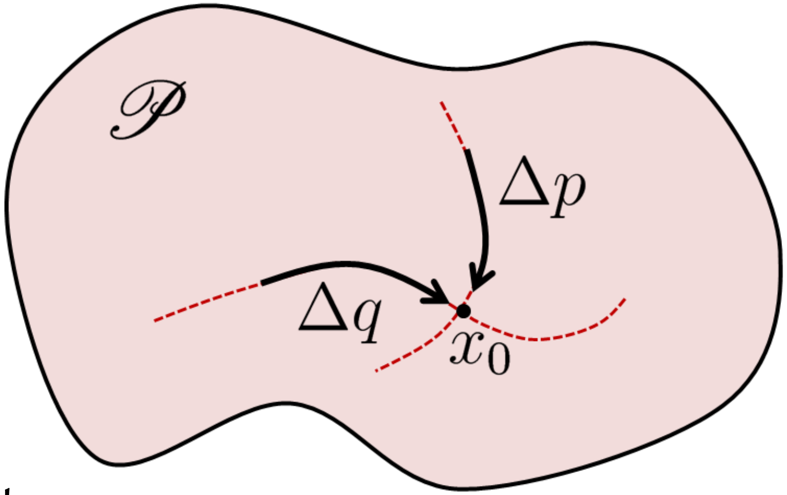

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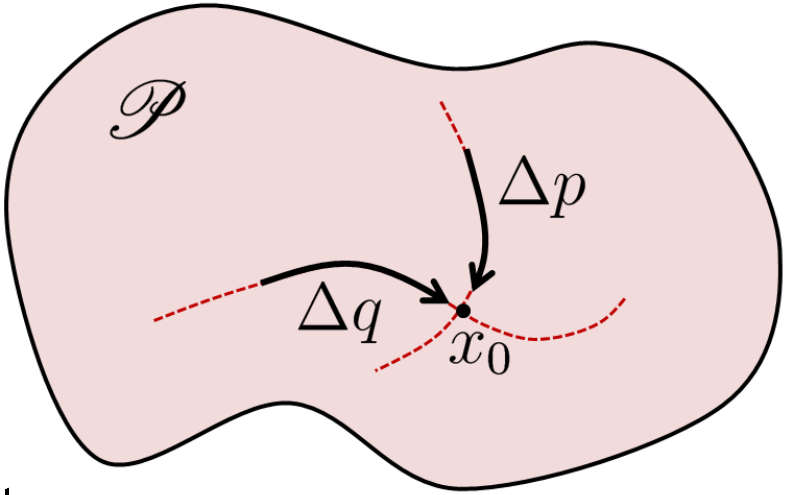

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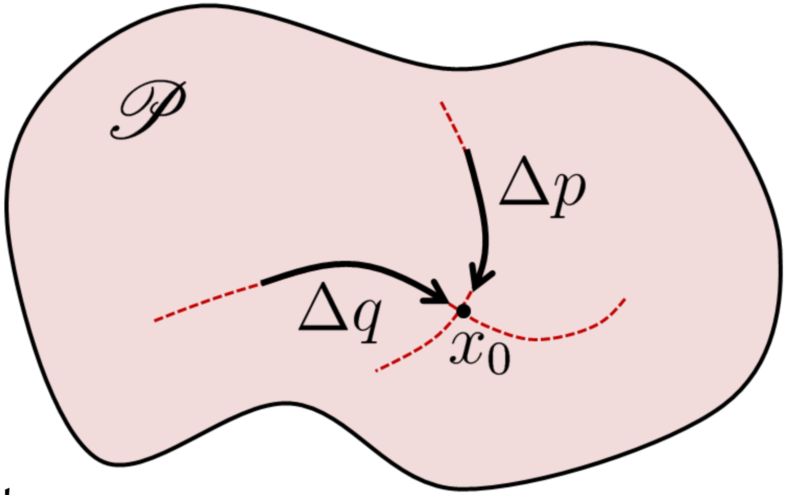

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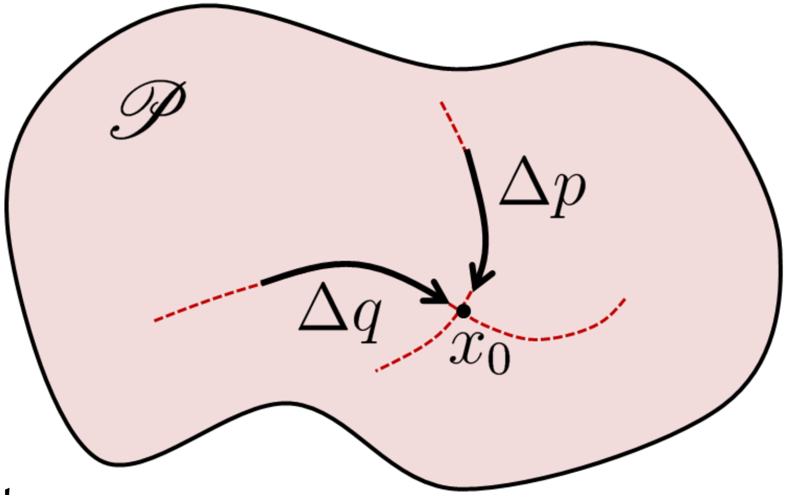

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- It *does not* work in GR because the second partials of the gravitational (vacuum) Hamiltonian w.r.t. (h_{ab}, π_{ab}) can change sign.

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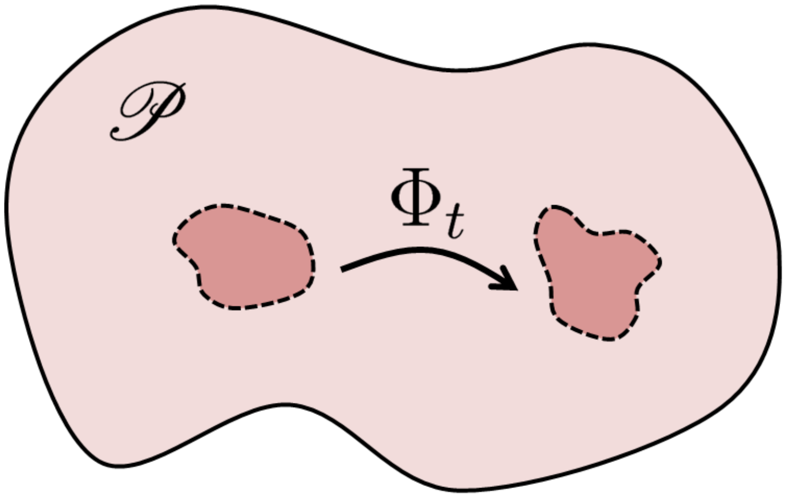

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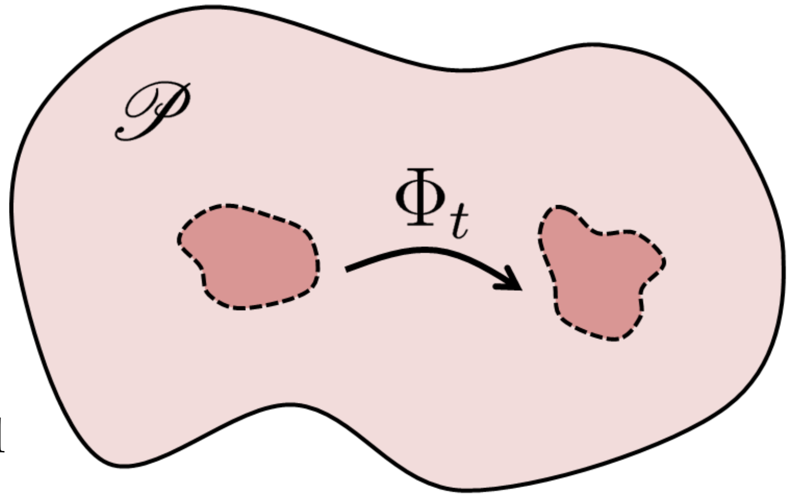

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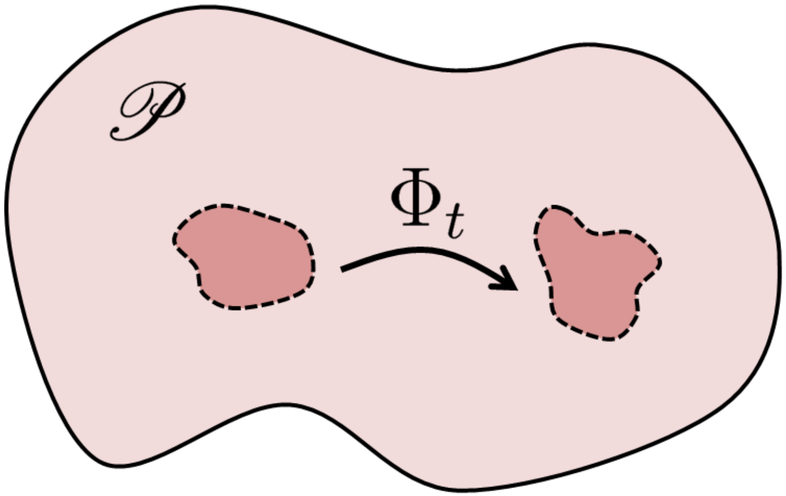

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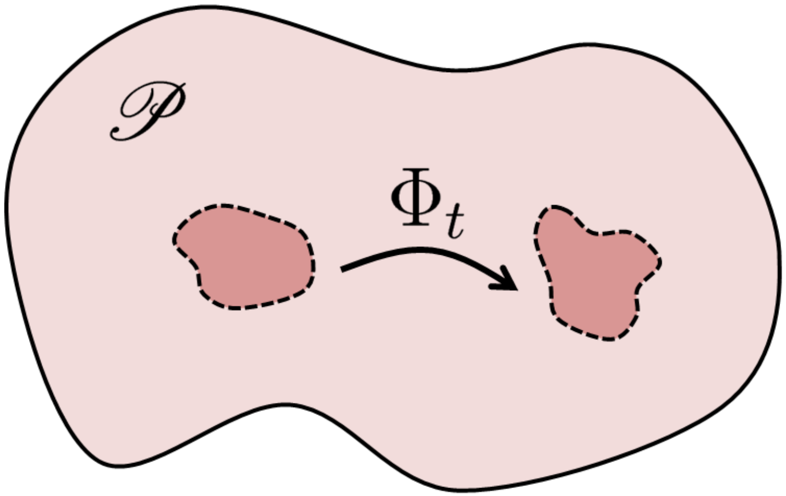

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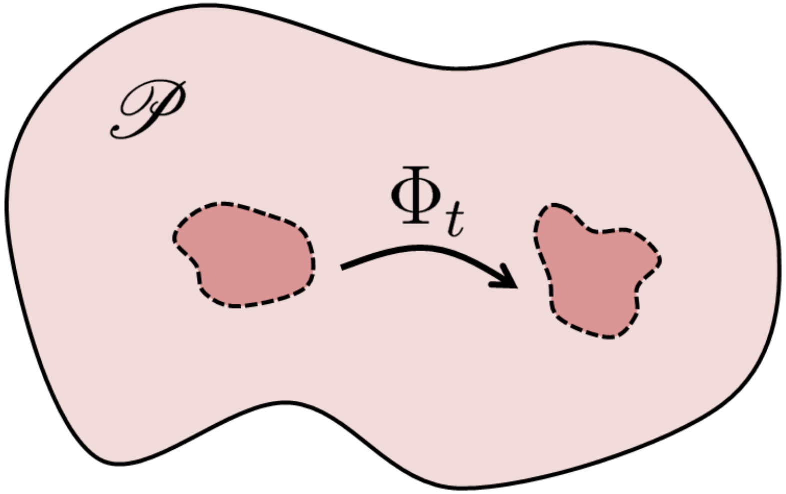

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- There is even a "no-return" theorem by Tipler [Nature, **280(5719)**, 203 (1979)], but only applicable when Cauchy surfaces are compact.

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We have shown that the total phase space volume (i.e. the integral of this form) diverges.

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Future work

- General definition of entropy in GR, and proof that it obeys the second law?
- Association of entropy increase in GR with the problem of motion (in a Lagrangian formulation)?

Thanks for your attention!