Renormalization for the self-potential of a charge in static space-times

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- Motivation
- Charge at Rest in Static Spacetime
- Expansion of Green's Function
- Results
- Examples
- Conclusion

Calculating the self-force one must evaluate the field that the point charge induces at the position of the charge.

$$\phi(x,x')^{;\mu}_{;\mu} = -4\pi q \int \delta^{(4)}(x - x'(\tau)) \frac{d\tau}{\sqrt{-g^{(4)}}},$$

This field diverges and must be renormalized.
There are different methods of such type of renormalization:

mod-sum ... zeta function method (Lousto, 2000)

Khusnutdinov, Bezerra, Bakhmatov (2007,2009)
Taylor (2012), ...
In an ultrastatic space-time

$$ds^2 = -dt^2 + g_{jk}(x^i)dx^j dx^k, \qquad i,j,k = 1,2,3$$

$$\phi^{;\mu}_{;\mu} - \xi R\phi = -4\pi q \int \delta^{(4)}(x - x'(\tau)) \frac{d\tau}{\sqrt{-g^{(4)}}},$$

$$\mu = 0,1,2,3$$

for a static charge $\Downarrow$

$$G_{\mathsf{E}}(x,x')^{;i}_{;i} - \xi R^{(3)} G_{\mathsf{E}}(x,x') = -4\pi q \frac{\delta^{(3)}(x^j - x^{j'})}{\sqrt{-g^{(3)}}},$$

$$\phi(x,x') = 4\pi q G_{\mathsf{E}}(x,x')$$

In 3D space

$$\phi_{ren}(x) = \lim_{x' \to x} \left[\phi(x;x') - \phi_{DS}(x;x') \right].$$

$$\phi_{DS}(x;x') = q \frac{\Delta^{1/2}}{4\pi\sqrt{2\sigma}},$$

Δ is the Van-Vleck Morrette determinant,
σ is one half the square of the distance between the points x and x' along the shortest geodesic connecting them.

"Massive field" renormalization ⇐ Rosenthal (2004)

$$f_\mu(x) = q \lim_{m\to\infty} \left\{ \lim_{x'\to x} \Delta\phi_{,\mu}(x,x') + \frac{1}{2} q \left[m^2 n_\mu(x) + m a_\mu(x) \right] \right\},$$

where

$$\Delta\phi(x,x') \equiv \phi(x,x') - \phi_m(x,x') ,$$

ϕ is the massless field of scalar charge,
ϕ_m is an auxiliary massive scalar field satisfying the inhomogeneous massive field equation, with the same charge density,
n^μ is a unit spatial vector, which is perpendicular to the object's world line $x^\mu(\tau)$ but is otherwise arbitrary
(i.e. $n^\mu n_\mu = 1$, $g_{\mu\nu} n^\mu dx^\nu / d\tau = 0$),
a^μ denotes the object's four-acceleration.

The equation for the scalar field with source is

$$\phi^{;\mu}_{;\mu} - \xi R\phi = -4\pi q \int \delta^{(4)}(x - x'(\tau))\frac{d\tau}{\sqrt{-g^{(4)}}},$$

$$\phi_{m;\mu}{}^{;\mu} - \left(m^2 + \xi R\right)\phi_m = -4\pi q \int \delta^{(4)}(x - x'(\tau))\frac{d\tau}{\sqrt{-g^{(4)}}},$$

q is the scalar charge, τ is its proper time,
$x^{\mu\prime}(\tau)$ is the world line of the charge ($\mu, \nu = 0, 1, 2, 3$),
ξ is a coupling of the scalar field with mass m to the scalar curvature R.

$c = G = 1$

Static space-time

$$ds^2 = g_{tt}(x^i)dt^2 + g_{jk}(x^i)dx^j dx^k, \qquad i,j,k = 1,2,3.$$

Field equation for the static charge

$$\frac{1}{\sqrt{g^{(3)}}}\frac{\partial}{\partial x^j}\left(\sqrt{g^{(3)}}g^{jk}\frac{\partial\phi_m(x^i;x^{i'})}{\partial x^k}\right) + \frac{g^{jk}}{2g_{tt}}\frac{\partial g_{tt}}{\partial x^j}\frac{\partial\phi_m(x^i;x^{i'})}{\partial x^k}$$
$$-\left(m^2+\xi R\right)\phi_m(x^i;x^{i'}) = -4\pi q\frac{\delta^{(3)}(x^i,x^{i'})}{\sqrt{g^{(3)}}},$$

where $g^{(3)} = \det g_{ij}$.

In the case

$$m \gg 1/L,$$

where L is the characteristic curvature scale of the background geometry, it is possible to construct the iterative procedure of the solution of this equation with small parameter $1/(mL)$ in the vicinity of x'.

$$\phi_m(x^i;x^{i'})=q\left(\frac{1}{\sqrt{2\sigma}}+\frac{g_{t't',i'}\sigma^{i'}}{4g_{t't'}\sqrt{2\sigma}}-m\right)+O\left(\frac{q\sqrt{\sigma}}{L^2}\right)$$
$$+\frac{q}{2m}\left[-\frac{g_{t't';i'}{}^{i'}}{12g_{t't'}}+\frac{5g_{t't',i'}g_{t't'}{}^{,i'}}{48{g_{t't'}}^2}-\left(\xi-\frac{1}{6}\right)R\right]$$
$$+O\left(\frac{q}{m^2L^3}\right)+O\left(\frac{q\sqrt{\sigma}}{mL^3}\right)+O\left(\frac{qm\sigma}{L^2}\right),$$
$$\frac{1}{mL}\ll 1,\ \sigma\to 0,$$

σ is one half the square of the distance between the points x^i and $x^{i'}$ along the shortest geodesic connecting them,

$\sigma_{i'}=\dfrac{\partial\sigma}{\partial x^{i'}}$,

L is the characteristic scale of variation of the background gravitational field,

m is the mass of scalar field.

$$ds^2 = g_{tt}(x^i)dt^2 + g_{jk}(x^i)dx^j dx^k, \qquad i,j,k = 1,2,3$$

$$\phi_{\mathbf{DS}}(x^i; x^{i'}) = q\left(\frac{1}{\sqrt{2\sigma}} + \frac{g_{t't',i'}\sigma^{i'}}{4g_{t't'}\sqrt{2\sigma}}\right),$$

σ is one half the square of the distance between the points x^i and $x^{i'}$ along the shortest geodesic connecting them,

$$\sigma_{i'} = \frac{\partial\sigma}{\partial x^{i'}} = -\left(x^i - x^{i'}\right) - \frac{1}{2}\Gamma^{i'}_{j'k'}\left(x^j - x^{j'}\right)\left(x^k - x^{k'}\right)$$
$$-\frac{1}{6}\left(\Gamma^{i'}_{j'm'}\Gamma^{m'}_{k'l'} + \frac{\partial\Gamma^{i'}_{j'k'}}{\partial x^{l'}}\right)\left(x^j - x^{j'}\right)\left(x^k - x^{k'}\right)\left(x^l - x^{l'}\right)$$
$$+O\left(\left(x - x'\right)^4\right),$$

Popov, Phys. Rev. D (2011)

$$ds^2 = g_{tt}(x^i)dt^2 + g_{jk}(x^i)dx^j dx^k, \qquad i,j,k = 1,2,3$$

$$\phi_m^{\mathit{ren}}(x) = \lim_{x' \to x} \left[\phi_m(x;x') - \phi_{\mathit{DS}}(x;x')\right]$$
$$= \frac{q}{2m}\left[-\frac{{g_{tt,i}}^{;i}}{12 g_{tt}} + \frac{5 g_{tt,i} {g_{tt}}^{,i}}{48 {g_{tt}}^2} - \left(\xi - \frac{1}{6}\right) R\right] + O\left(\frac{q}{m^2 L^3}\right),$$

$$\frac{1}{mL} \ll 1,$$

m is the mass of scalar field,
L is the characteristic scale of curvature of the background gravitational field

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

Wiseman, Phys. Rev. D (2000)

$$\phi(r;r') = \frac{q}{|r - r'|}\sqrt{1 - \frac{2M}{r}}$$

$$\phi_{\mathsf{DS}}(r;r') = \frac{q}{|r - r'|}\sqrt{1 - \frac{2M}{r'}}$$

$$\phi_{\mathsf{ren}}(x) = \lim_{r' \to r} \left[\phi(r;r') - \phi_{\mathsf{DS}}(r;r')\right] = const$$

Frolov and Zel'nikov (2012)

$$\phi_{\mathrm{DS}}(x,x') = \frac{\Delta^{1/2}(x,x')}{(2\pi)^{\frac{n}{2}+1}} \sum_{k=0}^{[n/2]} \frac{\Gamma\left(\frac{n}{2}-k\right)}{2^{k+1}\sigma^{\frac{n}{2}-k}} a_k(x,x'), \quad n = D-3.$$

D is the spacetime dimension,
Δ is the Van-Vleck Morrette determinant,
σ is one half the square of the distance between the points x and x' along the shortest geodesic connecting them,
a_k are the Schwinger–DeWitt coefficients.
When n is even the last term ($k = n/2$) in the sum should be substituted by

$$\left.\frac{\Gamma\left(\frac{n}{2}-k\right)}{2^{k+1}\sigma^{\frac{n}{2}-k}} a_k(\boldsymbol{x},\boldsymbol{x}')\right|_{k=n/2} \to -\frac{\ln\sigma(\boldsymbol{x},\boldsymbol{x}') + \gamma - \ln 2}{2^{\frac{n}{2}+1}} a_{n/2}(\boldsymbol{x},\boldsymbol{x}').$$

$$ds^2 = g_{tt}(x^i)dt^2 + g_{jk}(x^i)dx^j dx^k, \qquad i,j,k = 1,2,3.$$

Field equation for the static charge

$$\frac{1}{\sqrt{g^{(3)}}}\frac{\partial}{\partial x^j}\left(\sqrt{g^{(3)}}g^{jk}\frac{\partial A_t(x^i,x^{i'})}{\partial x^k}\right)$$
$$-m^2 A_t(x^i,x^{i'}) + \frac{g^{jk}}{2g_{tt}}\frac{\partial g_{tt}}{\partial x^j}\frac{\partial A_t(x^i,x^{i'})}{\partial x^k}$$
$$-\left(\frac{g^{ij}}{4g_{tt}^2}\frac{\partial g_{tt}}{\partial x^i}\frac{\partial g_{tt}}{\partial x^j} + R_t^t\right)A_t(x^i,x^{i'}) = -4\pi e\frac{\delta^{(3)}(x^i,x^{i'})}{\sqrt{g^{(3)}}}$$

where m is a field mass, $g^{(3)} = \det g_{ij}$.

$$A_t(x^i;x^{i'}) = e\left(\frac{1}{\sqrt{2\sigma}} + \frac{g_{t't',i'}\sigma^{i'}}{4g_{t't'}\sqrt{2\sigma}} - m\right) + O\left(\frac{e\sqrt{\sigma}}{L^2}\right)$$
$$+O\left(\frac{e}{mL^2}\right) + O\left(\frac{e\sqrt{\sigma}}{mL^3}\right) + O\left(\frac{em\sigma}{L^2}\right),$$
$$\frac{1}{mL} \ll 1,\ \sigma \to 0,$$

σ is one half the square of the distance between the points x^i and $x^{i'}$ along the shortest geodesic connecting them,

$\sigma_{i'} = \frac{\partial \sigma}{\partial x^{i'}}$,

L is the characteristic scale of curvature of the background gravitational field,

m is the mass of field.

$$ds^2 = g_{tt}(x^i)dt^2 + g_{jk}(x^i)dx^j dx^k, \qquad i, j, k = 1, 2, 3$$

$$A_t^{\mathsf{DS}}(x^i; x^{i'}) = e\left(\frac{1}{\sqrt{2\sigma}} + \frac{g_{t't',i'}\sigma^{i'}}{4g_{t't'}\sqrt{2\sigma}}\right)$$

The metric of the static spherically symmetric wormhole

$$ds^2 = -f(\rho)dt^2 + d\rho^2 + r(\rho)^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right),$$
$$\rho \in (-\infty, +\infty).$$

There is a sphere of the minimum area

$$\begin{cases} r(0) = r_0, & - \text{ radius of the wormhole throat} \\ r'(0) = 0, \\ r''(0) \geq 0. \end{cases}$$

The space-time is flat at $\rho \to \pm\infty$

$$\lim_{\rho \to \pm\infty} \frac{r^2(\rho)}{\rho^2} = 1, \lim_{\rho \to \pm\infty} f(\rho) = 1.$$

Bronnikov (1973), Ellis (1973)

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{8\pi}R-\varepsilon(\nabla\varphi)^2\right],$$

$\varepsilon=1$ - an ordinary scalar field,

$\varepsilon=-1$ - a ghost scalar field $\Rightarrow$ wormhole $\Rightarrow$

$$ds^2=-e^{-\alpha(\rho)}dt^2+e^{\alpha(\rho)}d\rho^2+r^2(\rho)d\Omega^2,$$

$$r^2(\rho)=(\rho^2+Q^2-M^2)e^{\alpha(\rho)},$$

$$\alpha(\rho)=2\frac{2M}{\sqrt{Q^2-M^2}}\left\{\frac{\pi}{2}-\arctan\left(\frac{\rho}{\sqrt{Q^2-M^2}}\right)\right\}$$

M is a mass of the wormhole,
Q is a scalar charge of the wormhole

The Maxwell equations under covariant Lorentz gauge (e is an electric charge of the particle)

$$g^{\alpha\beta}A_{\mu;\alpha\beta} - A_\nu R^\nu_\mu = -4\pi J_\mu = -4\pi e \int u_\mu(\tau)\delta^{(4)}(x - x'(\tau))\frac{d\tau}{\sqrt{-g}},$$

$$A_\mu = (A_t(\rho, \rho'), 0, 0, 0), \; u_\mu = (u_t, 0, 0, 0).$$

The tetrad component of the renormalized potential

$$A^{ren}_{(t)} = \lim_{\rho' \to \rho} \left(A_{(t)} - A^{\mathrm{DS}}_{(t)}\right) = \frac{e}{\rho^2 + Q^2 - M^2}\frac{Me^{-\alpha/2}}{\tanh \pi b}.$$

The tetrad nonzero component of the self-force

$$\mathcal{F}^{(\rho)} = \frac{e^2}{(\rho^2 + Q^2 - M^2)^2}\frac{M(M - \rho)e^{-\alpha}}{\tanh\left(\pi M/\sqrt{Q^2 - M^2}\right)}.$$

Khusnutdinov, Popov, Lipatova (2010)

$$ds^2 = -e^{-\alpha(\rho)}dt^2 + e^{\alpha(\rho)}d\rho^2 + r^2(\rho)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right),$$

$$r^2(\rho)=(\rho^2 + Q^2 - M^2)e^{\alpha(\rho)},$$

$$\alpha(\rho)=2\frac{2M}{\sqrt{Q^2-M^2}}\left\{\frac{\pi}{2} - \arctan\left(\frac{\rho}{\sqrt{Q^2-M^2}}\right)\right\},$$

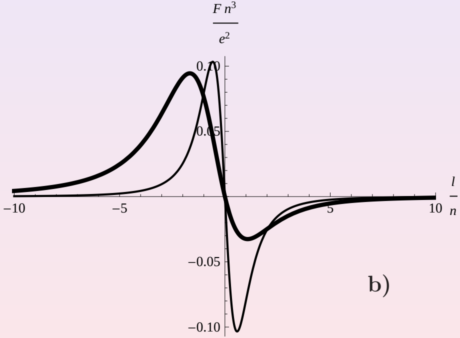

Рис. : The self-force for massless case (thin curves) and for massive wormhole case (thick curves) for $M/Q = 0.7$.

Taylor(2012)

$$ds^2 = -dt^2 + d\rho^2 + (\rho^2 + Q^2)d\Omega^2$$

$$\phi^{;\mu}_{;\mu} - \xi R\phi = -4\pi q \int \delta^{(4)}(x - x'(\tau)) \frac{d\tau}{\sqrt{-g^{(4)}}},$$

Self-force

$$f_\rho = q^2 |Q| \sqrt{2\xi} \cot(\pi\sqrt{2\xi}) \frac{\rho}{(\rho^2 + Q^2)^2}$$

ξ is a coupling of the scalar field to the scalar curvature R

- We've developed the renormalization procedure of the potential of scalar charge at rest in static space-times and found the term which have to be subtracted from the potential for renormalization.
- The application of this scheme gives well-known result for self-force of scalar charge at rest in the Schwarzschild space-time.
- We've evaluated self-force on a scalar charge at rest coupled with massive scalar field in static space-times.