# Second-order self-force: formulation and applications

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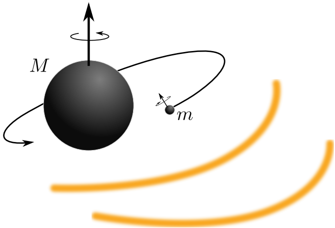
# Outline

1. Why second order?
2. Self-force theory: the local problem
3. Self-force theory: the global problem
4. Application: quasicircular orbits in Schwarzschild

# Outline

# Modeling EMRIs




- treat $m$ as source of perturbation of $M$'s metric $g_{\mu\nu}$:

$$\mathsf{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^1_{\mu\nu} + \epsilon^2 h^2_{\mu\nu} + \ldots$$

where $\epsilon \sim m/M$

- represent motion of $m$ via worldline $z^\mu$ satisfying

$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon F_1^\mu + \epsilon^2 F_2^\mu + \ldots$$

- force is small; inspiral occurs very slowly, on time scale $\tau \sim 1/\epsilon$
- suppose we neglect $F_2^\mu$; leads to error $\delta\left(\frac{D^2 z^\mu}{d\tau^2}\right) \sim \epsilon^2$

$\Rightarrow$ error in position $\delta z^\mu \sim \epsilon^2 \tau^2$

$\Rightarrow$ after inspiral time $\tau \sim 1/\epsilon$, error $\delta z^\mu \sim 1$

$\therefore$ accurately describing orbital evolution requires second order —see Moxon's talk for more details

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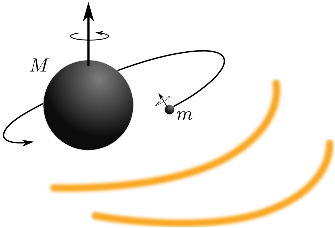


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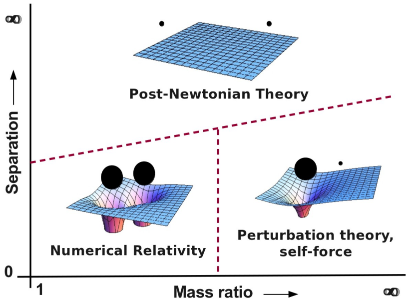
- at interface between models, SF data can fix high-order PN terms and calibrate EOB
- already done at first order
- second-order results will further improve these models
- also can use SF to *directly* model IMRIs

Binary parameter space

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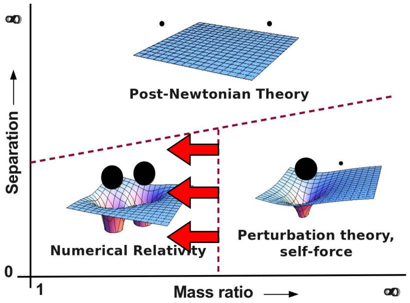



[Leor Barack]

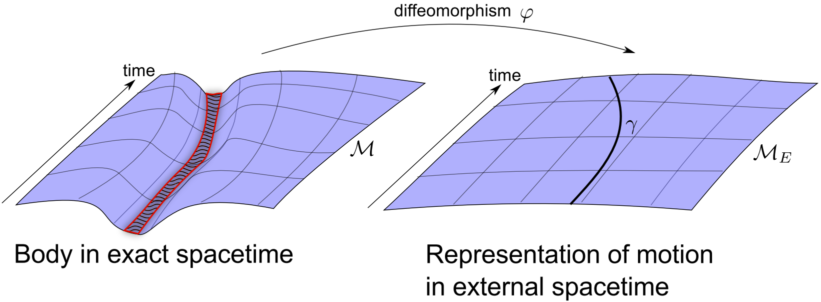
# Outline

# How do you replace an object with a worldline?

- we treat $m$ as source of perturbation of external background $g_{\mu\nu}$:

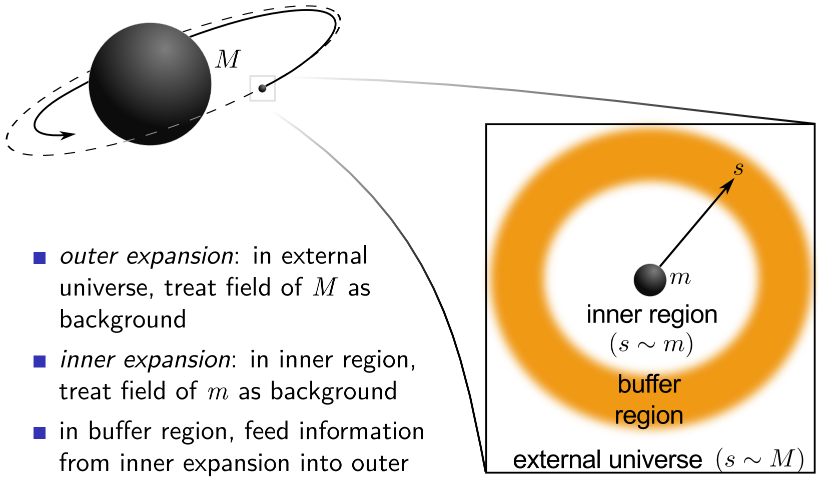
$$\mathrm{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^1_{\mu\nu} + \epsilon^2 h^2_{\mu\nu} + \dots$$

- we want to represent motion as worldline in background
- we want to encode all relevant information about object in multipole moments on worldline




- formalism due to Mino, Sasaki, Tanaka (1996), Quinn, Wald (1996), Detweiler, Whiting (2002-03), Gralla, Wald (2008, 2012), Pound (2009, 2012), Harte (2012)

# Matched asymptotic expansions

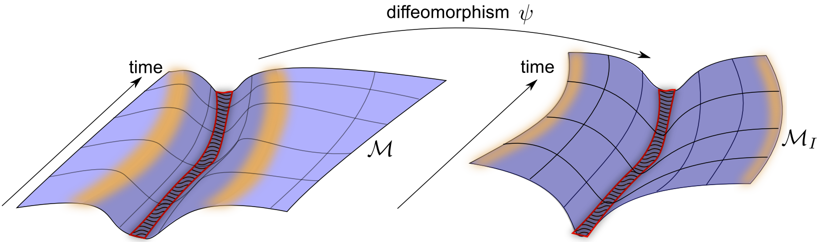



- *outer expansion*: in external universe, treat field of $M$ as background
- *inner expansion*: in inner region, treat field of $m$ as background
- in buffer region, feed information from inner expansion into outer expansion

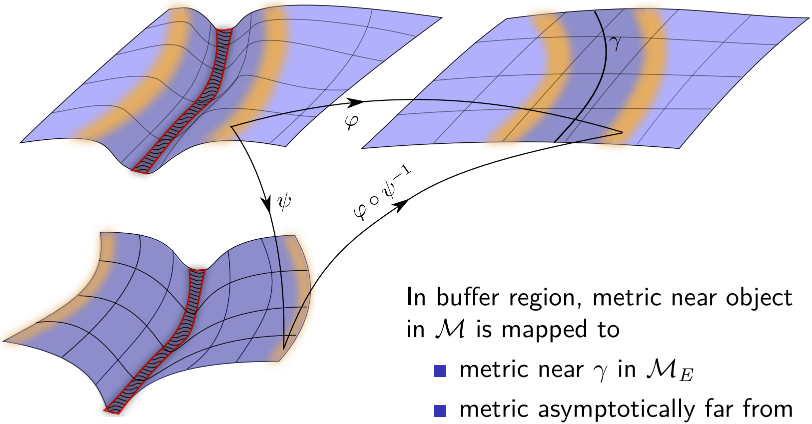
# The *inner expansion*

Zoom in on object

- use scaled distance $\tilde{s} \sim s/\epsilon$ to keep size of object fixed, send other distances to infinity as $\epsilon \to 0$
- unperturbed object defines background spacetime $g_{I\mu\nu}$ in inner expansion
- buffer region at asymptotic infinity $s \gg m$
  $\Rightarrow$ can define object's multipole moments as those of $g_{I\mu\nu}$



# Relating the expansions




In buffer region, metric near object in $\mathcal{M}$ is mapped to

- metric near $\gamma$ in $\mathcal{M}_E$
- metric asymptotically far from object in $\mathcal{M}_I$

# Expansion in the buffer region

- in coordinates centered on $\gamma$, reexpand outer expansion for small s:

$$\epsilon^n h^{(n)}_{\mu\nu} = \epsilon^n \left[ \frac{1}{s^n} h^{(n,-n)}_{\mu\nu} + s^{-n+1} h^{(n,-n+1)}_{\mu\nu} + s^{-n+2} h^{(n,-n+2)}_{\mu\nu} + \ldots \right]$$

- why no $1/s^{n+1}$?
  - would lead to $\epsilon^n h^{(n)}_{\mu\nu} \sim \frac{\epsilon^n}{s^{n+1}} = \frac{1}{\epsilon \tilde{s}^n}$
  - negative power of $\epsilon$ couldn't match anything in inner expansion
- more information from inner expansion:
  - $\epsilon^n / s^n = 1/\tilde{s}^n$ is zeroth-order in inner expansion
    $\Rightarrow h^{(n,-n)}_{\mu\nu}$ is determined by multipole moments of isolated object

# General solution in buffer region

What appears in the solution?

- put expansion into $n$th-order *vacuum* Einstein equation, solve order by order in $s$
- expand each $h^{(n,p)}_{\mu\nu}$ in spherical harmonics (wrt angles on sphere around $s = 0$)
- given a worldline $\gamma$, the solution at all orders is fully characterized by
  1. object's multipole moments (and corrections thereto): $\sim \frac{Y^{\ell m}}{s^{\ell+1}}$
  2. smooth solutions to vacuum wave equation: $\sim s^{\ell}\, Y^{\ell m}$
- everything else made of (linear or nonlinear) combinations of the above

Self field and regular field

- multipole moments define $h^{\mathrm{S}(n)}_{\mu\nu}$; interpret as bound field of object
- smooth homogeneous solutions define $h^{\mathrm{R}(n)}_{\mu\nu}$; free radiation, determined by global boundary conditions

# First and second order solutions

First order

- $h^{(1)}_{\mu\nu} = h^{S(1)}_{\mu\nu} + h^{R(1)}_{\mu\nu}$
- $h^{S(1)}_{\mu\nu} \sim 1/s + O(r^0)$ defined by mass monopole $m$
- $h^{R(1)}_{\mu\nu}$ is undetermined homogenous solution regular at $s = 0$
- evolution equations: $\dot{m} = 0$ and $a^{\mu}_{(0)} = 0$

  (where $\frac{D^2 z^\mu}{d\tau^2} = a^{\mu}_{(0)} + \epsilon a^{\mu}_{(1)} + \dots$)

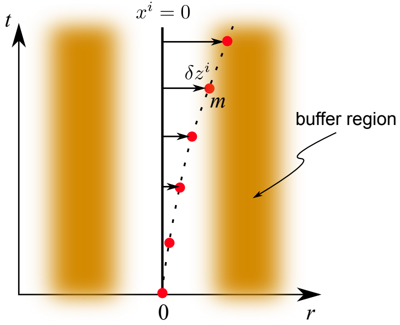
Second order

- $h^{(2)}_{\mu\nu} = h^{S(2)}_{\mu\nu} + h^{R(2)}_{\mu\nu}$
- $h^{S(2)}_{\mu\nu} \sim 1/s^2 + O(1/r)$ defined by
  1. monopole correction $\delta m$
  2. mass dipole $M^\mu$ (set to zero)
  3. spin dipole $S^\mu$
- evolution equations: $\dot{S}^\mu = 0$, $\delta\dot{m} = \dots$, and $a^{\mu}_{(1)} = \dots$

# Perturbed position at first order [Mino et al, Gralla-Wald, Pound]

Reminder: mass dipole moment $M^i$:

- small displacement of center of mass from origin of coordinates
- e.g., Newtonian field $\frac{m}{|x^i-\delta z^i|}\approx\frac{m}{|x^i|}+\frac{m\delta z^j n_j}{|x^i|^2} \;\Rightarrow\; M^i=m\delta z^i$




Definition of object's worldline:

- work in coordinates $(t,x^i)$ centered on a curve $\gamma$
- mass dipole is integral over small sphere:
  $$M^i=\frac{3}{8\pi}\lim_{s\to0}\oint h^2_{\mu\nu}u^\mu u^\nu n^i dS$$
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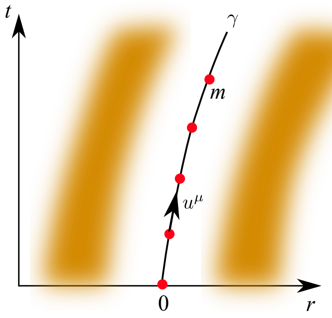


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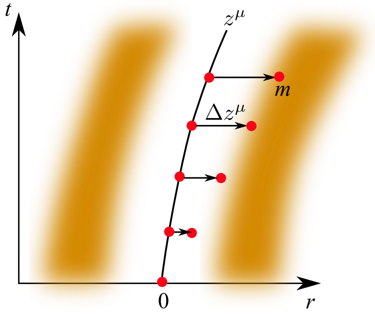
# Perturbed position at second order [Pound]

Problem:

- mass dipole moment defined for asymptotically flat spacetimes
- beyond zeroth order, inner expansion is not asymptotically flat




Solution:

- start in gauge mass-centered on $z^\mu$
- demand that transformation to practical (e.g., Lorenz) gauge does not move $z^\mu$
- i.e., insist $\Delta z^\mu = 0$
- ensures worldline in the two gauges is the same

# 0th-, 1st-, and 2nd-order equations of motion

0th order, arbitrary object: $\frac{D^2 z^\mu}{d\tau^2} = O(m)$ (geodesic motion in $g_{\mu\nu}$)

1st order, arbitrary object [MiSaTaQuWa]:

$$\frac{D^2 z^\mu}{d\tau^2} = -\tfrac{1}{2}\left(g^{\alpha\delta} + u^\alpha u^\delta\right)\left(2h^{\rm R1}_{\delta\beta;\gamma} - h^{\rm R1}_{\beta\gamma;\delta}\right) u^\beta u^\gamma + \tfrac{1}{2m} R^\alpha{}_{\beta\gamma\delta} u^\beta S^{\gamma\delta} + O(m^2)$$

(motion of spinning test body in $g_{\mu\nu} + h^{\rm R1}_{\mu\nu}$)

2nd-order, nonspinning, spherical object [Pound]:

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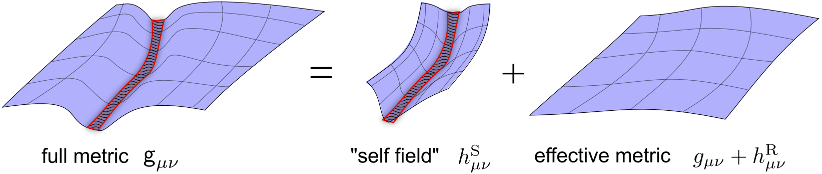


- at 1st order, can use this to *replace object with a point particle*
- beyond 1st order, point particles not well defined—but can replace object with a *puncture*, a local singularity in the field, moving on $\gamma$, equipped with the object’s multipole moments

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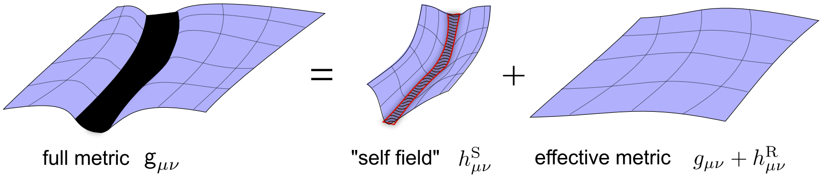


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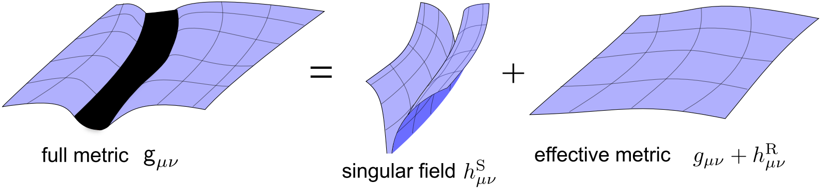


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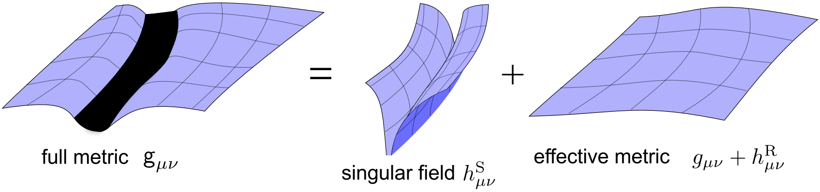


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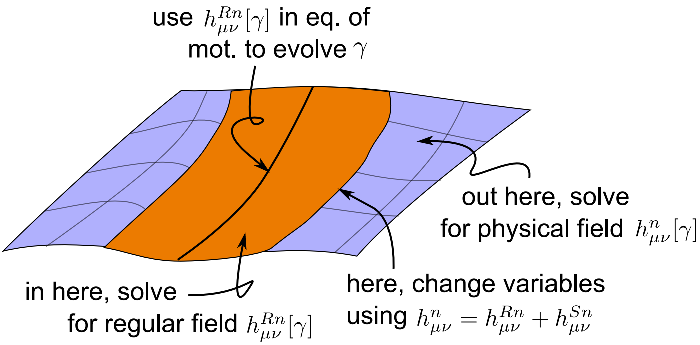
# How you replace an object with a worldline

- use a local expansion of $h^{Sn}_{\mu\nu}$ near $\gamma$ as a "puncture" that moves on $\gamma$
- solve field equations for $h^{n}_{\mu\nu}$ and $h^{Rn}_{\mu\nu}$
- move the puncture using equation of motion



# Outline

# Solving the perturbed Einstein globally

- solving the local problem told us how to replace the small object with a moving puncture in the field equations:

$$E_{\mu\nu}[h^{\mathcal{R}1}] = -E_{\mu\nu}[h^{\mathcal{P}1}] \quad \text{inside } \Gamma$$
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where $\Gamma$ is a tube around $z^\mu$, $E_{\mu\nu}[h] \sim \Box h_{\mu\nu}$, $h^{\mathcal{P}n}_{\mu\nu} \approx h^{Sn}_{\mu\nu}$, $h^{\mathcal{R}n}_{\mu\nu} = h^{n}_{\mu\nu} - h^{\mathcal{P}n}_{\mu\nu}$

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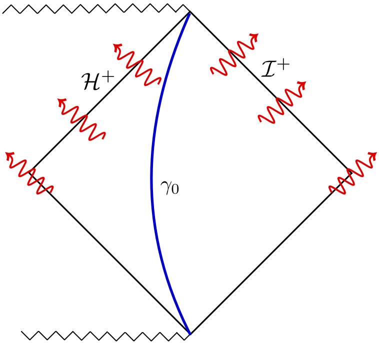
# Typical calculation at first order




- approximate the source orbit as a bound geodesic
- impose outgoing-wave BCs at $\mathcal{I}^+$ and $\mathcal{H}^+$
- solve field equation numerically, compute self-force from solution
- system radiates forever; at any given time, BH has already absorbed infinite energy
- but on short sections of time the approximation is accurate
- breaks down on *dephasing time* $\sim 1/\sqrt{\epsilon}$, when $|z^\mu - z_0^\mu| \sim M$

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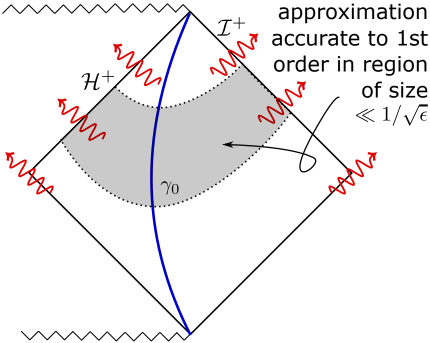


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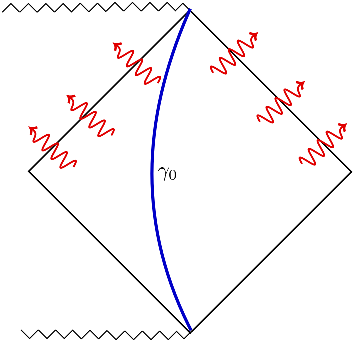


- suppose we try to use "typical" $h^1_{\mu\nu}$ to construct source for $h^2_{\mu\nu}$
- because $|z^\mu - z^\mu_0|$ blows up with time, $h^2_{\mu\nu}$ does likewise
- because $h^1_{\mu\nu}$ contains outgoing waves at all past times, the source $\delta^2 R_{\mu\nu}[h^1]$ decays too slowly, and *its retarded integral does not exist*
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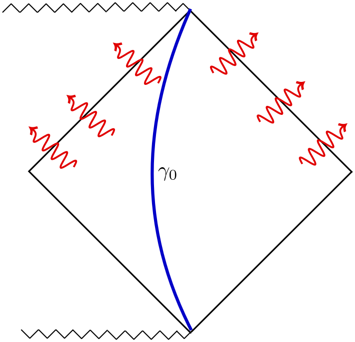


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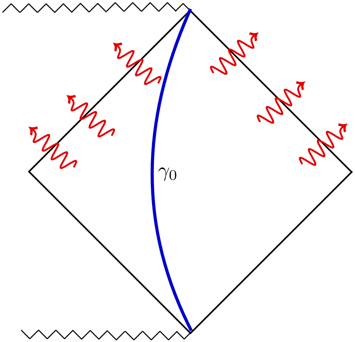


- suppose we try to use "typical" $h^1_{\mu\nu}$ to construct source for $h^2_{\mu\nu}$
- because $|z^\mu - z^\mu_0|$ blows up with time, $h^2_{\mu\nu}$ does likewise
- because $h^1_{\mu\nu}$ contains outgoing waves at all past times, the source $\delta^2 R_{\mu\nu}[h^1]$ decays too slowly, and *its retarded integral does not exist*
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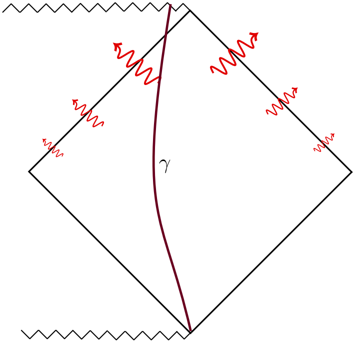


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# Resolutions of the infrared problem

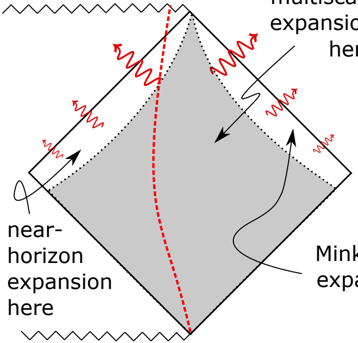
Option 1

- solve field equations and equation of motion simultaneously in the time domain
- problems:
  - limited accuracy
  - gauge instabilities
  - have to find good initial data

Option 2:

- again use matched expansions, use different expansions in different regions
- advantages:
  - allows calculations in frequency domain; high accuracy
  - no instabilities
  - better control over behavior in each region, easier to impose correct initial data

# Matched expansions [Pound, Moxon, Flanagan, Hinderer, Yamada, Isoyama, Tanaka]




Multiscale expansion

- multiscale expansion: expand orbital parameters and fields as

$$J = J_0(\tilde{t}) + \epsilon J_1(\tilde{t}) + \dots$$
$$h^n_{\mu\nu} \sim \sum_{kk'} h^n_{kk'}(\tilde{t}) e^{-ikq_r(\tilde{t}) - ik' q_\phi(\tilde{t})}$$

  where $(J, q)$ are action-angle variables for $z^\mu$, and $\tilde{t} \sim \epsilon t$ is a "slow time"

- solve for $h^n_{kk'}$ at fixed $\tilde{t}$ with standard frequency-domain techniques

Get boundary conditions from

- post-Minkowski expansion: expand $h^n_{\mu\nu}$ in powers of $M$
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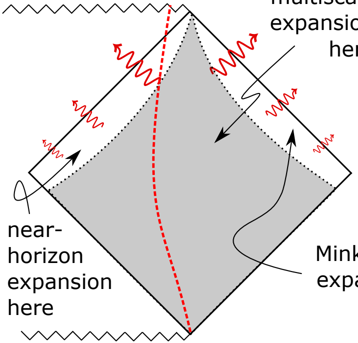


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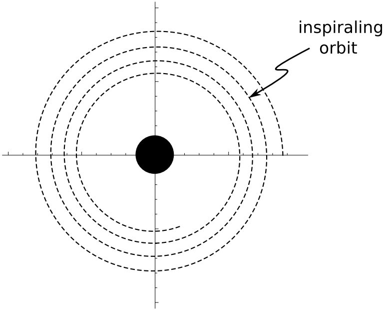
# Outline

# Quasicircular orbits in Schwarzschild [Pound, Wardell, Warburton, Miller, Barack]




- Multiscale expansion of the worldline:
  - radius $r_p = r_0(\tilde{t}) + \epsilon r_1(\tilde{t}) + \ldots$
  - frequency $\Omega = \Omega_0(\tilde{t}) + \epsilon\Omega_1(\tilde{t}) + \ldots$
  - orbital phase $\phi_p = \frac{1}{\epsilon}\int \Omega d\tilde{t}$

- Multiscale expansion of the field:

$$h^n_{\mu\nu} = \sum_{ilm} h^n_{ilm}(\tilde{t}, r) e^{-im\phi_p(\tilde{t})} Y^{ilm}_{\mu\nu}$$

- we take a “snapshot”, doing our calculations at some $\tilde{t} = \tilde{t}_0$

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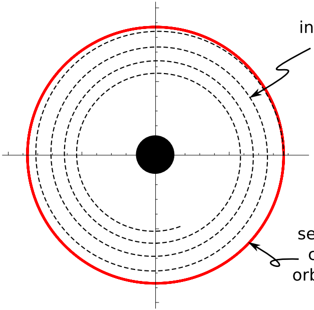


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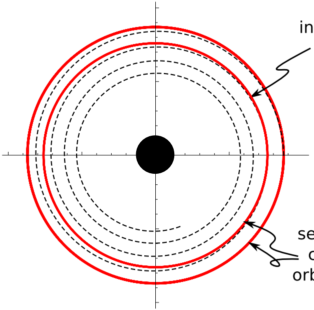


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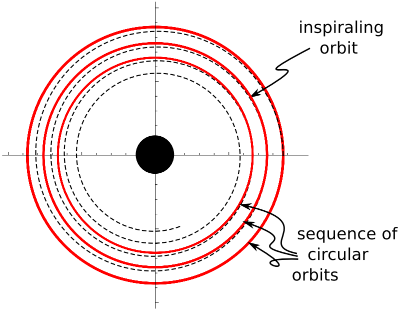


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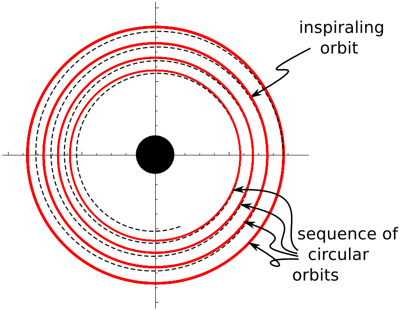


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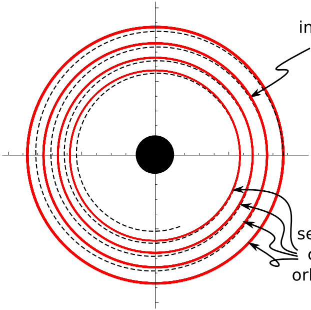


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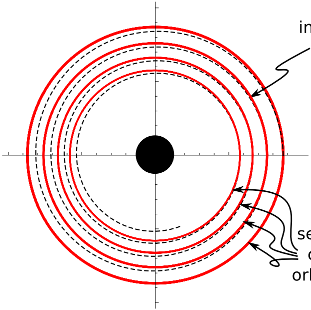


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# Field equations

$$E_{\mu\nu}[h^{\mathcal{R}1}] = -E_{\mu\nu}[h^{\mathcal{P}1}] \quad \text{inside } \Gamma$$
$$E_{\mu\nu}[h^{1}] = 0 \quad \text{outside } \Gamma$$

$$E_{\mu\nu}[h^{\mathcal{R}2}] = \delta^2 R_{\mu\nu}[h^1, h^1] - E_{\mu\nu}[h^{\mathcal{P}2}] \quad \text{inside } \Gamma$$
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$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon F_1^\mu + \epsilon^2 F_2^\mu$$

$$h^{\mathcal{P}1}_{\mu\nu} \sim \frac{m}{|x^\alpha - z^\alpha|}$$
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$$h^{\mathcal{P}1}_{ilm} \sim m|r - r_0|$$
$$h^{\mathcal{P}2}_{ilm} \sim m^2 \log|r - r_0| + (\delta m + m h^{\mathcal{R}1} + r_1 + \dot{r}_0)|r - r_0|$$

# Boundary conditions from PM/Near-Horizon expansions

At large $r$, adapt Blanchet-Damour PM methods

- The source behaves as $\delta^2 R^0_{il0} \sim \frac{S_{il0}}{r^2}$
- For $l = 0, 2$, hereditary terms arise:

$$h^2_{il0} \sim \ln(r/\epsilon) S_{il0} + \int_{-\infty}^{0} \frac{d}{d\tilde{t}} S_{il0}(\tilde{t} - \epsilon r + \tilde{z}) \ln \tilde{z} \, d\tilde{z}$$

At $r \approx 2M$, similar iteration using near-horizon retarded Green's function

- (Semi)hereditary terms arise:

$$h^2_{il0} \sim (r - 2M)\delta^2 R_{il0} + \int_{-\infty}^{0} \delta^2 R_{il0}(\tilde{t} + \epsilon r + \tilde{z}) d\tilde{z}$$

We use these asymptotic approximations as punctures $h^{\infty\mathcal{P}}_{il0}$ and $h^{\mathcal{HP}}_{il0}$ at infinity/horizon

# Specialization to $\ell = 0$

Advantages:

- Fewer fields $h^2_{i00}$ to deal with: $i = 1, 2, 3, 6$
- Clean split into dissipative and conservative sectors
  - Dissipative sector: $h^2_{200}$, $\partial_{\tilde{t}} h^1_{100}$, $\partial_{\tilde{t}} h^1_{300}$, $\partial_{\tilde{t}} h^1_{600}$, $\dot{r}_0$
  - Conservative sector: $h^2_{100}$, $h^2_{300}$, $h^2_{600}$, $r_1$

Things to mind:

- First-order perturbation must include slowly varying correction to BH mass: $h^{\delta M_{BH}}_{i00}$
- We absorb $\delta M_{BH}(\tilde{t}_0)$ (and hereditary integrals) into background mass $M$
- We take our "snapshot" at the preferred time when $\Omega(\tilde{t}_0) = \Omega_0(\tilde{t}_0)$

# Dissipative sector

- Wave equation:

$$\partial_r^2 h^2_{200} \sim \delta^2 R^0_{200} - \partial_r^2 h^{\mathcal{P}2}_{200} - \partial_{\tilde{t}} h^1_{100}$$

- Puncture:

$$h^{\mathcal{P}2}_{200} \sim m h^{R1} + \delta m + \dot{r}_0$$

- Gauge condition:

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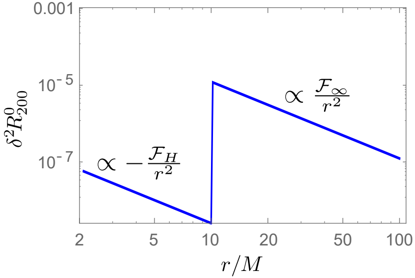


What comes out of the solution?

- The balance law! $\dot{E}_0 + \delta\dot{M}_{BH} = \mathcal{F}_\infty$
- First major result/consistency check of numerical implementation

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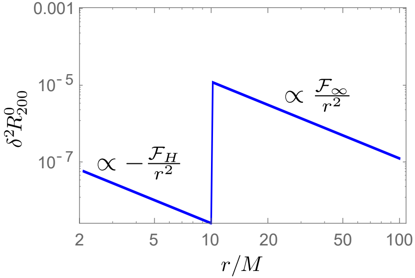


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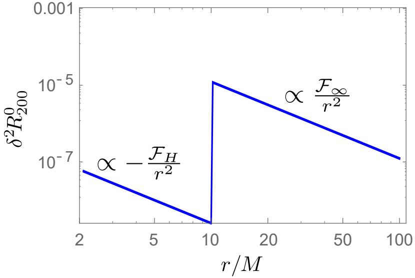


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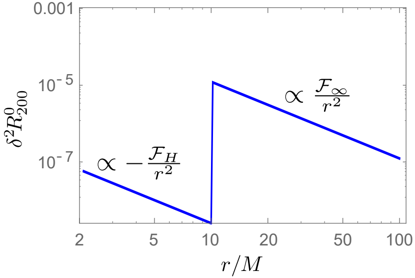


What comes out of the solution?

- The binding energy $E = M_{\rm Bondi} - m - M_{irr}$ (previously obtained from first law of binary mechanics)
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$$\partial_r^2 h_{i00}^2 \sim \delta^2 R_{i00}^0 - \partial_r^2 h_{i00}^{\mathcal{P}2} - \partial_r^2 (h_{i00}^{\infty\mathcal{P}} + h_{i00}^{\mathcal{HP}})$$

- Puncture:

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- Gauge condition:

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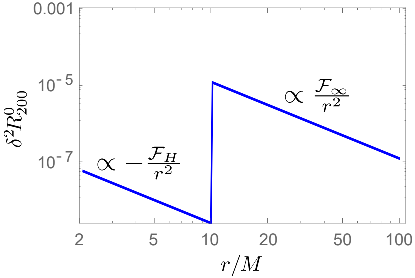


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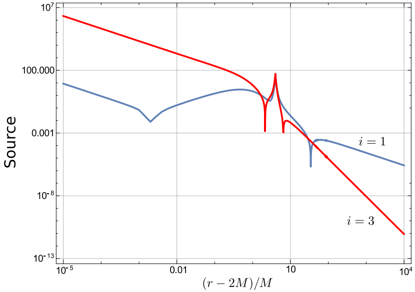


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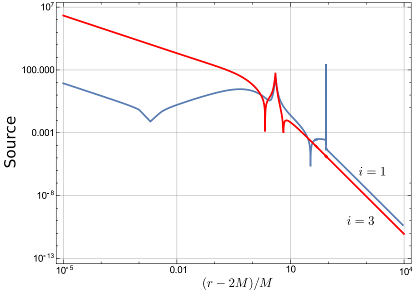


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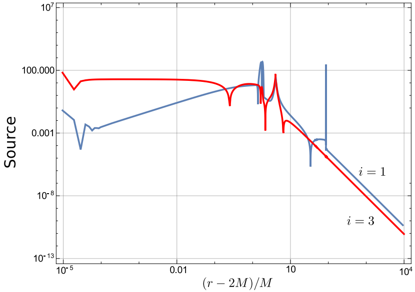
$$\partial_r^2 h^2_{i00} \sim \delta^2 R^0_{i00} - \partial_r^2 h^{\mathcal{P}2}_{i00} - \partial_r^2 (h^{\infty\mathcal{P}}_{i00} + h^{\mathcal{HP}}_{i00})$$

- Puncture:

$$h^{\mathcal{P}2}_{200} \sim m h^{R1} + \delta m + \dot{r}_0$$

- Gauge condition:

$$\partial_r h^2_{200} \sim \partial_{\tilde{t}} h^1_{100}$$




What comes out of the solution?

- The binding energy $E = M_{\rm Bondi} - m - M_{irr}$ (previously obtained from first law of binary mechanics)
- But computing the source accurately near $r = r_0$ is very difficult—see Wardell's talk

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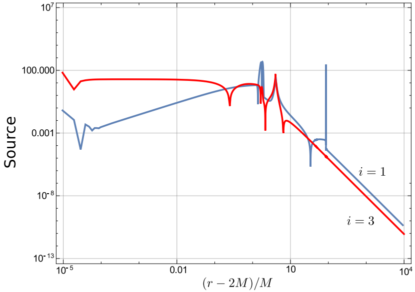


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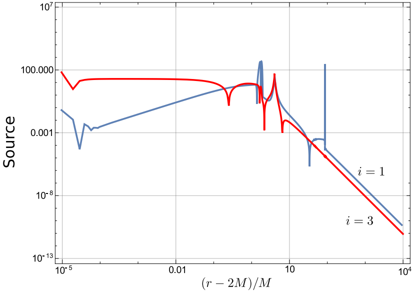


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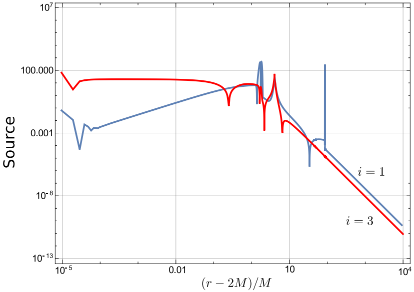


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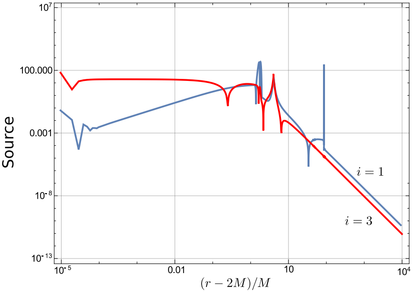


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# Conclusion

Status of formalism

- “local problem” solved, but still missing higher-moment effects at second order
- “global problem” under development, solved in some cases —see talks by Moxon and Wardell

Status of concrete computations for quasicircular orbits in Schwarzschild

- “snapshot calculation” essentially complete for $\ell = 0$ field —see talk by Wardell
- portions of calculation complete for $\ell > 0$
- long-term evolution straightforward after snapshot computations complete

# Hierarchy of self-force models [Hinderer and Flanagan]

- on an inspiral timescale $t \sim 1/\epsilon$, the phase of the gravitational wave has an expansion

$$\phi = \frac{1}{\epsilon}\left[\phi_0 + \epsilon\phi_1 + O(\epsilon^2)\right]$$

- a model that gets $\phi_0$ right is probably enough for signal detection in many cases
- a model that gets both $\phi_0$ and $\phi_1$ is enough for parameter extraction

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**Adiabatic order**

determined by

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# Hierarchy of self-force models [H

## Adiabatic order

determined by

- averaged dissipative piece of $F_1^\mu$

## Post-adiabatic order

determined by

- averaged dissipative piece of $F_2^\mu$
- conservative piece of $F_1^\mu$
- oscillatory dissipative piece of $F_1^\mu$

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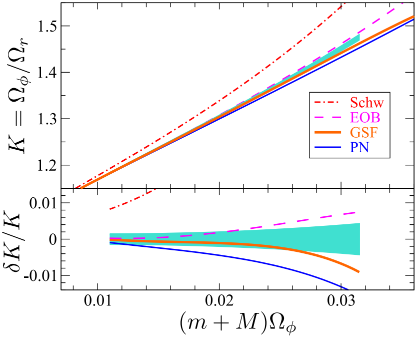
# Using SF to *directly* model IMRIs and similar-mass binaries

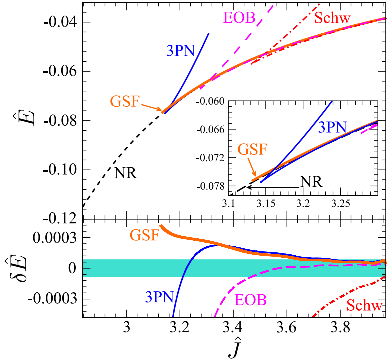
Comparisons for **equal-mass** binaries

## Orbital precession




[Le Tiec et al]

## Gravitational binding energy




- SF results use "mass symmetrized" model: $\frac{m}{M} \to \frac{mM}{(m+M)^2}$
- with mass-symmetrization, second-order self-force might be able to directly model even comparable-mass binaries