Preparations for Second-Order Self-Force Calculations

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Steve Detweiler: In Memoriam

Introduction

- Review of Perturbation Theory
- Setting Up the Second-Order Problem
- Detweiler's Second-Order Formalism
- Local Singular Field

Perturbation Theory Primer

We begin by assuming the small mass m is a compact object (point particle) moving along a geodesic of the background, γ_0.

- We model the physical spacetime in a perturbative manner,

$$g_{ab} = g^0_{ab} + h_{ab},$$

  where g^0_{ab} is the Schwarzschild metric and $h_{ab} \sim O(m)$.

- Expand $G_{ab}(g^0 + h)$ about the background g^0:

$$G_{ab}(g^0 + h) = G^{(1)}_{ab}(g^0, h) + G^{(2)}_{ab}(g^0, h) + \cdots ,$$

  with $G^{(n)}_{ab}(g^0, h) \sim O(m^n)$.

- Given the perturbing stress-energy tensor $T_{ab}(\gamma_0) \sim O(m)$, the Einstein equations may be written to first-order in m as

$$G^{(1)}_{ab}(g^0, h^{1,\mathrm{ret}}) = 8\pi T_{ab}(\gamma_0) + O(m^2),$$

Perturbation Theory Primer

- Decompose the retarded metric perturbation $h^{1,\mathrm{ret}}_{ab}$ into a locally-defined singular term, $h^{1\mathrm{S}}_{ab}$, and a non-local, regular term $h^{1\mathrm{R}}_{ab}$,

$$G^{(1)}_{ab}(g^0, h^{1\mathrm{S}}) = T_{ab}(\gamma_0), \qquad G^{(1)}_{ab}(g^0, h^{1\mathrm{R}}) = 0.$$

- Through mode-sum regularization techniques[1], we remove the singular behavior of the retarded field by subtraction; schematically this is written as a difference of the retarded and singular fields,

$$h^{\mathrm{R}}_{ab} = h^{\mathrm{ret}}_{ab} - h^{\mathrm{S}}_{ab}.$$

[1] L. Barack and A. Ori, Phys. Rev. D **61** 061502 (2000)

Preparations for Second-Order

Given h^{1R}_{ab}, the regularized vacuum solution to the Einstein equations at $O(m)$, we adopt the notion of geodesic motion in the "regularly perturbed" spacetime at first-order, since

$$G_{ab}(g^0 + h^{1R}) = O(m^2)$$

implies that an observer local to the particle will be unable to distinguish h^{1R}_{ab} from the background geometry.

When expressed on g^0, the worldline of the particle is perturbed away from the background geodesic by an $O(m)$ correction,

$$\gamma_0 \to \gamma_0 + \gamma_{1R},$$

which is determined by solving the first-order geodesic equation,

$$\frac{\mathrm{d}u_a}{\mathrm{d}s} = \frac{1}{2} u^b u^c \frac{\partial}{\partial x^a}\left(g^0_{ab} + h^{1R}_{ab}\right).$$

Preparations for Second-Order

In the case of circular orbits, the orbital frequency of the particle is adjusted in $g^0 + h^{1\mathrm{R}}$,

$$\Omega^2 = \frac{M}{r^3} - \frac{r-3M}{2r^2}\bar{u}^a\bar{u}^b\partial_r h^{1R}_{ab} + O(m^2).$$

Focusing on the conservative self-force effects, the orbit remains circular; the self-force effects shift the particle's orbital radius,

$$r_{\mathrm{new}} = r_{\mathrm{old}} - \frac{r^2(r-3M)}{6M}\bar{u}^a\bar{u}^b\partial_r h^{1R}_{ab},$$

bringing the orbital frequency into the expected form,

$$\Omega^2 = \frac{M}{r^3_{\mathrm{new}}} + O(m^2).$$

Second-Order Equations*

One anticipates that second-order perturbations will now follow (almost) exactly as the first-order:

$$G_{ab}(g^0 + h^{1\mathrm{R}}, h^{1\mathrm{S}} + h^2) = T_{ab}(\gamma_0 + \gamma_{1R}) + O(m^3).$$

We may even expand both sides,

$$\begin{aligned} &G^{(1)}_{ab}(g^0, h^1) + G^{(1)}_{ab}(g^0, h^2) + G^{(2)}_{ab}(g^0, h^1) \\ &\qquad = 8\pi T^{(1)}_{ab}(\gamma_0) + 8\pi T^{(2)}_{ab}(\gamma_0, \gamma_1) + \mathcal{O}(m^3). \end{aligned}$$

With the deviation to the worldline in hand, we can also visualize the stress-energy expansions,

$$\begin{aligned} T^{(1)}_{ab}(\gamma_0) &= \frac{m\, u^0_a u^0_b}{u^t_0 \sqrt{-g^0}} \delta^{(3)}[X^i - \gamma^i_0(T)], \\ T^{(2)}_{ab}(\gamma_0, \gamma_1) &= m\, \delta t_{ab}\, \delta^{(3)}[X^i - \gamma^i_0(T)] \\ &\qquad - \frac{m\, u^0_a u^0_b}{u^t_0 \sqrt{-g^0}} \gamma^j_{1\mathrm{R}} \frac{\partial}{\partial X^j} \delta^{(3)}[X^i - \gamma^i_0(T)]. \end{aligned}$$

Complications and Workarounds

- One term in the second-order expansion, $G^{(2)}_{ab}(g^0, h^1)$, is ill-defined (even distributionally) on the worldline of the particle.
- Unlike the first-order problem, we have no clear way of solving for $h^{2,\mathrm{ret}}$ directly.
- Instead, we start directly with the adoption of a regular/singular split,

$$h^2 = h^{2\mathrm{S}} + h^{2\mathrm{S}\dagger} + h^{2\mathrm{R}},$$

  with $h^{2\mathrm{S}\dagger}$ arising from the adjustments to $h^{1\mathrm{S}}$ from the addition of $h^{1\mathrm{R}}$ to the background.
- Schematically, you might think that $h^{2\mathrm{S}\dagger}$ belongs with $h^{1\mathrm{S}}$, but it is second-order in the mass:

$$h^{1\mathrm{S}} + h^{2\mathrm{S}\dagger} = \frac{m}{r}\left[1 + \frac{x^2}{\mathcal{R}^2}\left(1 + \frac{m}{\mathcal{R}}\right) + \cdots\right]$$

Introduce a Working Regular/Singular Split

- We understand that the singular field is known only as an asymptotic expansion of the true singular field h^{S}. In terms of locally inertial and Cartesian coordinates, x^i, one might expect,

$$h^{1\mathsf{S}} = h^{1\mathsf{s}} + O(mx^4/r\mathcal{R}^4)$$
$$h^{2\mathsf{S}} + h^{2\mathsf{S}\dagger} = h^{2\mathsf{s}} + h^{2\mathsf{s}\dagger} + O(m^2x^4/r^2\mathcal{R}^4)$$

- At first order, $h^{1\mathsf{s}}$ is known accurately enough to allow $h^{1\mathsf{r}} \equiv h^{1,\mathsf{ret}} - h^{1\mathsf{s}}$ to be C^2 on γ_0.
- For second-order calculations, we sidestep the definition of $h^{2,\mathsf{ret}}$ and solve directly for $h^{2\mathsf{r}}$:

$$\begin{aligned} G^{(1)}_{ab}(g^0, h^{2\mathsf{r}}) = &-G^{(2)}_{ab}(g^0, h^{1\mathsf{r}}) - [G^{(2)}_{ab}(g^0, h^{1\mathsf{r}}) + G^{(1)}_{ab}(g^0, h^{2\mathsf{s}})] \\ &+ [8\pi T_{ab}(\gamma_0 + \gamma_{1\mathsf{r}}) - G^{(1)}_{ab}(g^0 + h^{1\mathsf{r}}, h^{1\mathsf{s}})] \\ &- [8\pi T_{ab}(\gamma_0) - G^{(1)}_{ab}(g^0, h^{1\mathsf{s}})] \end{aligned}$$

Calculating h^{1s} and h^{2s}

Steve began considering local expansions of h^{1S} by looking at asymptotic expansions of a small Schwarzschild black hole. For an isolated Schwarzschild black hole of mass m, the local geometry may be written as,

$$g^{\text{schw}}_{ab} = \eta_{ab} + {}_0h^{\text{schw}}_{ab}$$

with

$${}_0h^{\text{schw}}_{ab} dx^a dx^b = \frac{2m}{r} dt^2 + \frac{2m}{r-2m} n_k n_l dx^k dx^l,$$

given $n_i \equiv \nabla_i r$. Expanding as $m/r \ll 1$ but remaining finite,

$${}_0h^{\text{schw}}_{ab} dx^a dx^b = \underbrace{\frac{2m}{r} dt^2 + \frac{2m}{r} n_k n_l dx^k dx^l}_{{}_0h^{\mathbf{1schw}}_{ab} dx^a dx^b} + \sum_{j=2} \underbrace{\left(\frac{2m}{r}\right)^j n_k n_l dx^k dx^l}_{{}_0h^{j\mathbf{schw}}_{ab} dx^a dx^b},$$

Calculating h^{1s} and h^{2s}

- At quadrupolar and higher orders, one considers perturbations to the Schwarzschild geometry, sourced by some external curvature (in this case, the large black hole M). These perturbations are matched in the buffer region to an expansion of the background geometry about the geodesic $\gamma_0 + \gamma_{1R}$.[2]
- After matching, the singular pieces may be identified by taking the $m/r \ll 1$ limit:

$$
{}_2h^{1s}_{ab}\mathrm{d}x^a\mathrm{d}x^b = \frac{2m}{r}[(1 + r^2\mathcal{E}^{(2)})\,\mathrm{d}t^2 + (1 - 3r^2\mathcal{E}^{(2)})\,(\mathrm{d}r^2 + \sigma_{AB}\,\mathrm{d}x^A\mathrm{d}x^B)] \\
- 4mr\mathcal{E}^{(2)}_A\,\mathrm{d}r\mathrm{d}x^A - 2mr\mathcal{E}^{(2)}_{AB}\,\mathrm{d}x^A\mathrm{d}x^B + 2\frac{m}{3r}\mathcal{B}^{(2)}_A\,\mathrm{d}t\mathrm{d}x^A
$$

[2] K. S. Thorne and J. B. Hartle, Phys. Rev. D **31**, 1815 (1985).

Calculating h^{1s} and h^{2s}

In particular, some of Steve's last work was in computing the A-K pieces of the local singular fields, for use in the Einstein field equation expansions:

$$
\begin{aligned}
2G^{(1)}(g^0, h)|_{\mathbf{A}} = {} & \frac{2(r-2M)^2}{r^2}\left(\frac{\partial^2}{\partial r^2}\mathbf{E}\right) + \frac{2(r-2M)(3r-5M)}{r^3}\left(\frac{\partial}{\partial r}\mathbf{E}\right) - \frac{(r-2M)(\ell+2)(\ell-1)}{r^3}\mathbf{E} \\
& - \frac{\ell(\ell+2)(\ell+1)(\ell-1)(r-2M)}{2r^3}\mathbf{F} + \frac{2\ell(\ell+1)(r-2M)^2}{r^3}\left(\frac{\partial}{\partial r}\mathbf{H}\right) + \frac{2\ell(\ell+1)(r-2M)(2r-3M)}{r^4}\mathbf{H} \\
& - \frac{2(r-2M)^3}{r^4}\left(\frac{\partial}{\partial r}\mathbf{K}\right) - \frac{(2r+4M+r\ell+r\ell^2)(r-2M)^2}{r^5}\mathbf{K}
\end{aligned}
$$

In Summary

- Steve was very passionate about the second-order self-force problem!
- His formalism gives us a different view of some of the challenges faced at second-order.