Self-force: Implementations and Results

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Motivation: extreme mass ratio binaries

- Major goal of eLISA is to study Extreme Mass Ratio Inspirals.
- Binary black hole/neutron star systems with a mass ratio ~1:1,000,000.
- Many (>10,000) orbits.
- Generic (eccentric, inclined) orbits.
- Larger black hole spinning.
- Ultimate goal: ~10^5 **accurate** evolved generic orbits in Kerr with gravitational self-force.

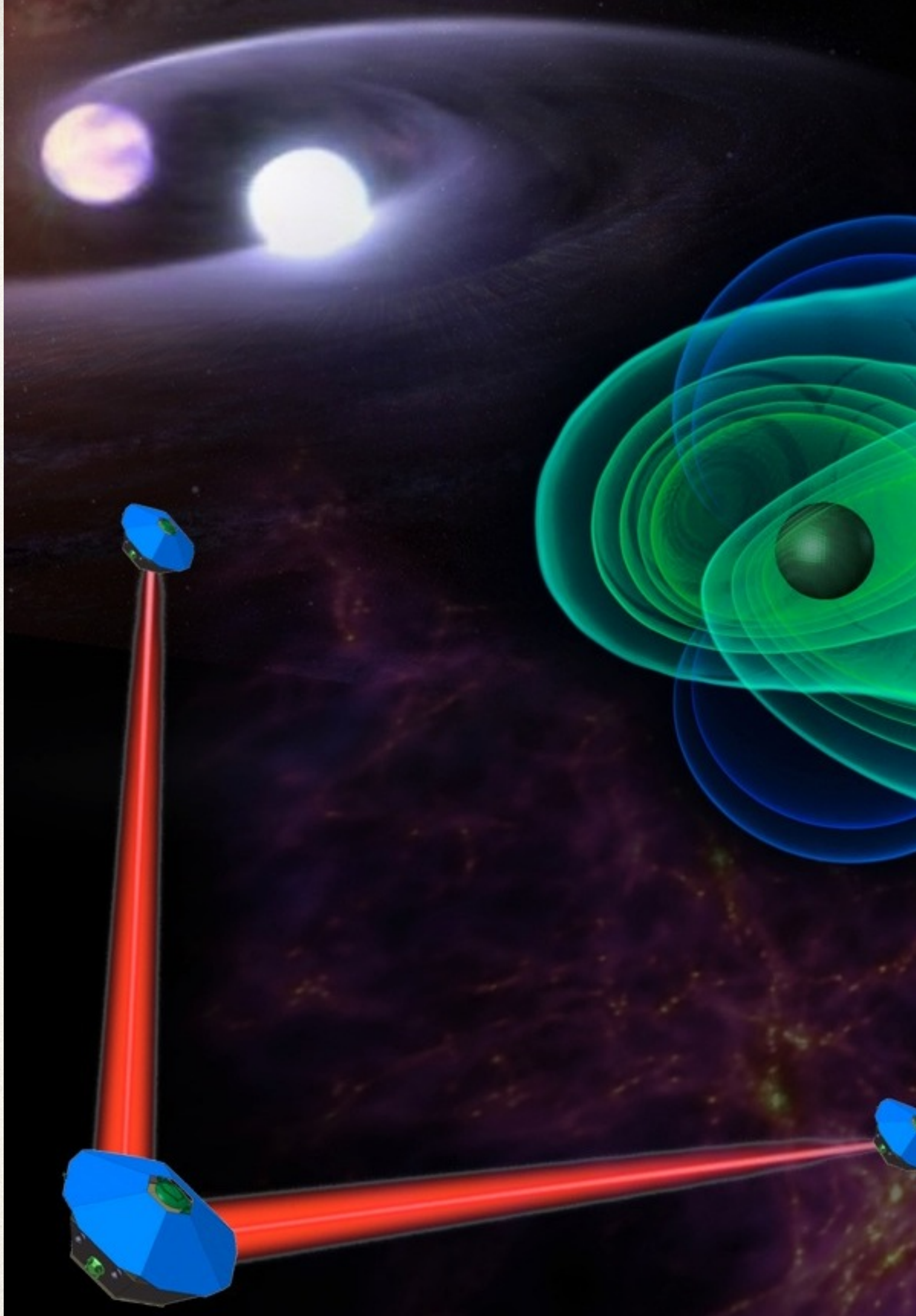
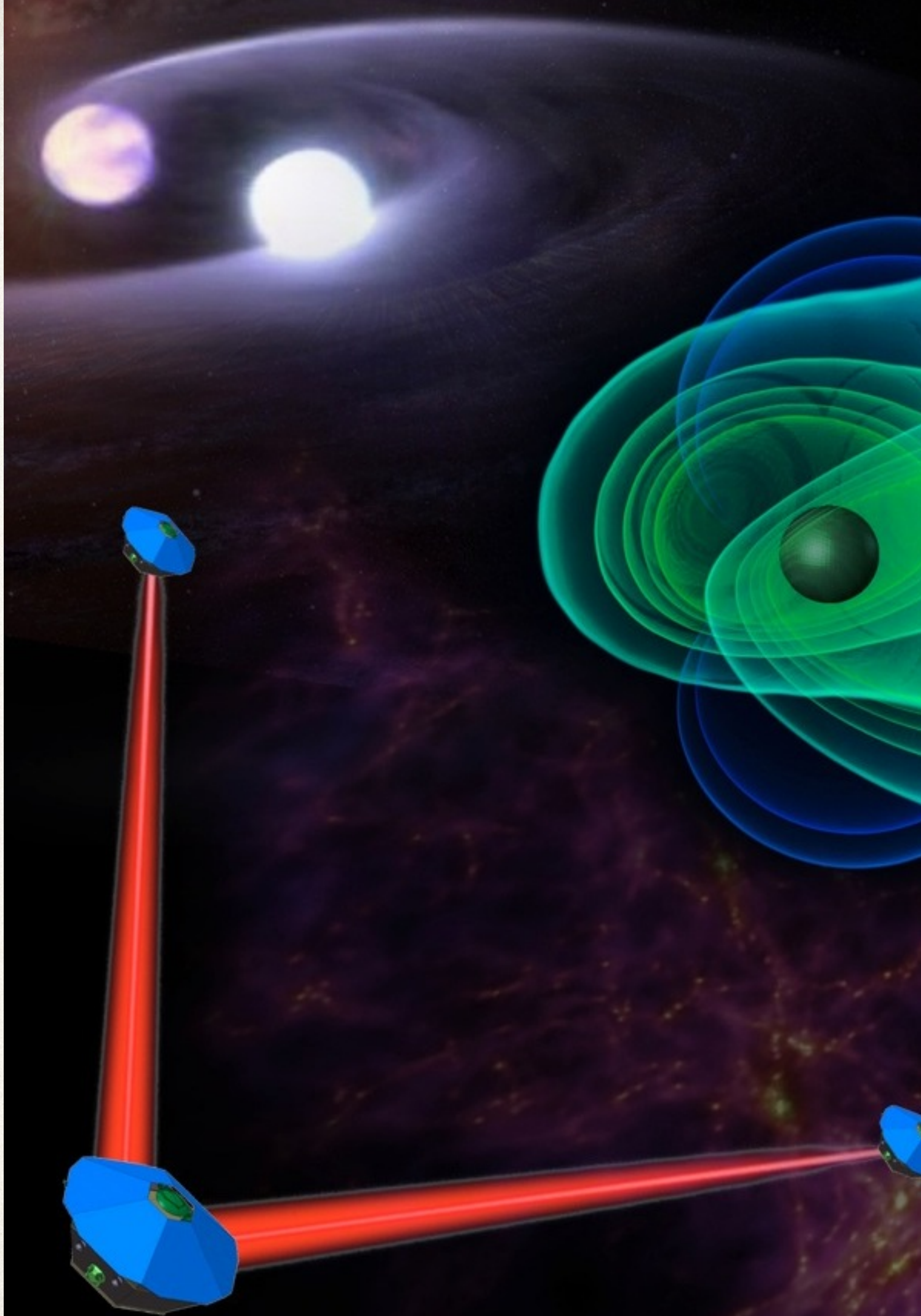


Image credit: eLISA/NGO Yellow book (ftp://ftp.rssd.esa.int/pub/ojennric/NGO_YB/NGO_YB.pdf)

Motivation: extreme mass ratio binaries

Corrections to orbital phase have contributions at adiabatic ($1/\varepsilon$) order involving (time averaged) first-order metric perturbation. Post-1-adiabatic order includes contributions from remaining first-order perturbation and from second-order metric perturbation.
[Hinderer & Flanagan, Phys. Rev. D78, 064028]

$$\begin{aligned} q(t,\varepsilon) &= \frac{1}{\varepsilon}\psi^{(0)}(\tilde{t}) + \left[\psi^{(1)}(\tilde{t}) + \bar{q}^{(0)}(\Psi,\tilde{t})\right] \\ &\quad + \varepsilon\left[\psi^{(2)}(\tilde{t}) + q^{(1)}(\Psi,\tilde{t})\right] + O(\varepsilon^2) \end{aligned}$$



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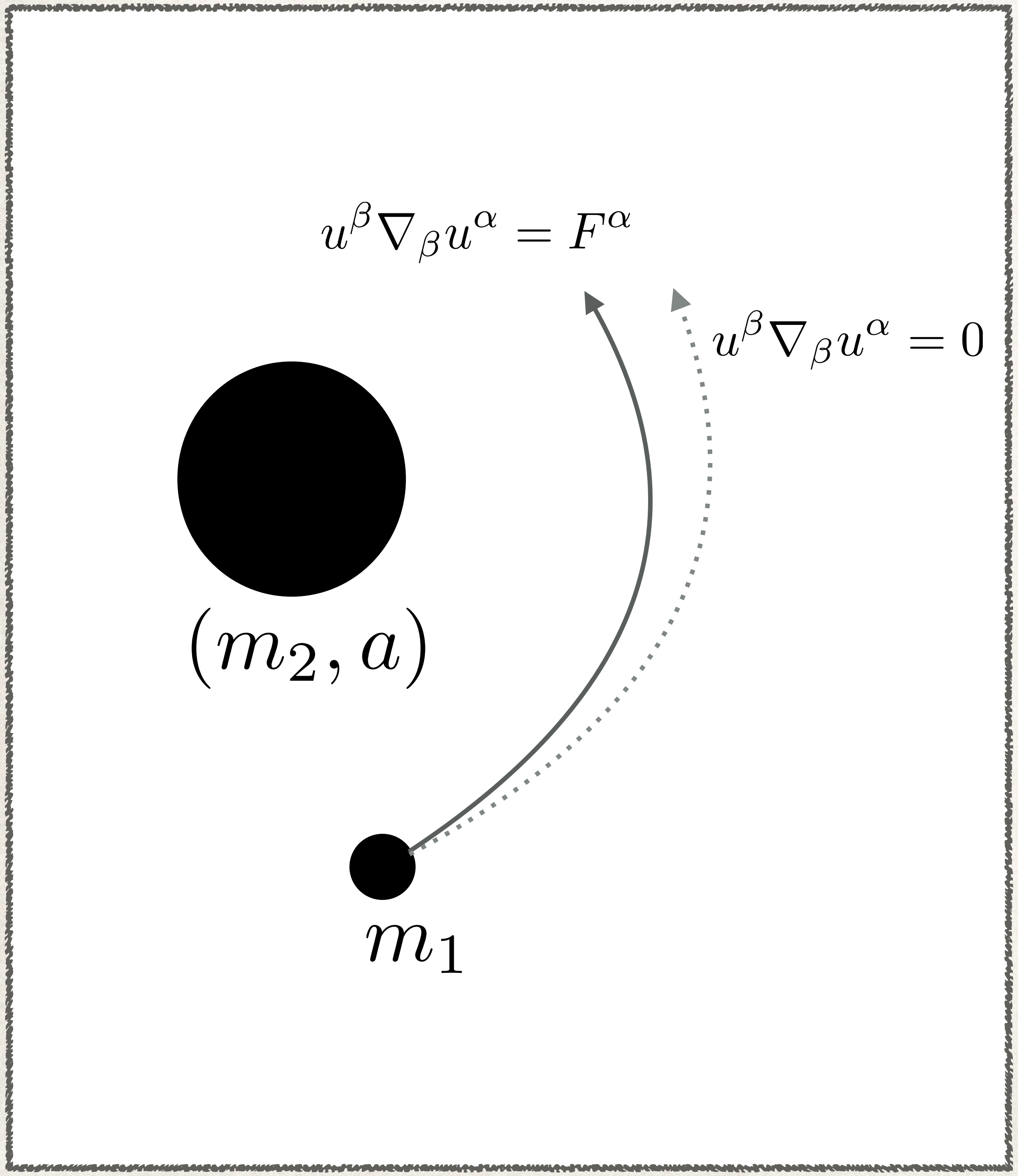
Self-force approach



Expand the metric into a background plus a perturbation

$$\boxed{\begin{aligned} g_{\alpha\beta} &= \mathring{g}_{\alpha\beta} + m_1 h^{(1)}_{\alpha\beta} + m_1^2 h^{(2)}_{\alpha\beta} \\ G_{\alpha\beta}[g_{\alpha\beta}] &= 8\pi T_{\alpha\beta} \end{aligned}}$$

Regularise

$$\boxed{h^{\text{ret}}_{\alpha\beta} = h^R_{\alpha\beta} + h^S_{\alpha\beta}}$$

Motion described by geodesic in perturbed spacetime or equivalently by accelerated motion in background.

Formal prescription at first order

- Foundations and formalism by now well understood (talks by A. Harte, P. Taylor).
- Solve the coupled system of equations for the motion of a point particle and its retarded field.
- Regularise retarded field to obtain finite regular field.

Scalar

$$\Box\Phi^{\rm ret} = -4\pi q \int \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} d\tau$$

$$\Phi^{\rm R} = \Phi^{\rm ret} - \Phi^{\rm S}$$

$$f_a = \nabla_a \Phi^{\rm R}$$

Electromagnetic

$$\Box A_a^{\rm ret} - R_a{}^b A_b^{\rm ret} = -4\pi e \int g_{aa'} u^{a'} \sqrt{-g}\delta_4(x, z(\tau)) d\tau$$

$$A_a^{\rm R} = A_a^{\rm ret} - A_a^{\rm S}$$

$$f^a = g^{ab} u^c A^{\rm R}{}_{[c,b]}$$

Gravitational

$$\Box\bar{h}_{ab}^{\rm ret} + 2C_a{}^c{}_b{}^d \bar{h}_{cd}^{\rm ret} + g_{ab} Z_d{}^{;d} - 2Z_{(a;b)} = -16\pi\mu \int g_{a'(a} u^{a'} g_{b)b'} u^{b'} \sqrt{-g}\delta_4(x, z(\tau)) d\tau$$

$$\bar{h}_{ab}^{\rm R} = \bar{h}_{ab}^{\rm ret} - \bar{h}_{ab}^{\rm S}$$

$$f^a = k^{abcd} \bar{h}^{\rm R}_{bc;d}$$

$$a^\alpha = (g^{\alpha\beta} + u^\alpha u^\beta) f_\beta \qquad \frac{dm}{d\tau} = u^\beta f_\beta$$

Formal prescription at second order

$$\mathcal{D}_{\mu\nu}[h] \equiv \Box h_{\mu\nu} + 2R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} h^{\alpha\beta}$$

$$\mathcal{D}_{\mu\nu}[h^{\mathrm{R}1}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S}1}]$$

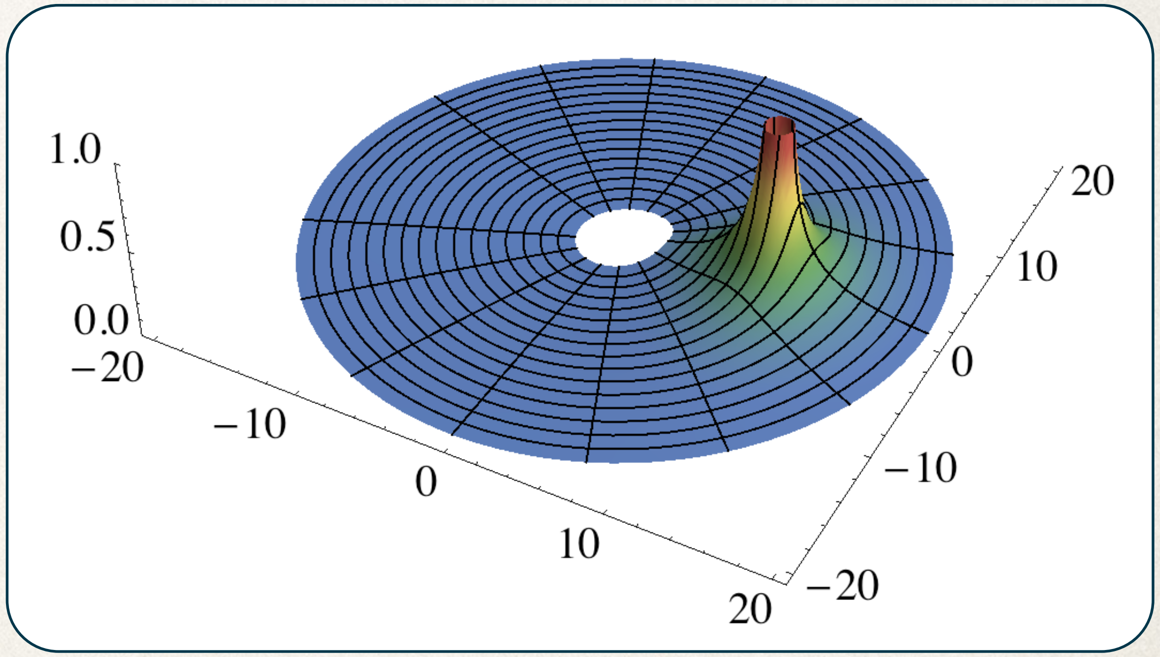
$$\mathcal{D}_{\mu\nu}[h^{\mathrm{R}2}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S}2}] + 2\delta^2 R_{\mu\nu}[h^1, h^1]$$

$$\begin{aligned} \delta^2 R_{\mu\nu}[h,h] \equiv & -\tfrac{1}{2} h^{\mu\nu}\left(2h_{\mu(\alpha;\beta)\nu} - h_{\alpha\beta;\mu\nu} - h_{\mu\nu;\alpha\beta}\right) \\ & + \tfrac{1}{4} h^{\mu\nu}{}_{;\alpha} h_{\mu\nu;\beta} + \tfrac{1}{2} h^{\mu}{}_{\beta}{}^{;\nu}\left(h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}\right) \\ & - \tfrac{1}{2} \bar{h}^{\mu\nu}{}_{;\nu}\left(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}\right) \end{aligned}$$

Why is calculating the self-force so hard?

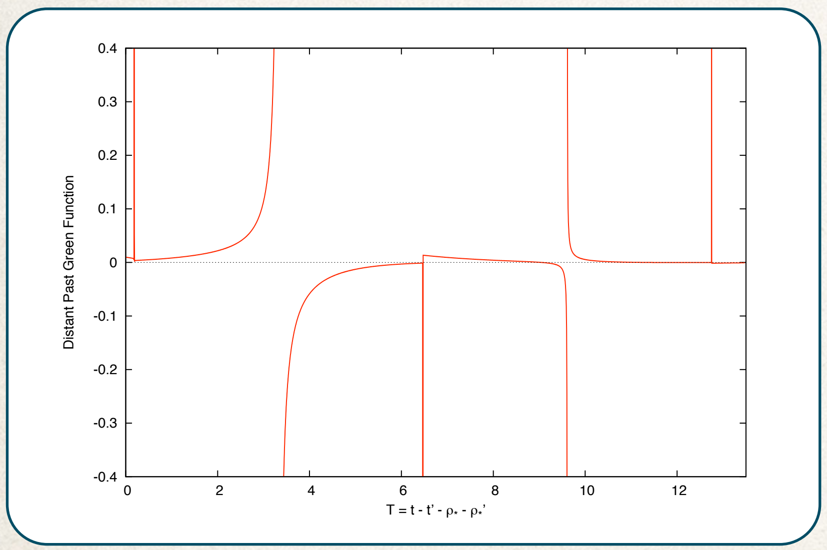
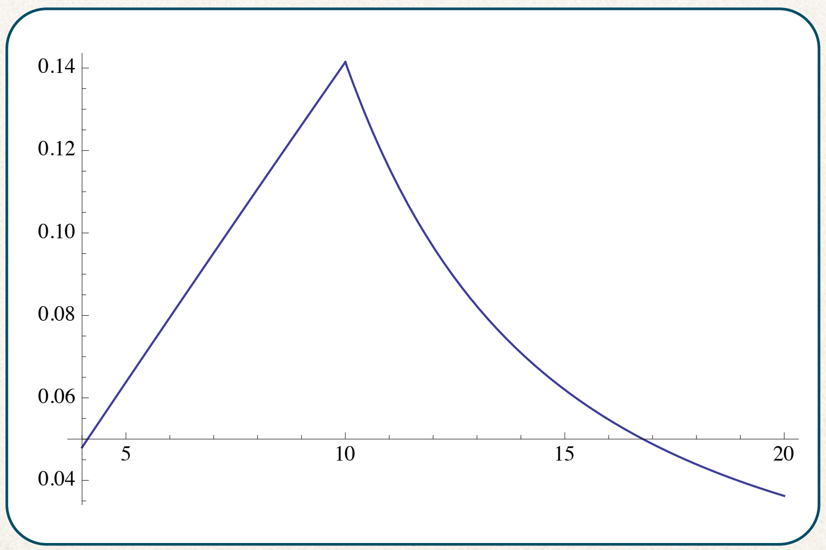
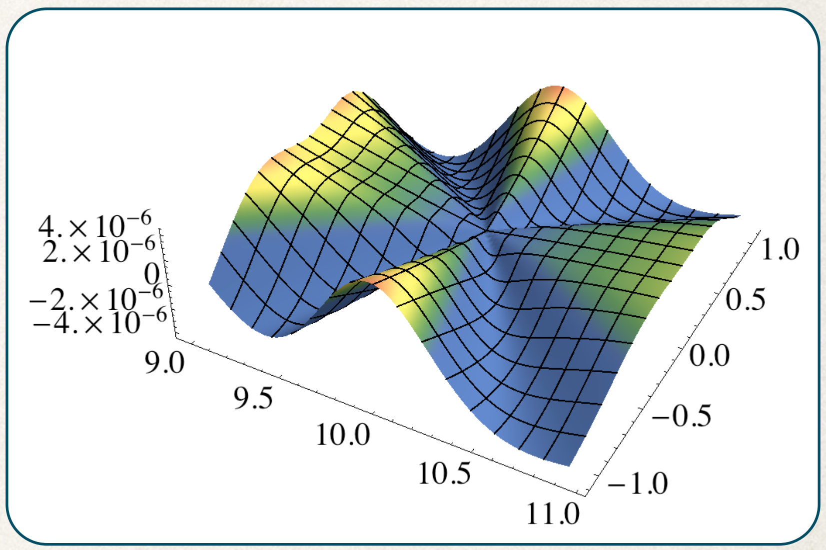
Several considerations arise when trying to turn this formal prescription into a practical numerical scheme:

- System is **coupled**: the field is sourced by the past worldline and the worldline accelerates due to the field => delay differential equation.
- Self-force is **gauge dependent**.
- δ-function sources difficult to handle numerically; retarded field **diverges** like r^{-1}.
- Second order field sourced by first order field **and** more singular (r^{-2}).



Self-force computation strategies

- Several methods have emerged for computing h^{1R}_{ab}, dealing with the numerical issues of point sources, singular fields.
- These broadly fall into three different categories (+ dissipative approx)

Worldline convolution	Mode-sum	Effective source
		
Earliest proposed, latest implemented Arbitrary motion Geometric interpretation	First implemented Most accurate Easiest to implement	Latest proposed Well suited to evolving orbits Well-defined at second order

Worldline convolution

Worldline convolution

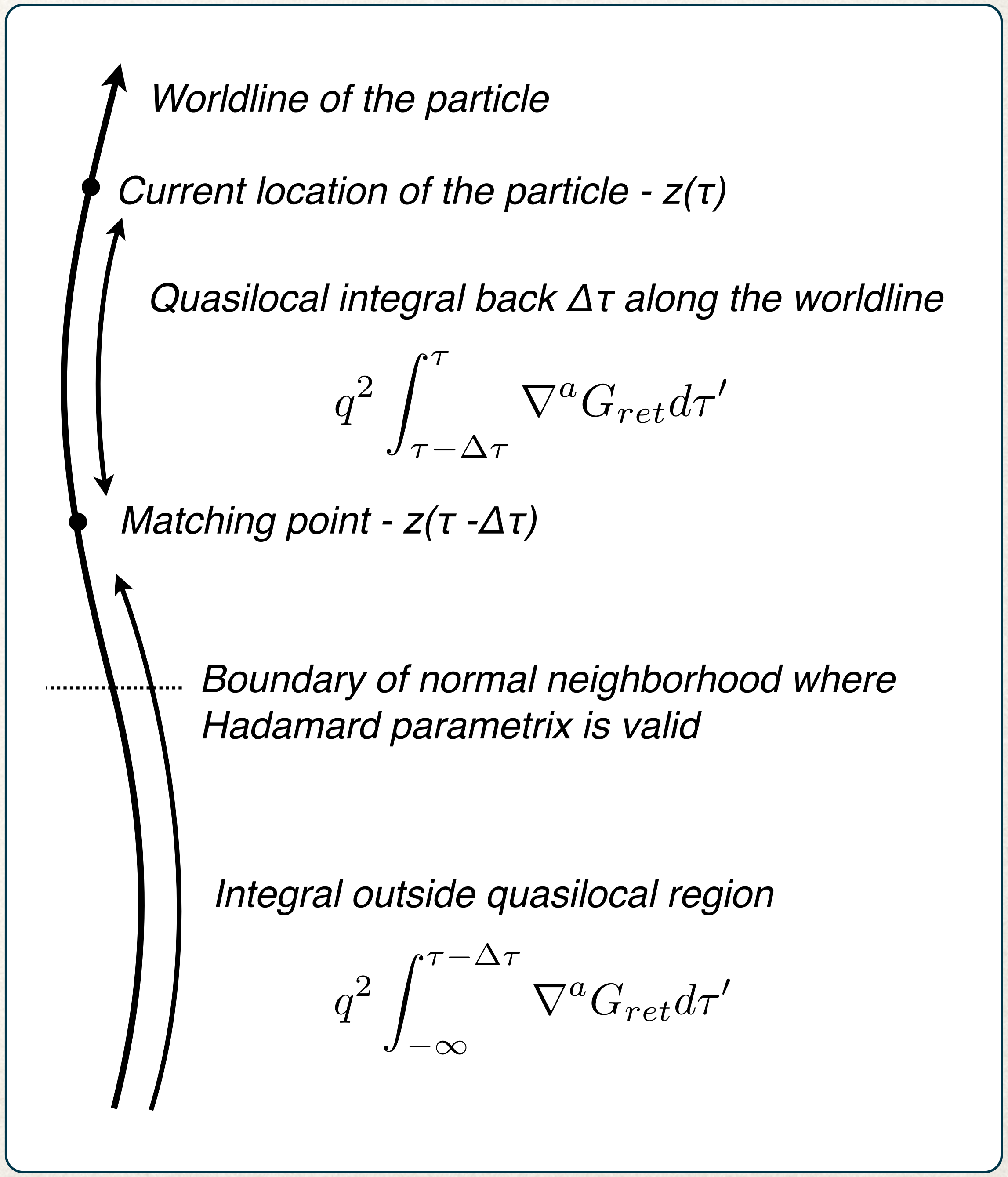
- MiSaTaQuWa equation gives the regularised self-force in terms of local components and a *tail* term.

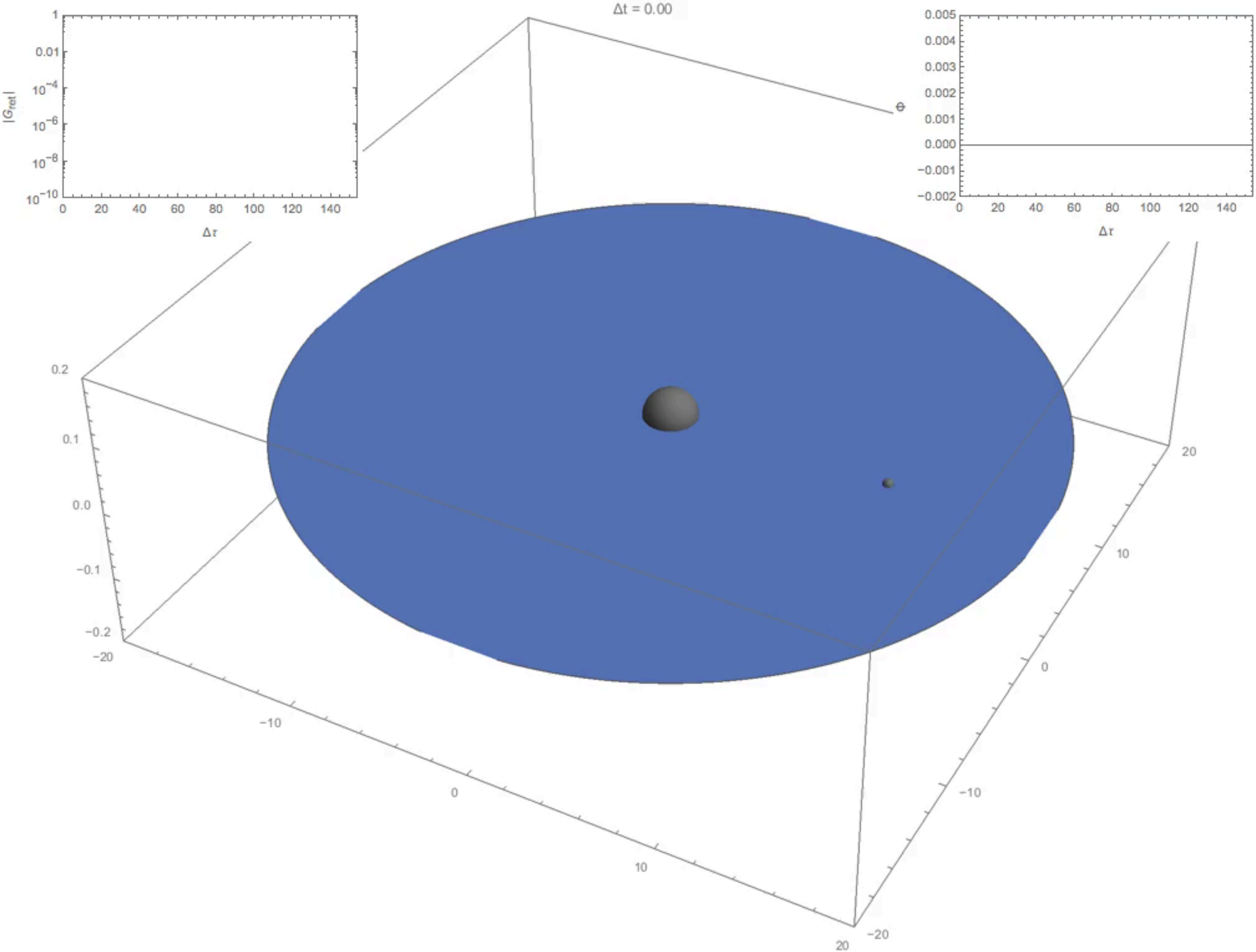
$$\boxed{f^a = (\text{local terms}) + \lim_{\epsilon \to 0} q^2 \int_{-\infty}^{\tau - \epsilon} \nabla^a G_{\text{ret}}(x, x') d\tau'}$$

- **Local terms** are easily calculated.
- **Tail** contains contribution to the self-force from the past.
- If we can compute the Green function along the world-line, then we're done: just integrate this to get the regularised self-force for any orbit.

Matched expansions

- Compute Green function using matched asymptotic expansions.
- Expansions for **recent past** (quasilocal) and **distant past**.
- Recent past - series expansion of Hadamard form; distant past - quasi-normal modes + branch-cut or "numerical Gaussian" or real-frequency integration.
- Stitch expansions in overlapping matching region.





Δt = 0.00
|G_ret|
1
0.01
10^-4
10^-6
10^-8
10^-10
0 20 40 60 80 100 120 140
Δτ
Φ
0.005
0.004
0.003
0.002
0.001
0.000
-0.001
-0.002
0 20 40 60 80 100 120 140
Δτ
0.2
0.1
0.0
-0.1
-0.2
-20
-10
0
10
20
20
10
0
-10
-20

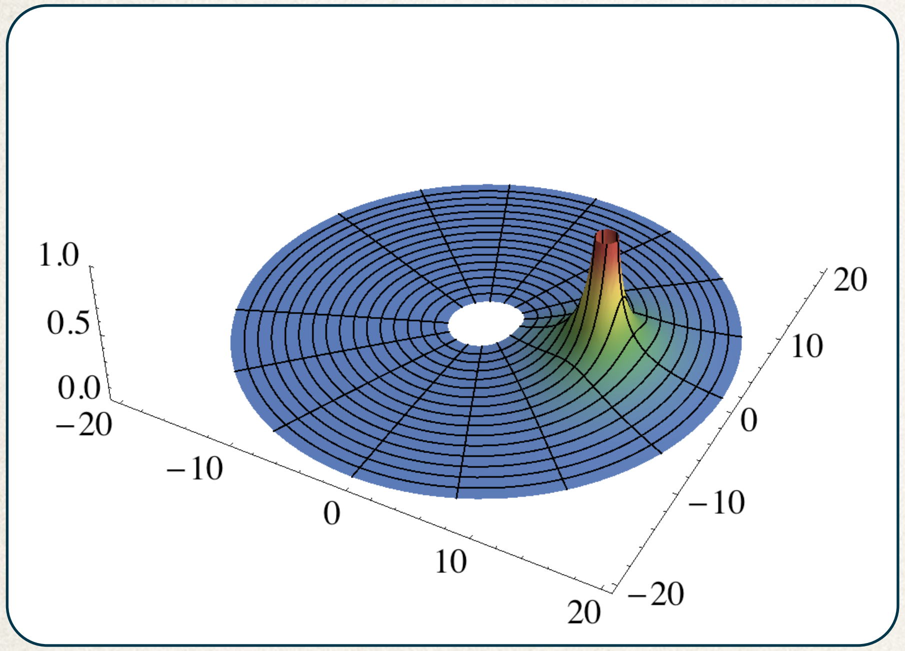
Worldline convolution

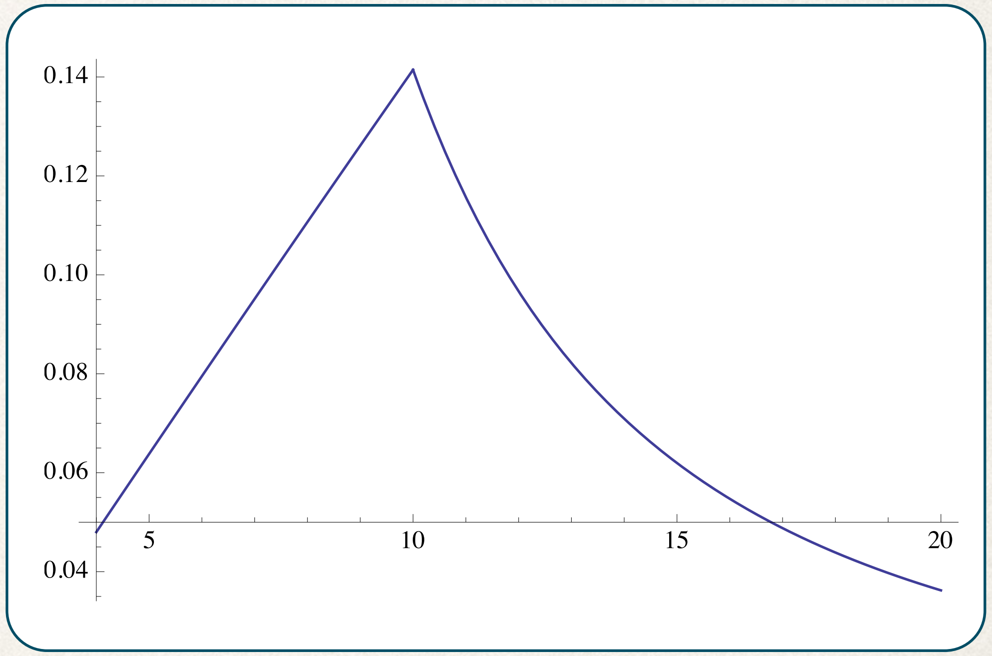
- Advantages:
  - Only need to compute the Green function once and we have the self-force for all orbits.
  - Avoids numerical cancellation by directly computing the regularised field.
  - May yield geometric insight (see talk by J. Thornburg).
  - Green function can be applied to other problems.
- Disadvantages:
  - Computing the Green function can be hard.
  - Have to compute the Green function for all pairs of points x and x' (see talk by C. Galley).
  - Not naturally suited to self-consistent evolution (see talk by C. Galley).
  - Second order not so well understood (see talk by C. Galley).

Mode-sum regularisation

Mode-sum regularisation

- Retarded field diverges close to the world-line.
- Decompose into spherical harmonic modes, the singularity is "smeared out" over a 2-sphere and each *l,m* mode is finite.





- Need to subtract "regularisation parameters" to render the sum over modes finite. [L. Barack and A. Ori, Phys. Rev. D 61, 061502]

Mode-sum regularisation

- Solve a 2D wave equation for each *l,m* mode

$$\boxed{\left[\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - V_l\right]\Phi^{\rm ret}_{lm} = S_{lm}\delta(r - r_0(t))}$$

- Similar equations for electromagnetic and gravitational cases.
- Solution can be found in time domain as either 1+1D or characteristic evolution. δ-function needs careful treatment through particular finite differencing schemes/multi-domain methods.
- In the frequency domain this becomes an ordinary differential equation for each *l,m,ω*. This is particularly convenient for orbits where the number of frequencies is small (e.g. circular orbits). δ-function appears as matching condition between two homogeneous solutions.

Mode-sum regularisation

- In order to regularise, decompose Φ^S into spherical/spheroidal/ spin-weighted spherical/spin-weighted spheroidal harmonic modes

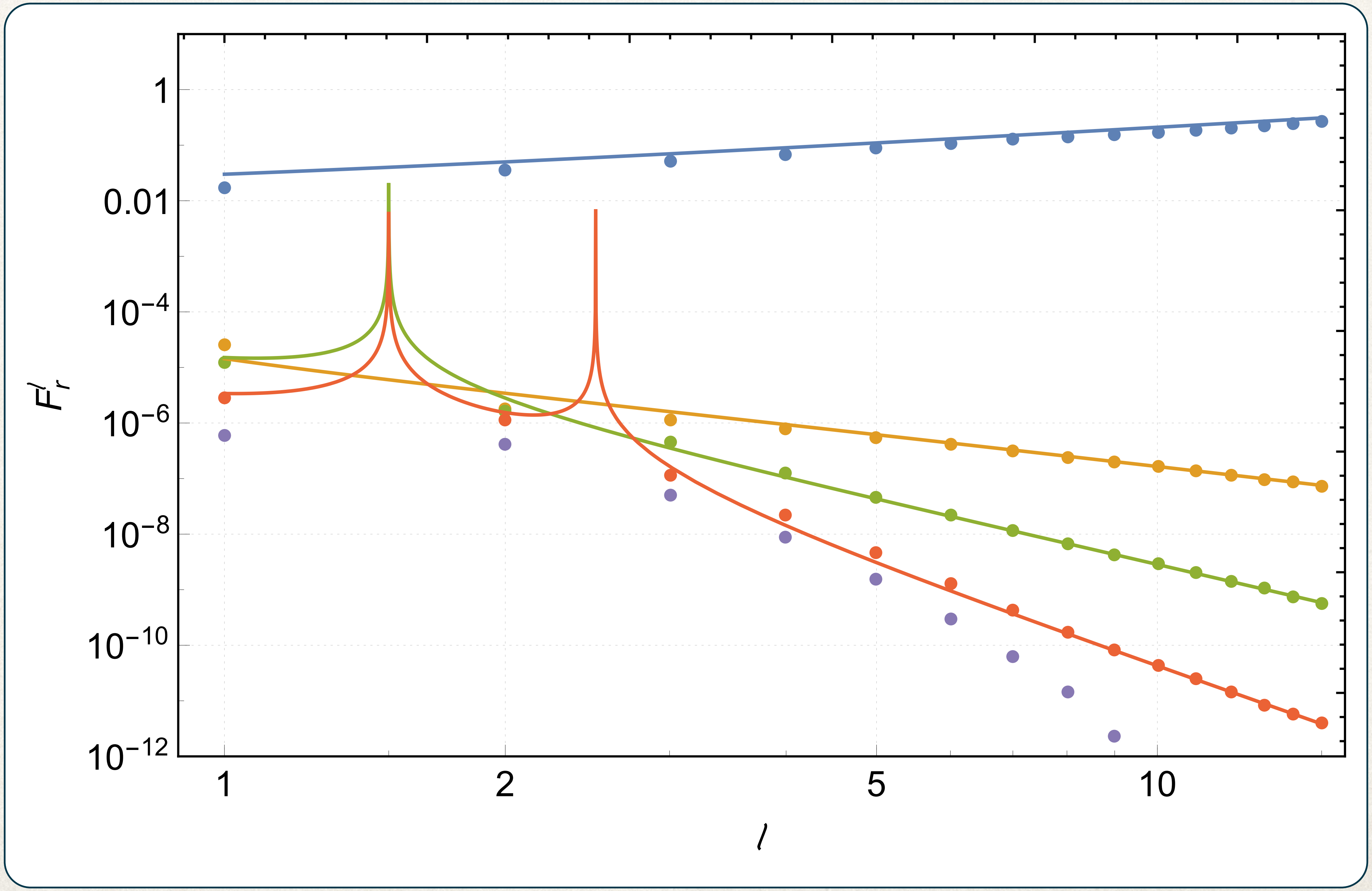
$$\boxed{\Phi^{\mathrm{S}}_{lm}(t,r) = \int \Phi^{\mathrm{S}} Y^{lm*}(\theta,\phi) d\Omega}$$

  and subtract mode by mode.

- Typically only know Φ^S approximately as an expansion for large *l*.
- Coefficients of this expansion are known as *regularisation parameters.*
- Compute a regularised self-force by subtracting regularisation parameters from unregularised self-force

$$\boxed{[f^a_l]^{\mathrm{R}} = \sum_{l=0}^{\infty} f^a_l - A_l(l+\tfrac{1}{2}) - B_l - \cdots}$$

Mode-sum regularisation



Mode-sum regularisation

- Advantages:
  - Suitable for fast, high-accuracy frequency domain calculations (talks by M. van de Meent, L. Barack, P. Giudice, D. Bini, C. Kavanagh).
  - In time domain leads to fast, accurate 1+1D evolutions.
  - Relatively easy to implement.
- Disadvantages:
  - Ill-suited to unbound or highly eccentric orbits (see talk by S. Hopper).
  - Ill-suited to Kerr due to use of spherical harmonics (talks by C. Kavanagh, M. van de Meent).
  - No clear extension to second order (see talk by J. Moxon).
  - Not naturally suited to self-consistent evolution (see talk by J. Moxon).

Effective source regularisation

Effective source regularisation

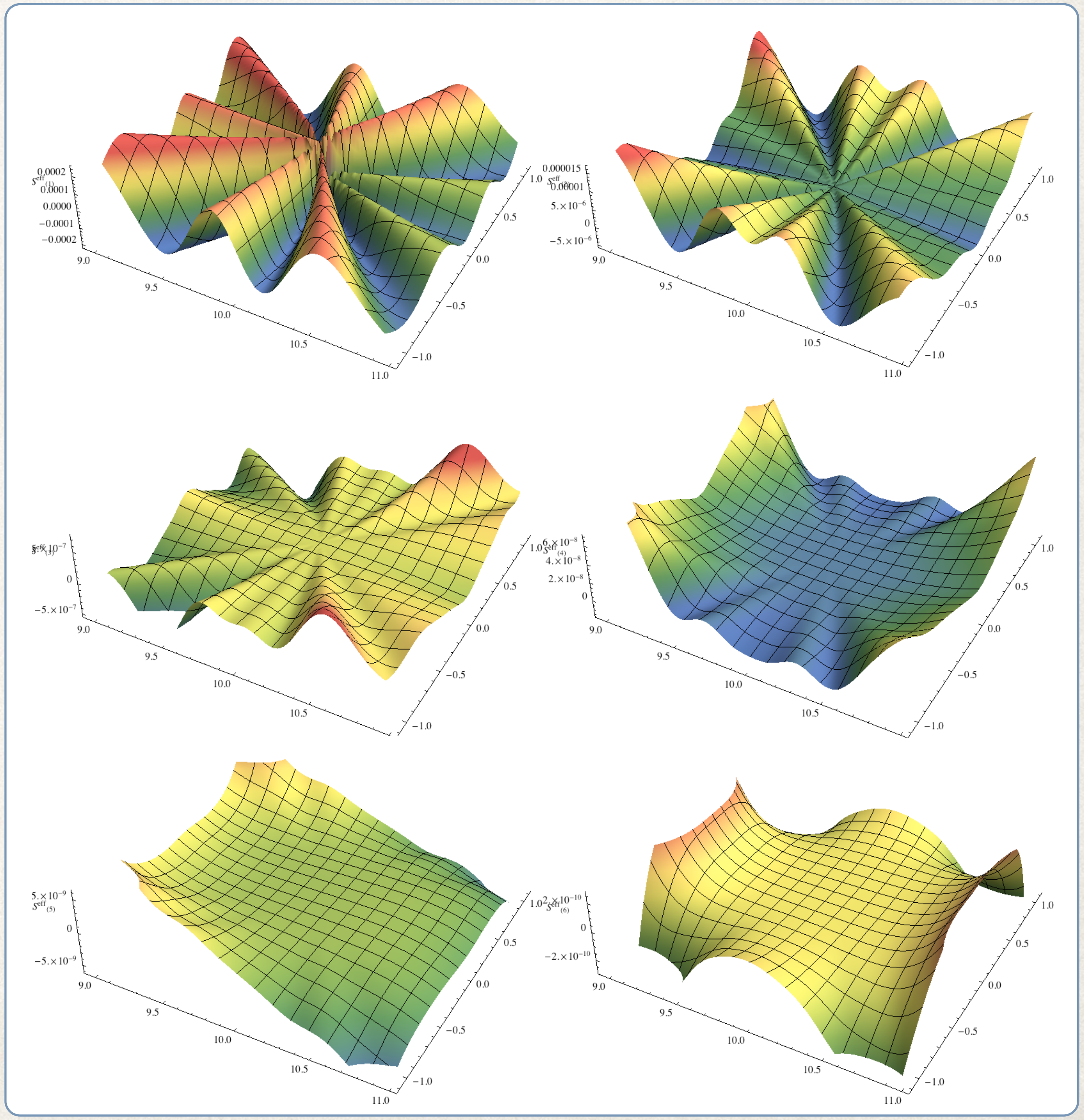
- Derive an evolution equation for Φ^{R}
  [Barack and Golbourn (2007), Detweiler and Vega (2008)]
- Always work with Φ^{R} instead of Φ^{ret}.
- No distributional sources and no singular fields.
- If Φ^{S} is chosen appropriately, then we can directly use Φ^{R} in the equations of motion.

$$\begin{aligned}\Box\Phi^{\mathrm{R}} &= \Box\Phi^{\mathrm{ret}} - \Box\Phi^{\mathrm{S}} \\ &= -4\pi q \int \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} d\tau - \Box\Phi^{\mathrm{S}}\end{aligned}$$

$$\begin{aligned}\frac{Du^\alpha}{d\tau} &= a^\alpha = \frac{\bar{q}}{m(\tau)}(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\beta \Phi^{\mathrm{R}} \\ \frac{dm}{d\tau} &= -\bar{q} u^\beta \nabla_\beta \Phi^{\mathrm{R}}\end{aligned}$$

Effective source regularisation

- If Φ^S is exactly the Detweiler-Whiting singular field, Φ^R is a solution of the homogeneous wave equation.
- If Φ^S is only approximately the Detweiler-Whiting singular field, then the equation for Φ^R has an effective source, *S*.
- *S* typically finite, but of limited differentiability on worldline.



Effective source regularisation

- Advantages:
  - Everything is finite. No distributional sources or singular fields.
  - Does not rely on any underlying symmetry. Can be applied to generic orbits in generic spacetimes (see talk by J. Thornburg).
  - Naturally suited to self-consistent evolution (see talk by P. Diener).
  - Works at second order (see talk by A. Pound)
- Disadvantages:
  - Relatively costly computationally when evolved in 2+1D or 3+1D (see talks by P. Diener, J. Thornburg).
  - Effective source is often a very complicated expression (see talk by J. Thornburg).
  - Problems with evolving Lorenz gauge metric perturbations in time domain.

Results

Case		Worldline	Mode-sum	Effective Source
Scalar	Schwarzschild	circular (apprx) [59]; generic (quasilocal) [67, 68]; generic [69–71]; static [72]; accelerated [73];	radial [74]; circular [75–78]; eccentric [79–83]; static [72];	circular [56, 57, 65, 84–86]; eccentric [87]; evolving [88];
	Kerr	generic [68]; accelerated [73];	circular [89]; equatorial [90, 91]; inclined circular [92]; accelerated [93]; static [94, 95];	circular [96]; eccentric [97];
EM	Schwarzschild	static [72];	static [72]; eccentric [82, 98]; static (Schwarzschild-de Sitter) [99]; radial (Reissner-Nordström) [100];	—
	Kerr	—	equatorial [90]; accelerated [93];	—
Gravity	Schwarzschild	generic (quasilocal) [101];	radial [102]; circular [103–111]; eccentric [82, 112–120]; osculating [121];	circular [122];
	Kerr	circular (quasilocal) [59]; branch cut [123];	equatorial [90]; accelerated [93]; circular [119, 124];	circular [125]; generic [126];

Table from: B. Wardell, Prog. Theor. Phys. 179

Results from self-force calculations

- Self-force methods produce highly accurate (>20 significant digits) results
- Probe strong-field regime, highly eccentric orbits (talks by J. Thornburg, M. Colleoni)
- Potential for “exact” results: “experimental mathematics”
- “Exact” functional methods
- Highly spinning black holes.

Inspiral into Gargantua

Samuel E. Gralla,[1,2] Scott A. Hughes,[3,4] and Niels Warburton[4,5]

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[2] *Center for the Fundamental Laws of Nature, Harvard University, Cambridge, MA 02138, USA*
[3] *Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*
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We model the inspiral of a compact object into a more massive black hole rotating very near the theoretical maximum. We find that once the body enters the near-horizon regime the gravitational radiation is characterized by a constant frequency, equal to (twice) the horizon frequency, with an exponentially damped profile. This contrasts with the usual “chirping” behavior and, if detected, would constitute a “smoking gun” for a near-extremal black hole in nature.

I. INTRODUCTION

General relativity imposes a hard upper limit on how fast a black hole can rotate. For a black hole of mass M, the angular momentum J must satisfy

$$J \leq GM^2/c, \tag{1}$$

where G is Newton’s constant and c is the speed of light (both hereafter set to unity). Above this value, the event horizon disappears and the spacetime contains a naked singularity. It is impossible to spin up a black hole above this limit with any continuous process featuring reasonable matter [1], and there is much evidence in favor of the “cosmic censorship conjecture” [2] that *no* generic initial data can produce a naked singularity.

Black holes that saturate the bound (1) are known as extremal. More generally, extremal black holes are defined as those with zero Hawking temperature. Extremal black holes play a key role in many theoretical arguments investigating the nature of classical and quantum gravity, such as cosmic censorship [3] and the quantum nature of black hole entropy [4]. They have near-horizon regions that possess additional emergent symmetries [5] and may be governed by a holographic duality [6] in the spirit of AdS/CFT [7]. At least in parameter space, they are a hair’s breath from being naked singularities, the existence

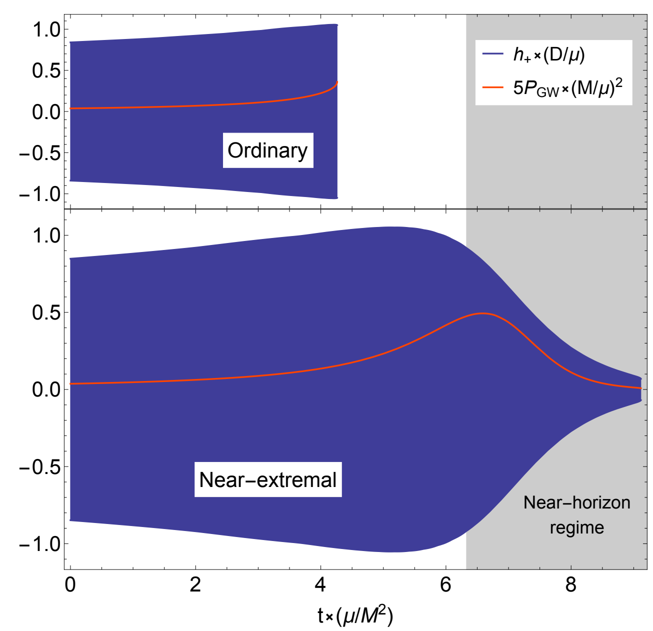


FIG. 1. Gravitational waveforms from equatorial, quasi-circular inspiral into ordinary and near-extremal black holes. The black hole spins are $a/M = 0.97$ and $a/M = 1 - 10^{-9}$, respectively. We show the h_+ component for a system viewed face-on. The waveform begins when the particle crosses $r = 3.3M$ and ends when the particle reaches the ISCO; we do

Invariants of a perturbed black hole

- 2008 — Redshift invariant (Detweiler ’08; Akcay, et. al. ’15)
- 2009 — Shift in the innermost stable circular orbit (Barack & Sago)
- Periastron advance, mildly-eccentric orbit (Barack & Sago)
- 2014 — Geodetic spin-precession (Dolan, et al.)
- 2015 — Tidal eigenvalues (Dolan, et al.)
- Octupolar invariants (Nolan, et al.)
- 2016? — Second order redshift invariant (Pound, et al.; Detweiler, et al. - see talks by A. Pound, A. Heffernan, H. Chen, J. Thompson)

Classification: conservative and dissipative invariants

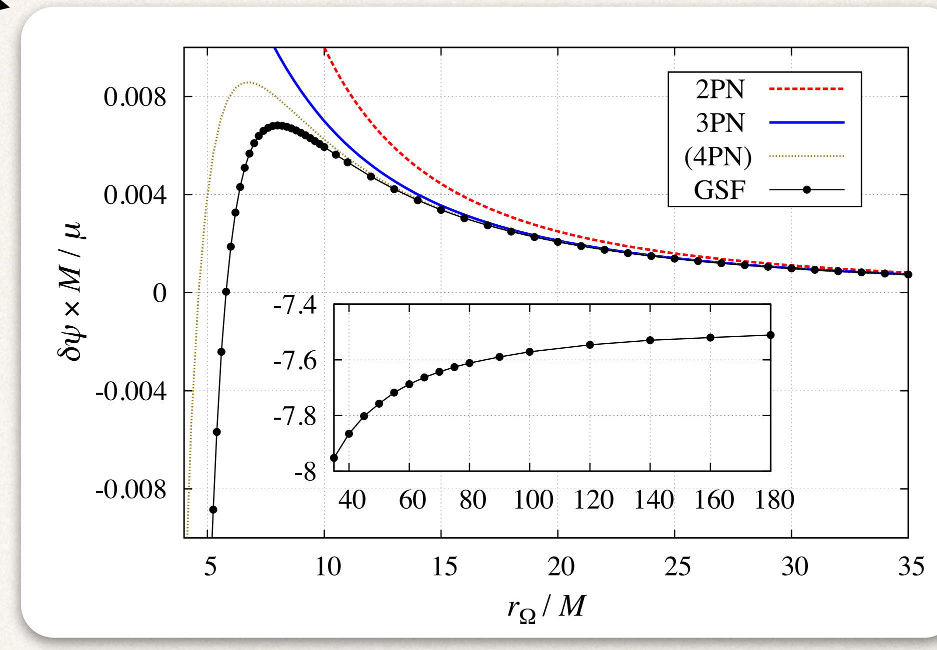
∇^n	Conservative	Dissipative
0th	ΔU *(redshift)*	
1st	$\Delta\psi$ *(geodetic spin precession)*	F_3 *(fluxes of radiation)*
2nd	E_{11}, E_{22}, B_{12} *(tidal tensor)*	B_{23} *(tidal tensor)*
3rd	E_{111}, E_{122}, B_{211}, B_{222} *(octupoles)*	E_{311}, E_{322}, B_{123} *(octupoles)*

Extracting information from self-force calculations

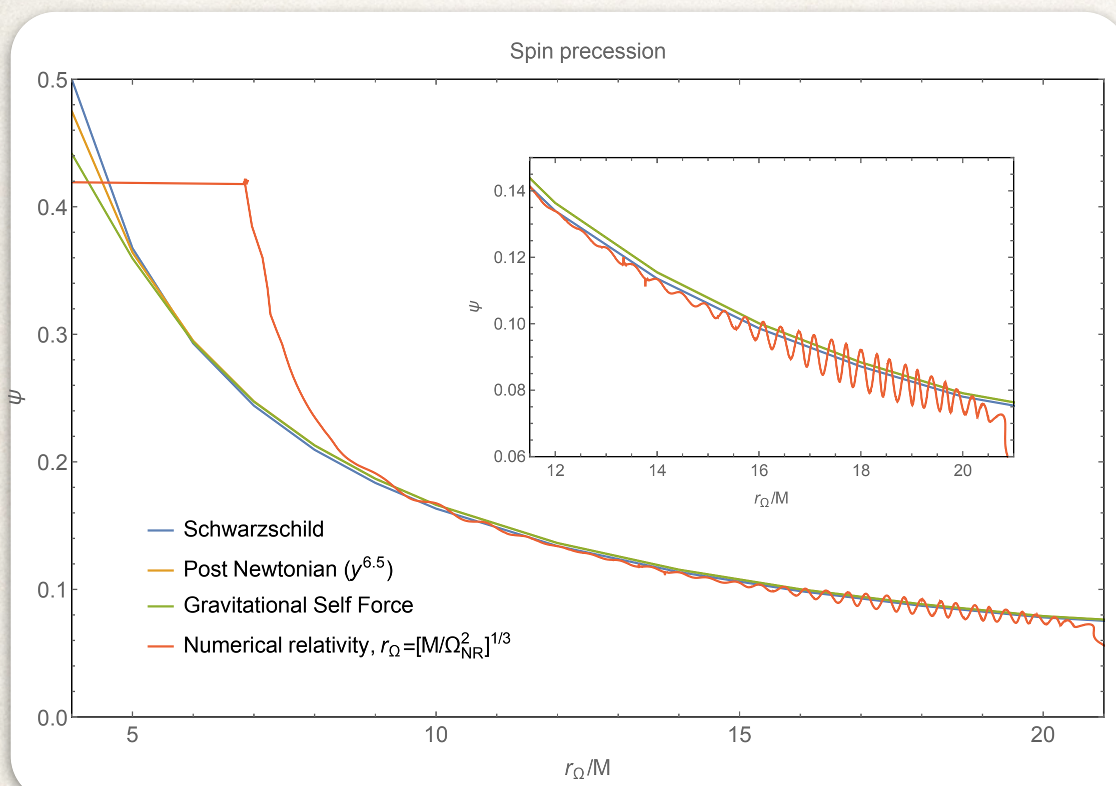
Comparison with post-Newtonian theory - talk by C. Evans, J. Vines.

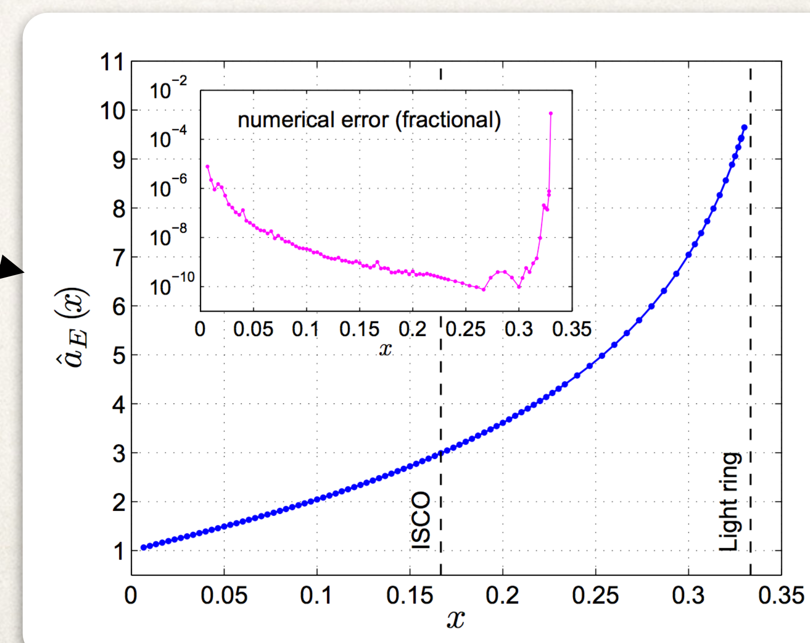
Comparison with Numerical Relativity - see talk by A. Zimmerman.

Calibrate EOB - see talks by T. Damour, D. Bini, C. Kavanagh, T. Hinderer, J. Steinhoff.



Dolan, et. al.



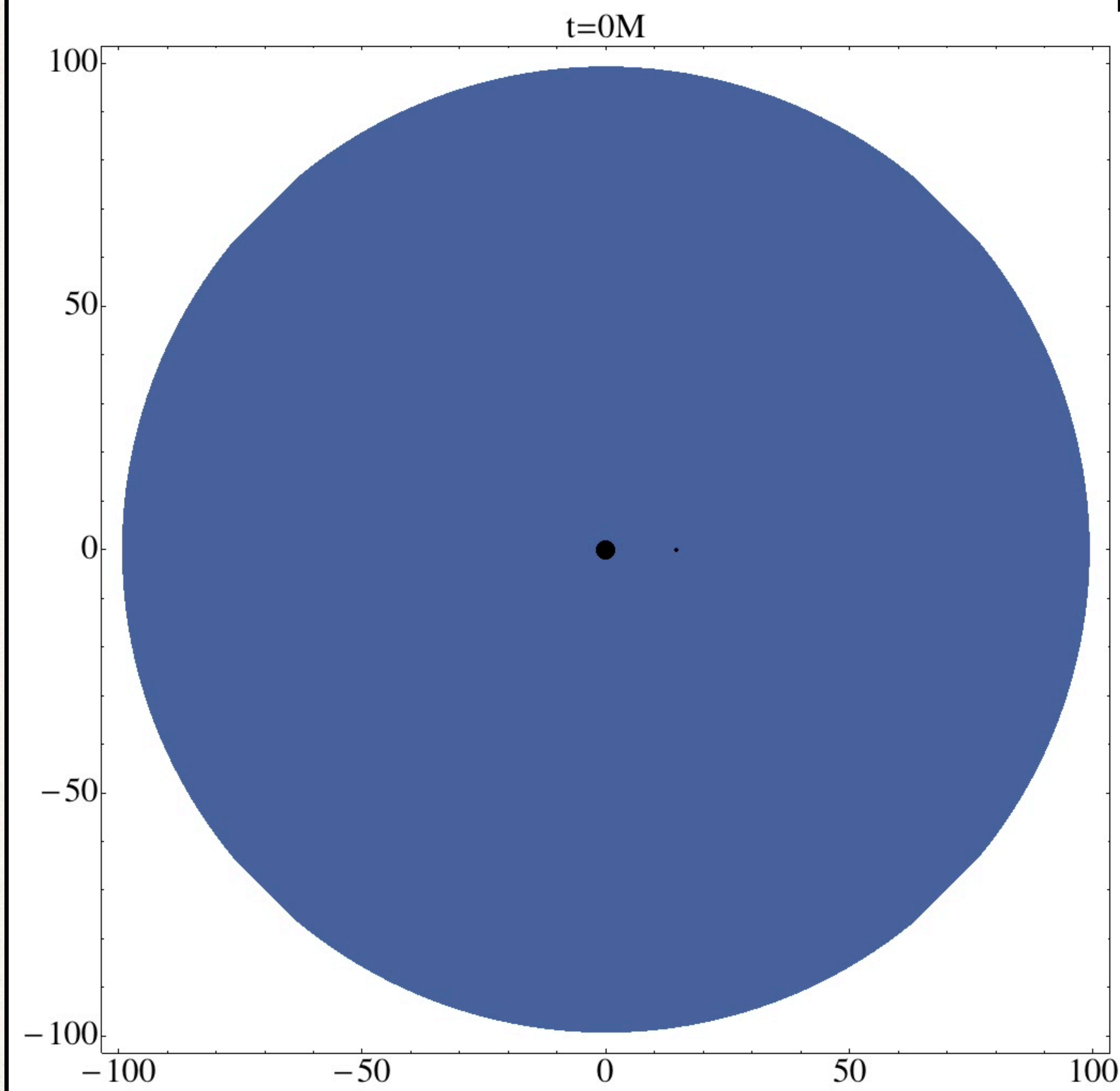


Bini & Damour

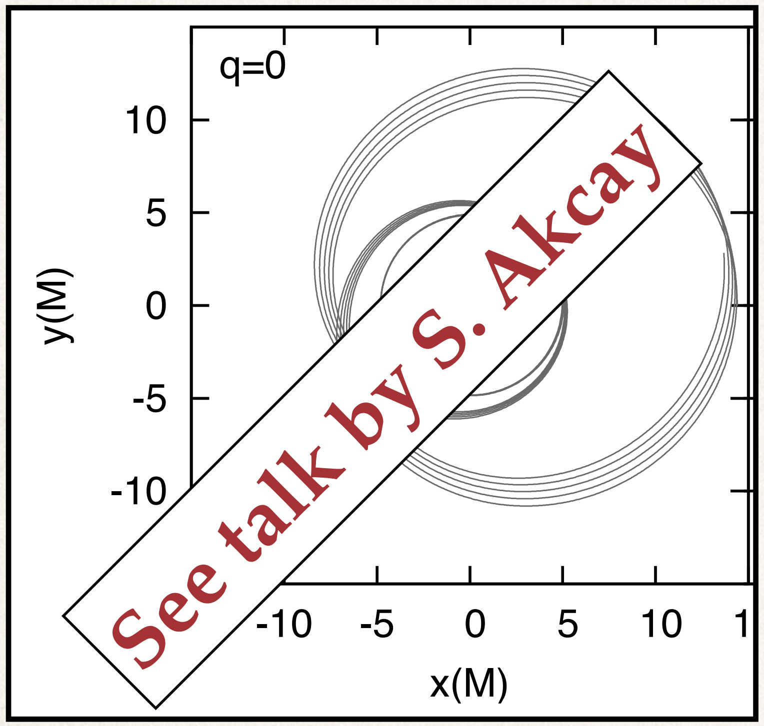
Akcay. et al.

Orbital evolution

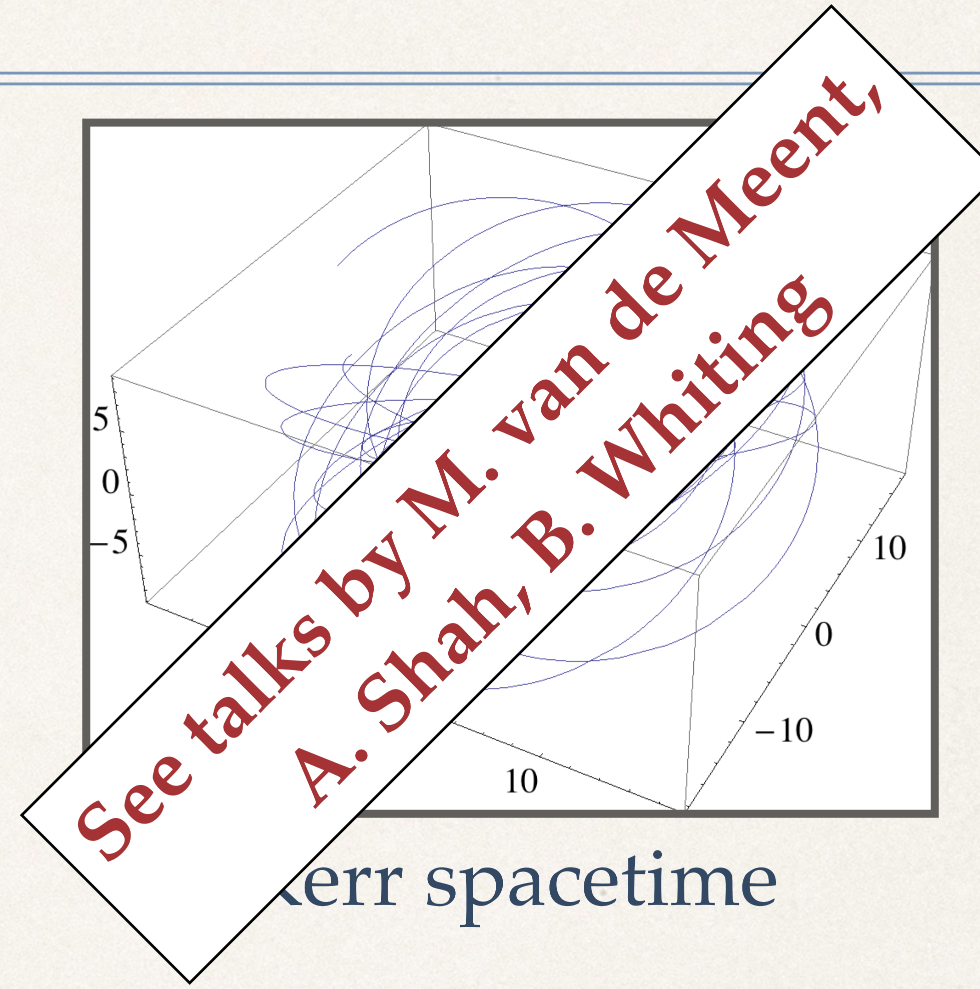
- Once we have the self-force, we need to use it to evolve orbits.
- "Geodesic" approximation vs. self-consistent evolution.
- Inclusion of spin effects.
- Talks by T. Osburn, P. Diener, C. Galley, S. Isoyama, N. Warburton, R. Fujita.



A lot done, more to do…



Invariants for eccentric orbits



Kerr spacetime

$$g_{ab} = \mathring{g}_{ab} + h^1_{ab} + h^2_{ab}$$

Second order perturbation theory

Second order conservative effects

Generalised redshift invariant for circular orbits
[Pound, Phys. Rev. D90, 084039]

$$U_0(\Omega) = \left(1 - 3M/r_\Omega\right)^{1/2} \qquad \Omega = \sqrt{\frac{M}{r_\Omega^3}}$$

$$\begin{aligned}\tilde{U} =& U_0(\Omega)\Big[1 + \tfrac{1}{2}\epsilon h^{\mathrm{R}1}_{u_0 u_0} + \epsilon^2\Big(\tfrac{1}{2}h^{\mathrm{R}2}_{u_0 u_0} + \tfrac{3}{8}(h^{\mathrm{R}1}_{u_0 u_0})^2 \\ & - \tfrac{r_\Omega^3}{6M^2}(F_{1r})^2\left(1 - 3M/r_\Omega\right)\Big)\Big]\end{aligned}$$

Second order field equations

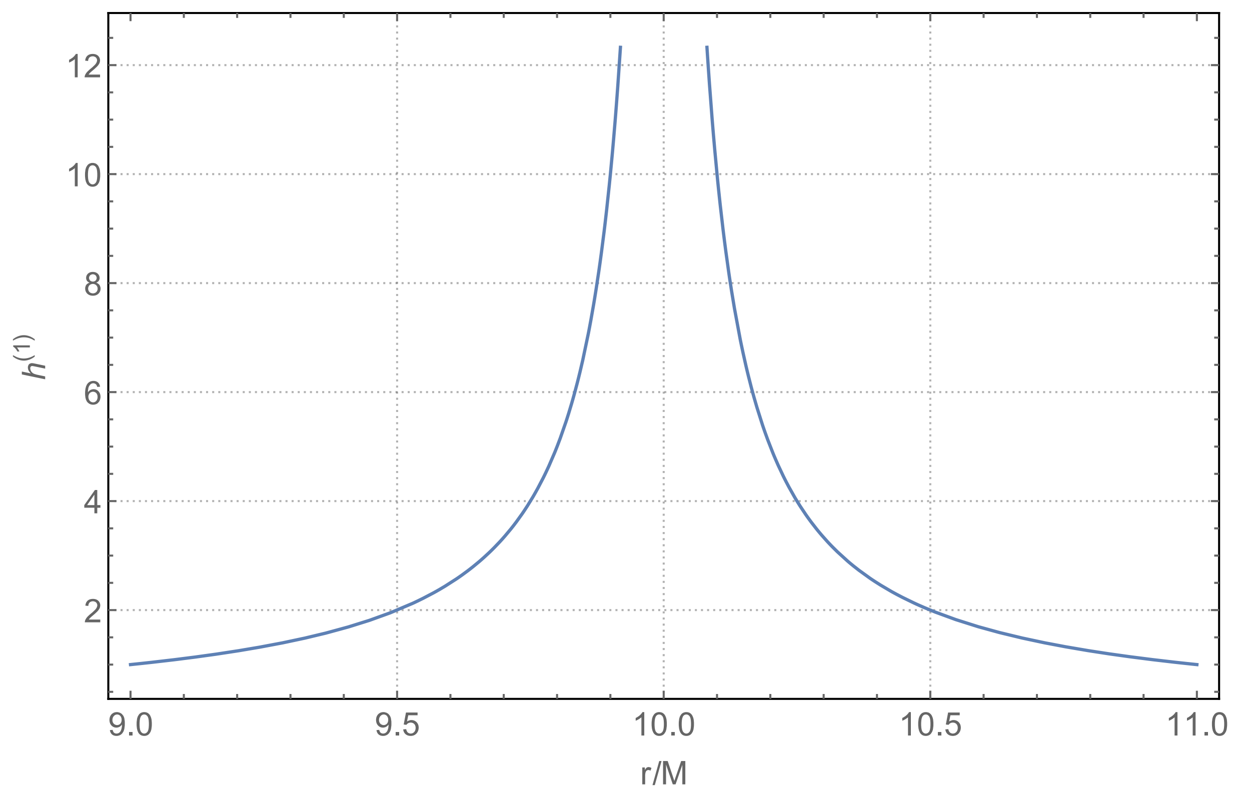
$$\boxed{\mathcal{D}_{\mu\nu}[h] \equiv \Box h_{\mu\nu} + 2R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} h_{\alpha\beta}}$$

$$\boxed{\begin{gathered} \mathcal{D}_{\mu\nu}[h^{\mathrm{R1}}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S1}}] \\ \mathcal{D}_{\mu\nu}[h^{\mathrm{R2}}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S2}}] + \delta^2 R_{\mu\nu}[h^1, h^1] \end{gathered}}$$

$$\boxed{\begin{aligned} \delta^2 R_{\alpha\beta}[h,h] \equiv & -\tfrac{1}{2} h^{\mu\nu}\left(2h_{\mu(\alpha;\beta)\nu} - h_{\alpha\beta;\mu\nu} - h_{\mu\nu;\alpha\beta}\right) \\ & + \tfrac{1}{4} h^{\mu\nu}{}_{;\alpha} h_{\mu\nu;\beta} + \tfrac{1}{2} h^{\mu}{}_{\beta}{}^{;\nu}\left(h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}\right) \\ & - \tfrac{1}{2} \bar{h}^{\mu\nu}{}_{;\nu}\left(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}\right) \end{aligned}}$$

Challenges at Second order

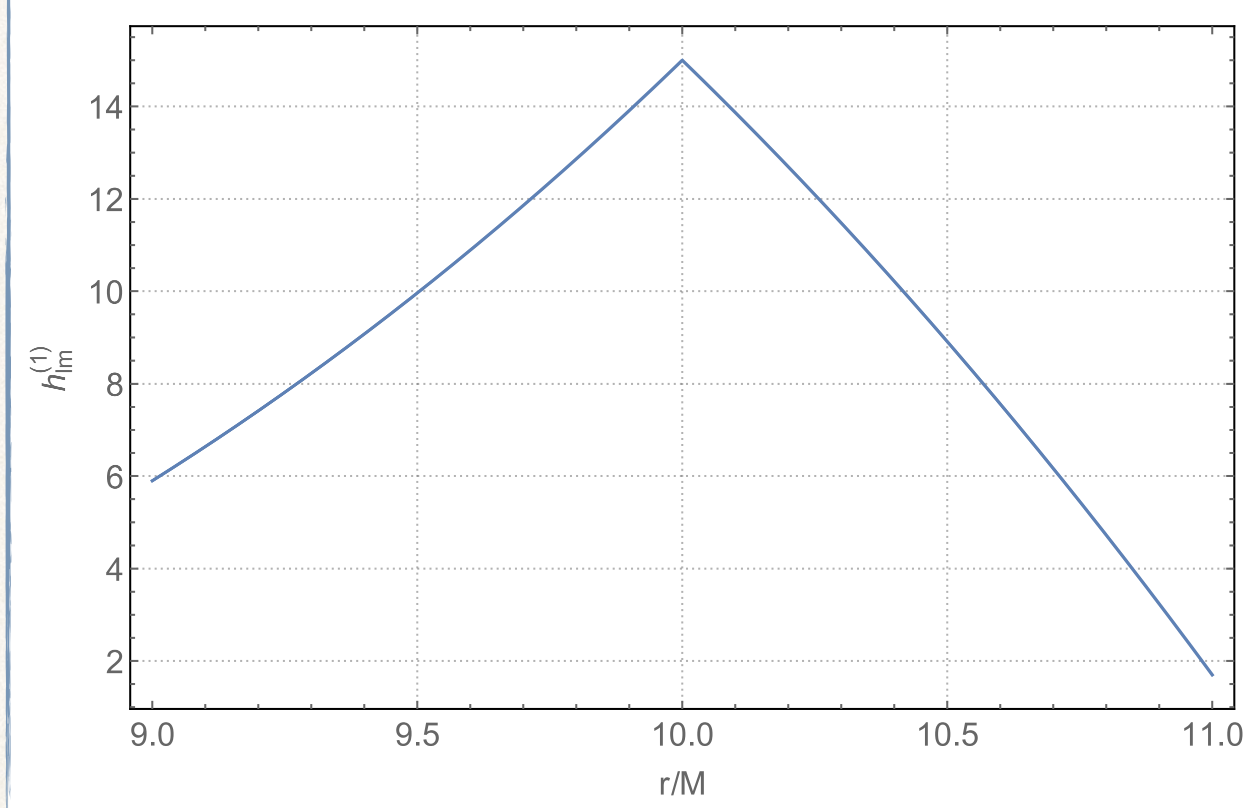
- Second order gravitational self-force will require high accuracy $\Rightarrow$ Frequency domain.



First order metric perturbation ~ $1/(r-r_0)$

Challenges at Second order

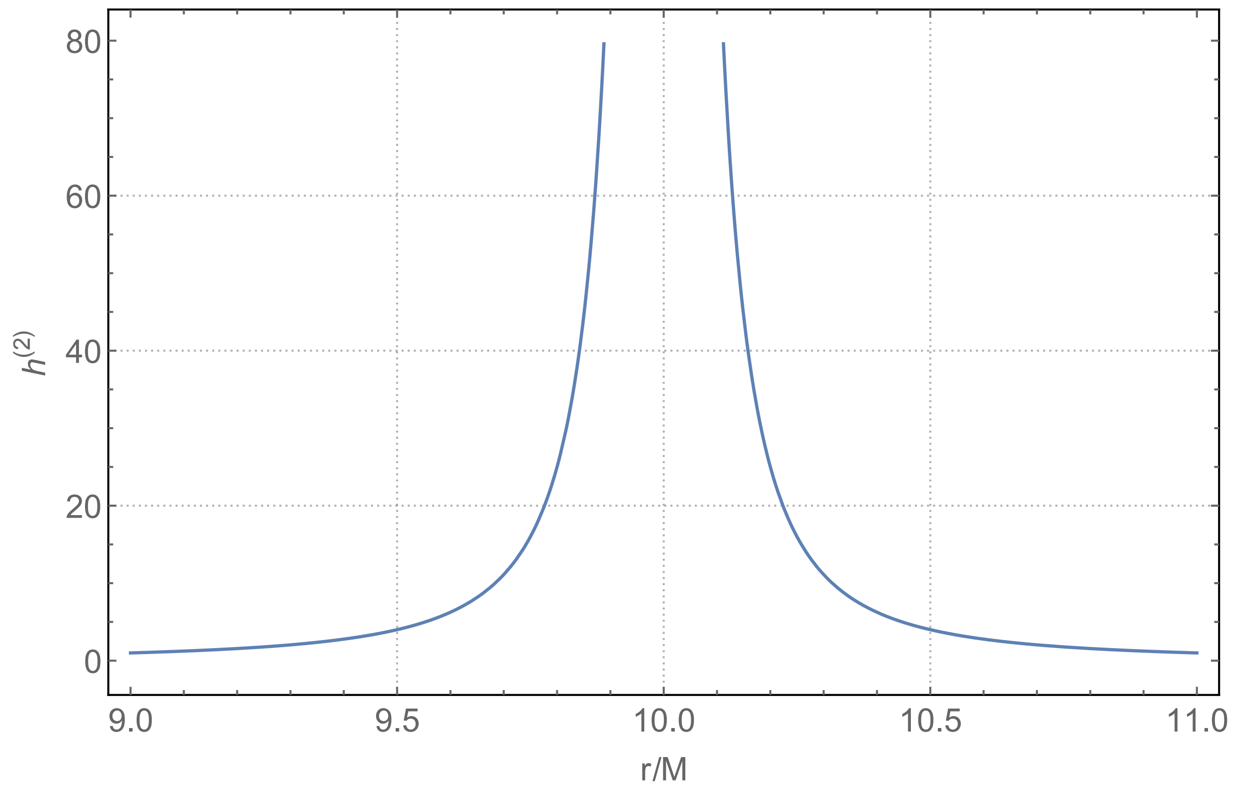
- Second order gravitational self-force will require high accuracy ⇒ Frequency domain.
- Spherical harmonic modes at first order finite on world line ⇒ mode-sum regularisation.



First order modes ~ $|r-r_0| / (r-r_0)$

Challenges at Second order

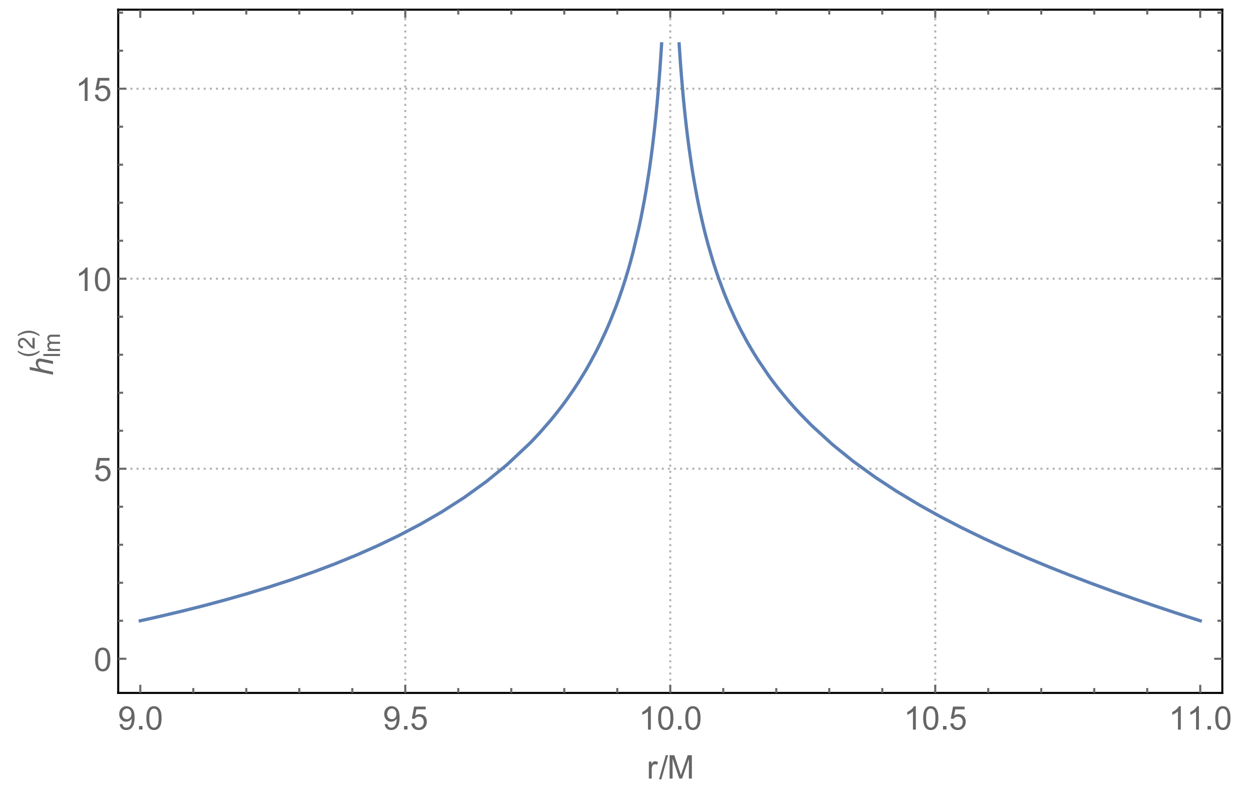
- Second order gravitational self-force will require high accuracy $\Rightarrow$ Frequency domain.
- Spherical harmonic modes at first order finite on world line $\Rightarrow$ mode-sum regularisation.
- Second order metric more singular.



Second order perturbation ~ $1/(r-r_0)^2$

Challenges at Second order

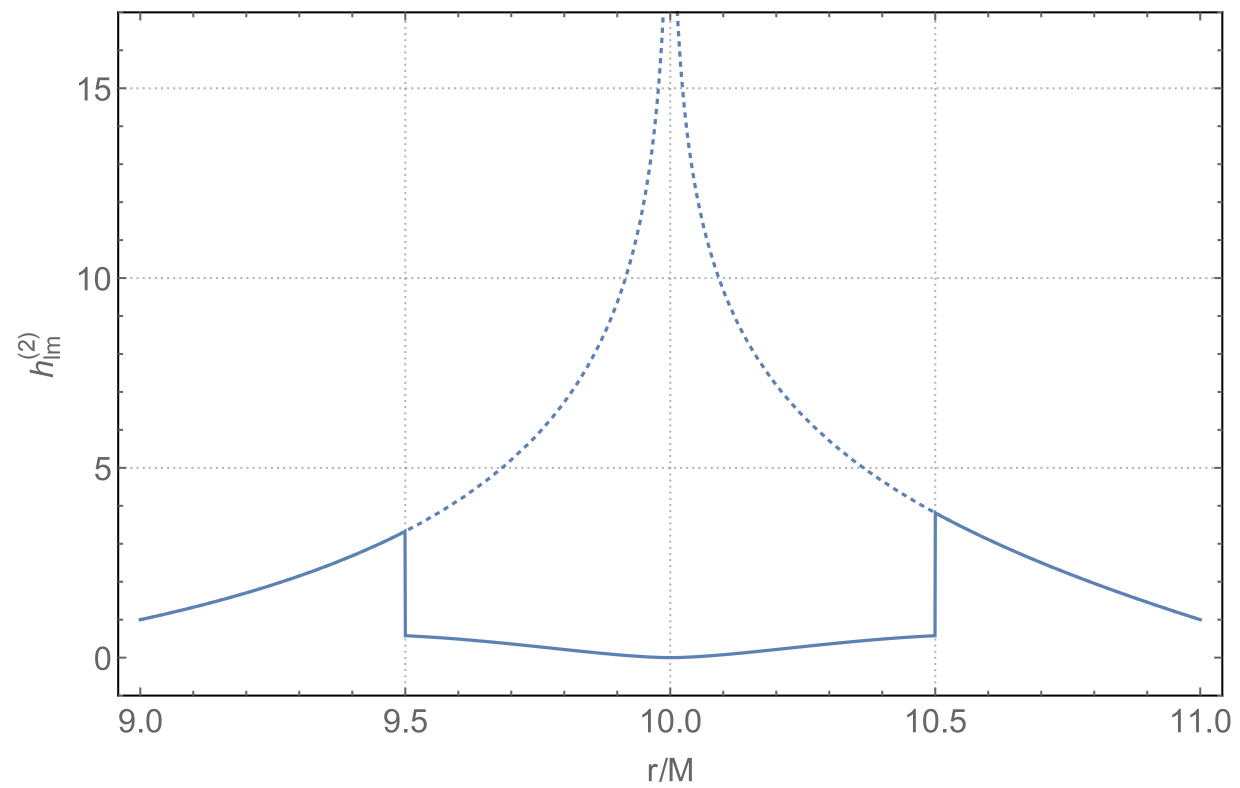
- Second order gravitational self-force will require high accuracy $\Rightarrow$ Frequency domain.
- Spherical harmonic modes at first order finite on world line $\Rightarrow$ mode-sum regularisation.
- Second order metric more singular.
- Second order modes diverge logarithmically.



Second order modes ~ $\log|r-r_0|$

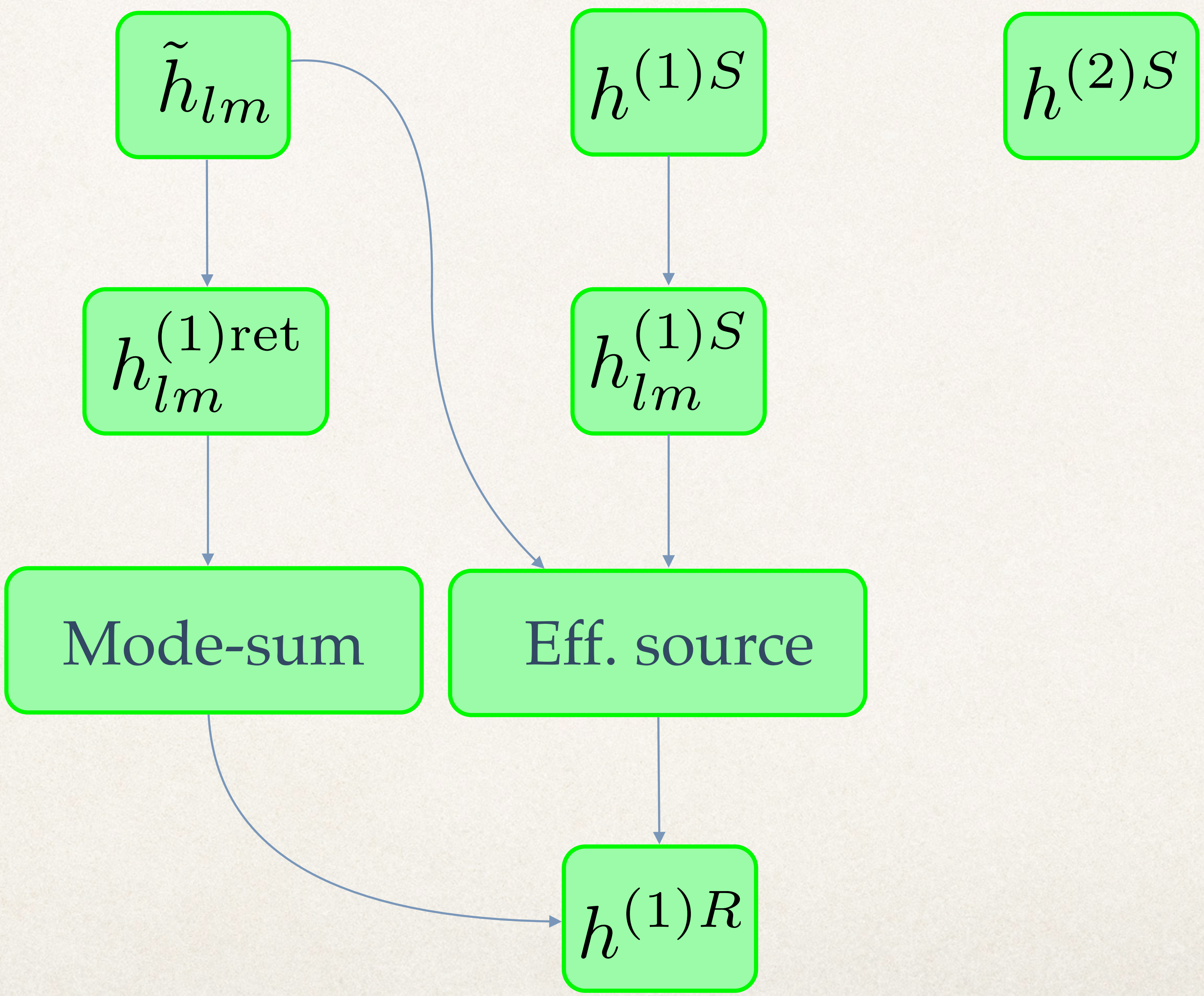
Challenges at Second order

- Second order gravitational self-force will require high accuracy $\Rightarrow$ Frequency domain.
- Spherical harmonic modes at first order finite on world line $\Rightarrow$ mode-sum regularisation.
- Second order metric more singular.
- Second order modes diverge logarithmically.
- Avoid computing retarded field on world line $\Rightarrow$ effective source.



Second order modes ~ $\log|r-r_0|$

Towards second order self-force



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Practical, covariant puncture for second-order self-force calculations

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Accurately modeling an extreme-mass-ratio inspiral requires knowledge of the second-order gravitational self-force on the inspiraling small object. Recently, numerical puncture schemes have been formulated to calculate this force, and their essential analytical ingredients have been derived from first principles. However, the "puncture," a local representation of the small object's self-field, in each of these schemes has been presented only in a local coordinate system centered on the small object, while a numerical implementation will require the puncture in coordinates covering the entire numerical domain. In this paper we provide an explicit covariant self-field as a local expansion in terms of Synge's world function. The self-field is written in the Lorenz gauge, in an arbitrary vacuum background, and in forms suitable for both self-consistent and Gralla-Wald-type representations of the object's trajectory. We illustrate the local expansion's utility by sketching the procedure of constructing from it a numerically practical puncture in any chosen coordinate system.

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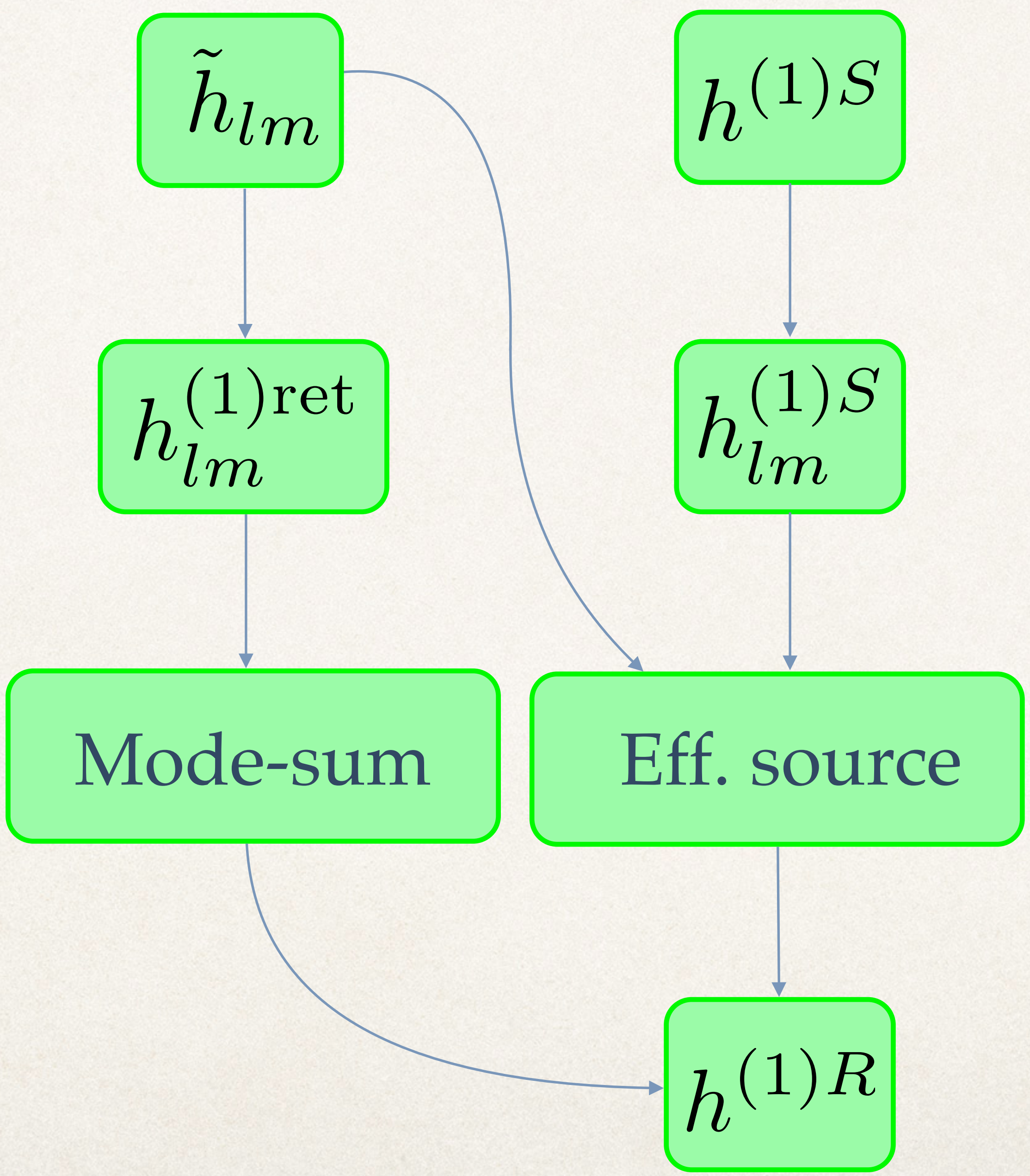
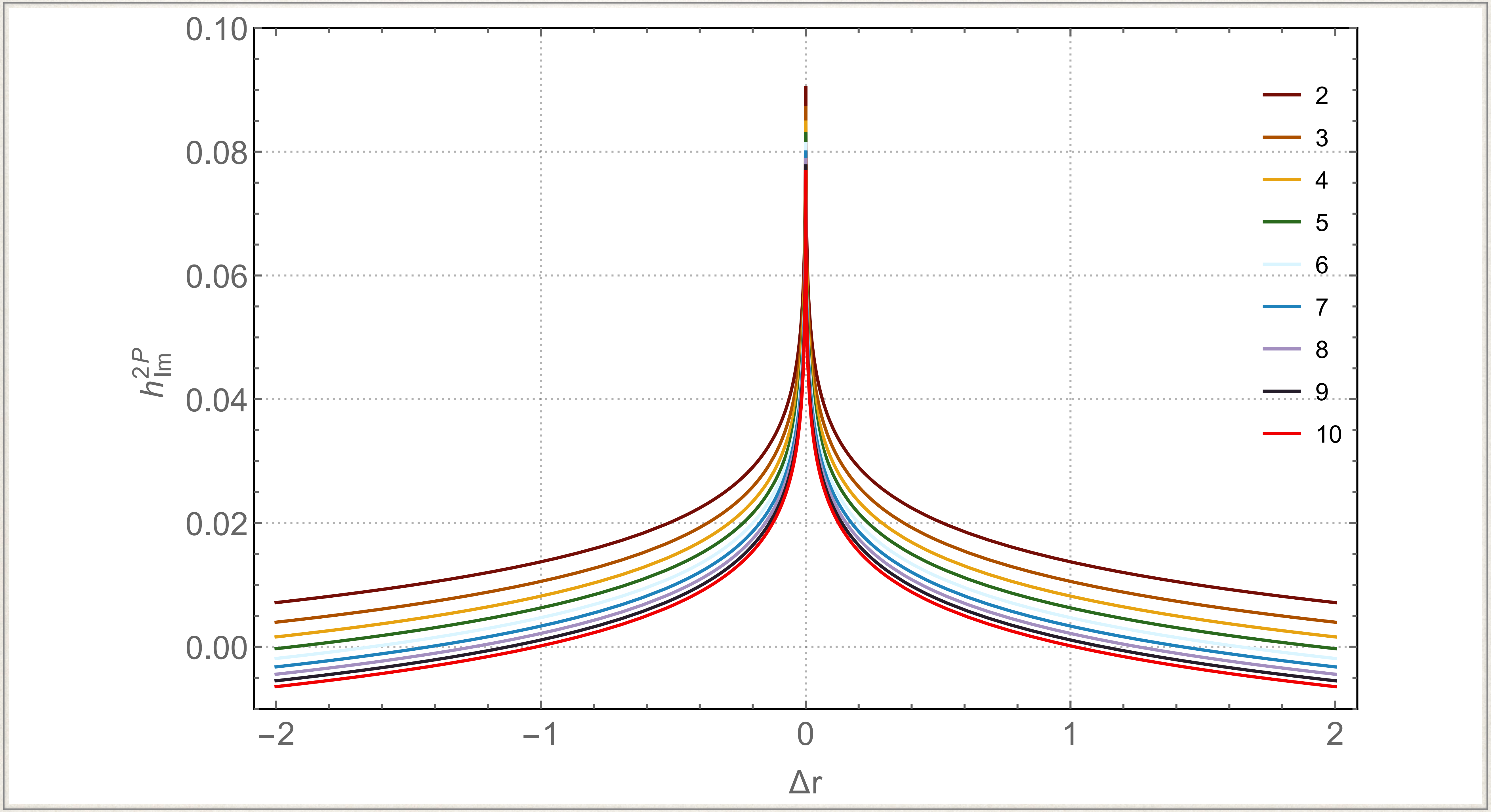
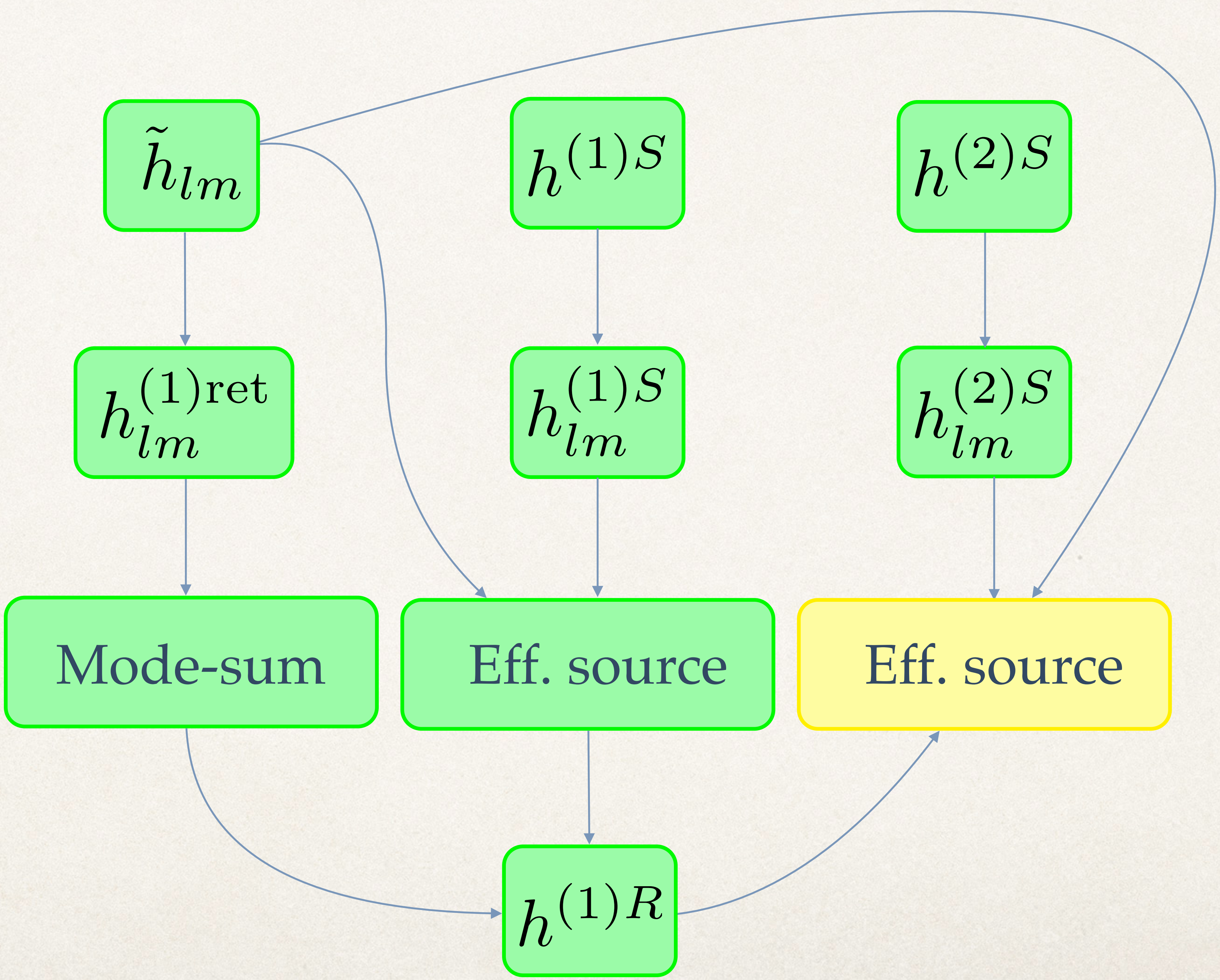
Towards second order self-force



$h_{\ell m}^{S2}$



Towards second order self-force



Second order effective source

$$\mathcal{D}_{\mu\nu}[h^{\mathrm{R}1}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S}1}]$$

$$\mathcal{D}_{\mu\nu}[h^{\mathrm{R}2}] = -\mathcal{D}_{\mu\nu}[h^{\mathrm{S}2}] + \delta^2 R_{\mu\nu}[h^1, h^1]$$

$$\begin{aligned}\delta^2 R_{\alpha\beta}[h,h] \equiv &- \tfrac{1}{2} h^{\mu\nu}\left(2h_{\mu(\alpha;\beta)\nu} - h_{\alpha\beta;\mu\nu} - h_{\mu\nu;\alpha\beta}\right)\\ &+ \tfrac{1}{4} h^{\mu\nu}{}_{;\alpha} h_{\mu\nu;\beta} + \tfrac{1}{2} h^{\mu}{}_{\beta}{}^{;\nu}\left(h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}\right)\\ &- \tfrac{1}{2}\bar{h}^{\mu\nu}{}_{;\nu}\left(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}\right)\end{aligned}$$

Second order Ricci tensor

$$\delta^2 R_{\alpha\beta}[h^{1\mathrm{ret}}, h^{1\mathrm{ret}}] =$$

$$\delta^2 R_{\alpha\beta}[h^{1\mathrm{R}}, h^{1\mathrm{R}}] \Longleftarrow$$ mode coupling

$$+\, 2\delta^2 R_{\alpha\beta}[h^{1\mathrm{R}}, h^{1\mathrm{S}}] \Longleftarrow$$ mode coupling

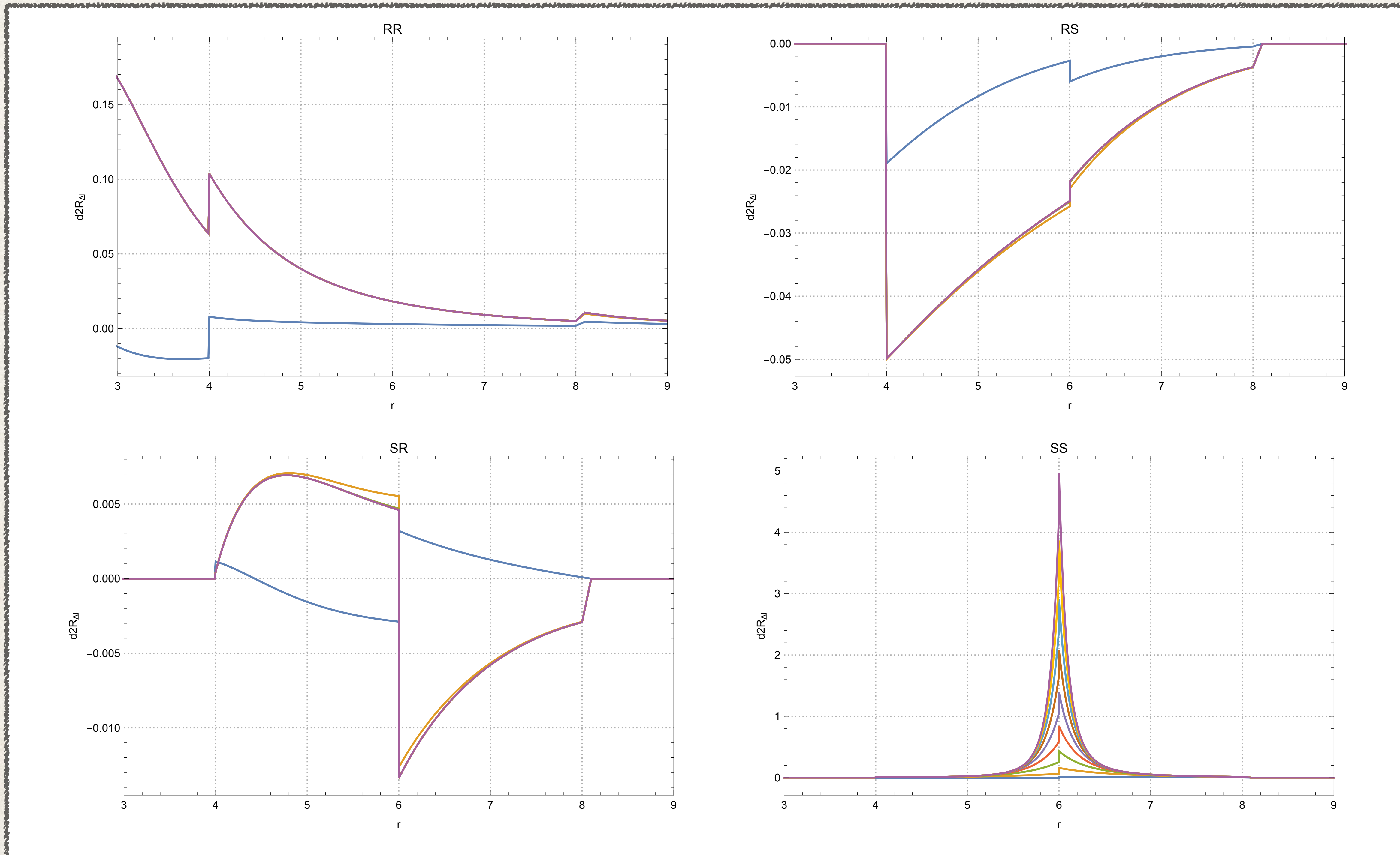
$$+\, \delta^2 R_{\alpha\beta}[h^{1\mathrm{S}}, h^{1\mathrm{S}}] \Longleftarrow$$ mode decomposition (c.f. $h^{\mathrm{S}2}$)

Mode coupling

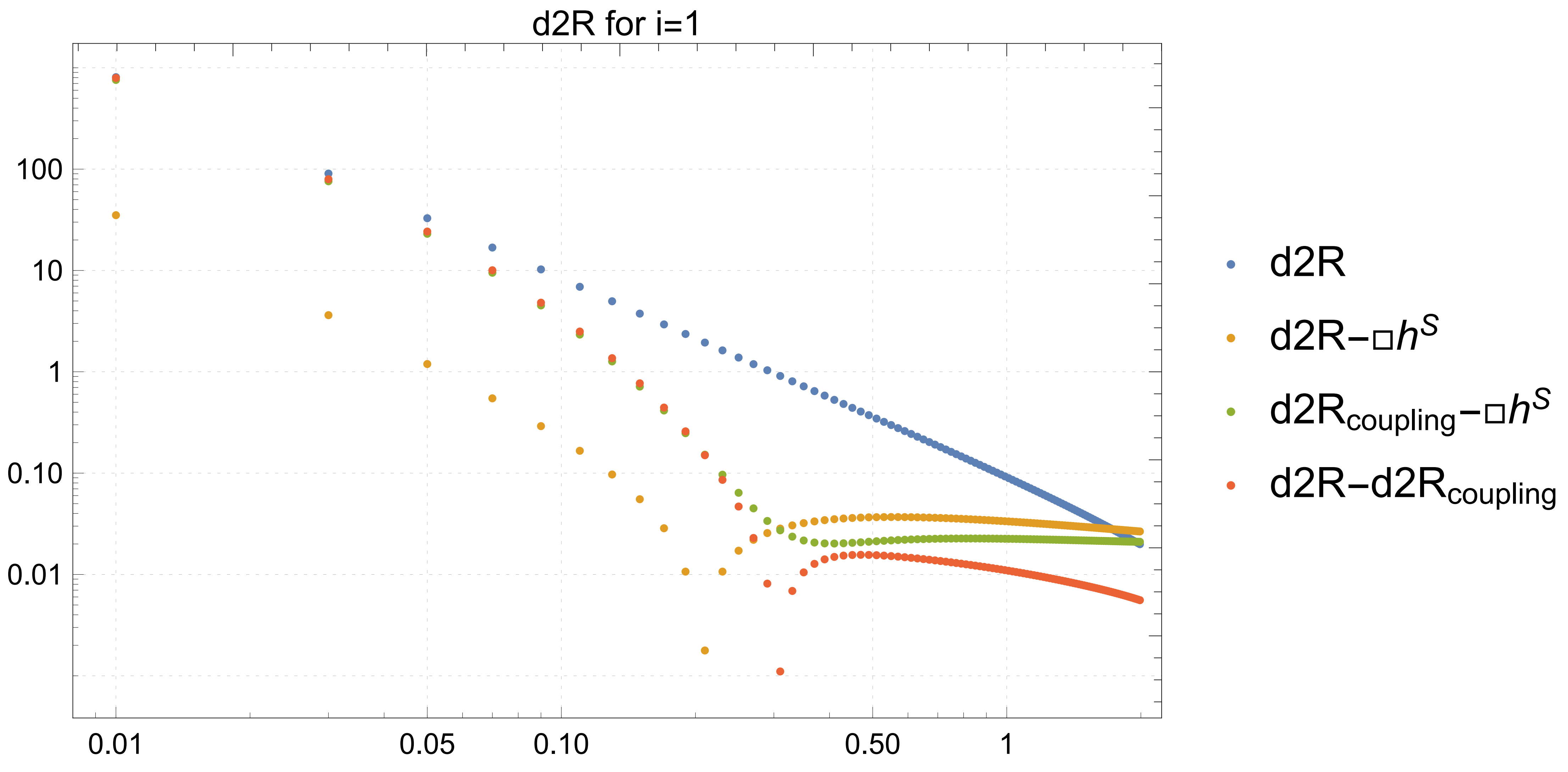
$$\delta^2 R_{\mu\nu}[h^1, h^1] = \sum_{i\ell m} \delta^2 R_{i\ell m}(r;\hat{r}) e^{-im\Omega t} Y^{i\ell m}_{\mu\nu}(r, \theta^A)$$

$$\delta^2 R_{i\ell m} = \sum_{\substack{i'\ell' m'\\ i''\ell'' m''}} \mathcal{D}^{i'\ell' m' i''\ell'' m''}_{i\ell m}\left[h_{1i'\ell' m'}, h_{1i''\ell'' m''}\right]$$

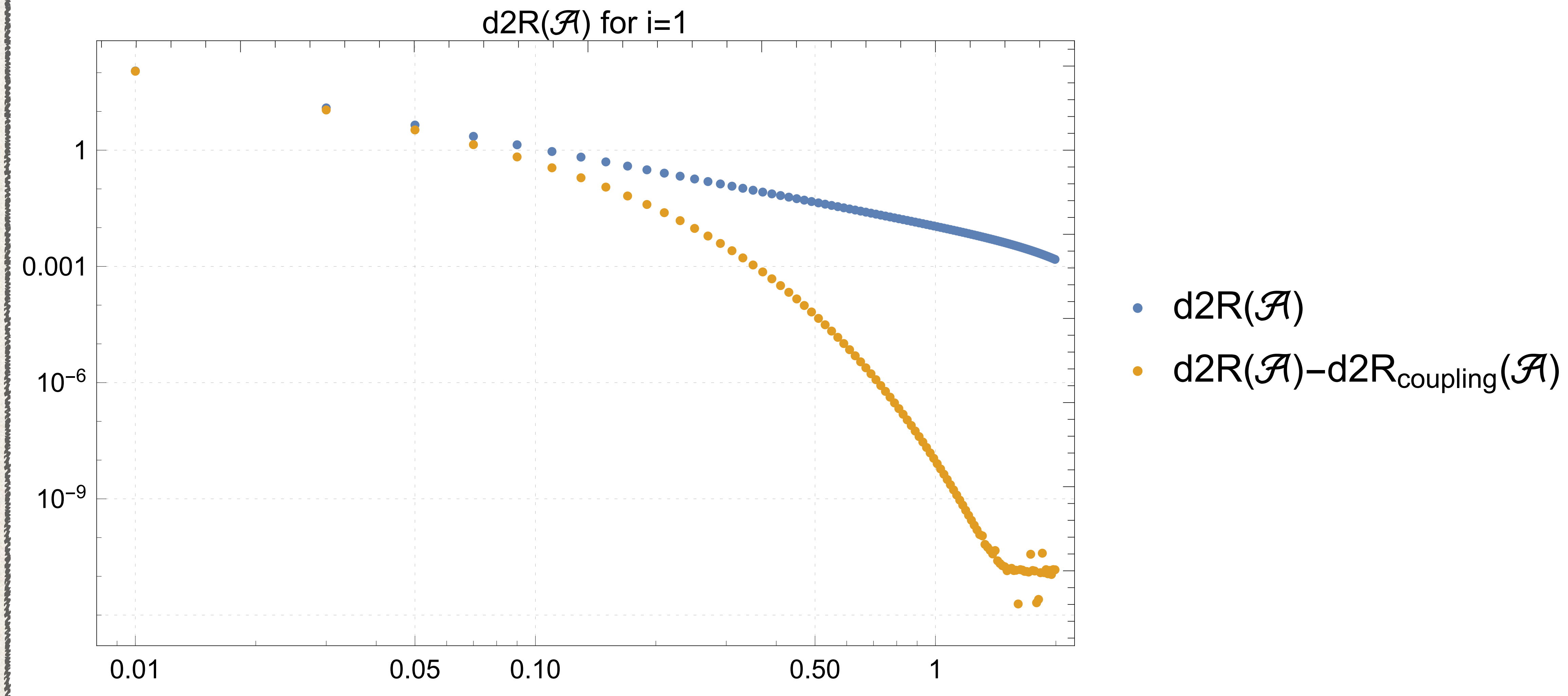
Mode coupling: $\delta^2 R[h^1,h^1]$



Mode decomposition: $\delta^2 R[h^{1S}, h^{1S}]$



Mode decomposition: $\delta^2 R[h^{1S}, h^{1S}]$



Towards second order self-force