Time-domain metric reconstruction for self-force applications

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Structure of talk

- Why another method for self-force calculations?
- Formulation of method in Kerr
- Implementation for circular orbits in Schwarzschild
- Going forward to Kerr

Why another method for self-force calculations?

Method for obtaining $h_{\alpha\beta}$	frequency domain	time domain
direct: $E_{\mu\nu}(h_{\alpha\beta}) = T_{\alpha\beta}$	✓	✓
metric reconstruction: $\text{Teuk}(\Psi) = \mathcal{T}, \quad \Psi \Rightarrow h_{\alpha\beta}$	✓	this talk

Reconstruction method is computationally cheaper, but so far only formulated and implemented via frequency-domain decomposition.

Reasons to want to have a time-domain method for metric reconstruction:

- Highly eccentric or unbound (incl. high-energy) orbits
- Orbital evolution under the self-force
- Test of f-domain calculations

Review of metric reconstruction

vacuum spacetime

- Let $h_{\alpha\beta}$ be a vacuum perturbation of the Kerr metric, with ψ_4, ψ_0 associated Weyl scalars.
- Find a "Hertz potential" Φ that satisfies

$$\boxed{\text{Teuk}_s \Phi = 0} \quad \text{and} \quad \boxed{D_s^4 \Phi = \psi_{-s} \quad \text{or} \quad \mathcal{L}_s^4 \Phi = \psi_s}.$$

- Then the original perturbation can be reconstructed via

$$h_{\alpha\beta} = \mathrm{Re}\left(e_{\mathbf{a}(\alpha} e_{\mathbf{b}\beta)} \mathcal{D}^{\mathbf{ab}} \Phi \right) + h_{\alpha\beta}^{\text{gauge}} + \delta M_{\alpha\beta} + \delta J_{\alpha\beta}.$$

Review of metric reconstruction

vacuum spacetime

Two variants:

- *Ingoing Radiation Gauge* (IRG)

$$\boxed{\text{Teuk}_{-2}\Phi = 0} \quad \text{and} \quad \boxed{D^4\Phi = \psi_0 \quad \text{or} \quad \mathcal{L}^4\Phi = \psi_4}.$$

$$h^{\rm rec}_{\alpha\beta} = \mathrm{Re}\left(e_{\mathbf{a}(\alpha}e_{\mathbf{b}\beta)}\mathcal{D}^{\mathbf{ab}}\Phi\right), \qquad \ell^\alpha h^{\rm rec}_{\alpha\beta} = 0.$$

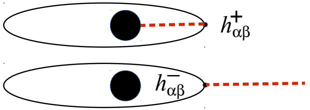
- *Ongoing Radiation Gauge* (ORG)

$$\boxed{\text{Teuk}_{+2}\Phi = 0} \quad \text{and} \quad \boxed{\tilde{D}^4\Phi = \psi_4 \quad \text{or} \quad \tilde{\mathcal{L}}^4\Phi = \psi_0}.$$

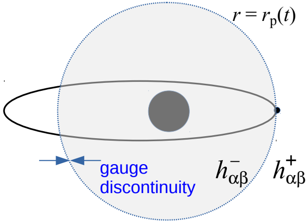
$$h^{\rm rec}_{\alpha\beta} = \mathrm{Re}\left(e_{\mathbf{a}(\alpha}e_{\mathbf{b}\beta)}\tilde{\mathcal{D}}^{\mathbf{ab}}\Phi\right), \qquad n^\alpha h^{\rm rec}_{\alpha\beta} = 0.$$

Review of metric reconstruction
point-particle source

- In presence of matter sources, the procedure fails to yield a valid solution even in vacuum away from sources [LB & Ori 2001; Price & Whiting 2007]
- Reconstruction for bound orbits, with string-like gauge singularities [Ori 2003]
- "No-string" reconstruction with gauge discontinuity on a sphere [Keidl, Friedman etal 2007-12; vdMeent 2015–]
- Self-force from a reconstructed metric [Pound, Merlin & LB (2014)]
- Determination of $\delta M_{\alpha\beta} + \delta J_{\alpha\beta}$ [Merlin etal 2016; vdMeent 2017]



Ori's "half-string" solutions



Friedman's "no-string" solution

New method: metric reconstruction in the t domain

Basic idea: Obtain $\Phi^{\pm}$ by solving the appropriate Teukolsky equation in 1+1D, with suitable boundary conditions at infinity and on the horizon, and with suitable junction conditions along the particle's worldline. From it obtain the "no-string" perturbation via

$$h^{\pm}_{\alpha\beta} = \mathrm{Re}\left(e_{\mathbf{a}(\alpha}e_{\mathbf{b}\beta)}\mathcal{D}^{\mathbf{ab}}\Phi^{\pm}\right) + \delta M^{\pm}_{\alpha\beta} + \delta J^{\pm}_{\alpha\beta}$$

or an ℓ-by-ℓ application thereof.

Need 3 things:

1. 1+1D decomposition of the Teukolsky equation
2. Boundary conditions for 1+1D solutions
3. Junction conditions along the particle's worldline

1+1D decomposition of the Teukolsky equation

Decompose into spin-weighted *spherical* harmonics (even in Kerr):

$$\Phi_s^{\pm} = (r\Delta^s)^{-1} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \phi_{s\ell m}^{\pm}(t,r)\, {}_sY_{\ell m}(\theta,\tilde{\varphi}).$$

Substitution into Teukolsky's master equation (in vacuum) gives

$$\sum_{\ell m} {}_sY_{\ell m}(\theta,\tilde{\varphi}) \left[\tilde{\Box}\phi_{s\ell m}^{\pm} - a^2 \sin^2\theta\, (\phi_{s\ell m}^{\pm})_{,tt} + 2ias\cos\theta\, (\phi_{s\ell m}^{\pm})_{,t}\right] = 0.$$

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After reexpanding in ${}_sY_{\ell m}$ we get, for each ℓ,

$$\boxed{\phi^{\ell}_{,uv} + U(r)\phi^{\ell}_{,u} + V(r)\phi^{\ell}_{,v} + W(r)\phi^{\ell} + K(r)\left[a^2 C_0^{\ell}\, \phi^{\ell}_{,tt} + \mathcal{I}(\phi^{\ell\pm1}, \phi^{\ell\pm2})\right] = 0,}$$

with the ℓ-mode coupling term

$$\mathcal{I} = -a^2 \left(C^{\ell}_{++}\phi^{\ell+2}_{,tt} + C^{\ell}_{+}\phi^{\ell+1}_{,tt} + C^{\ell}_{-}\phi^{\ell-1}_{,tt} + C^{\ell}_{--}\phi^{\ell-2}_{,tt}\right) + 2ias\left(c^{\ell}_{+}\phi^{\ell+1}_{,t} + c^{\ell}_{-}\phi^{\ell-1}_{,t}\right).$$

Boundary conditions for the fields $\phi^{\pm}_{s\ell m}(v,u)$

behavior at $r \gg M$

- time-dependent modes:

$$\begin{aligned} \text{physical:} \quad \phi^{+} &\sim e^{-i\omega u} && \hookrightarrow \text{bounded}, \\ \text{nonphysical:} \quad \phi^{+} &\sim r^{2s} e^{-i\omega v} && \hookrightarrow \text{blows up for ORG}. \end{aligned}$$

- static modes:

$$\begin{aligned} \text{physical:} \quad \phi^{+} &\sim r^{-\ell+s} && \hookrightarrow \text{bounded}, \\ \text{nonphysical:} \quad \phi^{+} &\sim r^{\ell+s+1} && \hookrightarrow \text{blows up}. \end{aligned}$$

In an ORG reconstruction, all nonphysical modes blow up at infinity

Boundary conditions for the fields $\phi^{\pm}_{s\ell m}(v,u)$

behavior near the event horizon

- time-dependent modes:

$$\begin{array}{rlll} \text{physical:} & \phi^- \sim e^{-i\omega v}, & \hookrightarrow \text{bounded}, \\ \text{nonphysical:} & \phi^- \sim \Delta^s e^{-i\omega u} e^{-2im\Omega_{\rm H} r_*} & \hookrightarrow \text{blows up for IRG}. \end{array}$$

- static modes:

$$\begin{array}{rlll} \text{physical:} & \phi^- \sim \mathrm{const} & \hookrightarrow \text{bounded}, \\ \text{nonphysical:} & \phi^- \sim \Delta^s & \hookrightarrow \text{blows up for IRG}. \end{array}$$

In an IRG reconstruction, all nonphysical modes blow up on the horizon

Boundary conditions for the fields $\phi^{\pm}_{s\ell m}(v,u)$

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In an IRG reconstruction, all nonphysical modes blow up on the horizon

This motivates a "mixed-gauge" approach where $h^{+}_{\alpha\beta}$ is reconstructed in ORG while $h^{-}_{\alpha\beta}$ is reconstructed in IRG. (We won't implement it here.)

Junction conditions on the particle's worldline

- Radial inversion relations decompose into ${}_sY_{\ell m}$ modes without any mode coupling, even in Kerr! E.g., for IRG:

$$8r\Delta^2\mathcal{D}_l^4\left(\Delta^2\bar{\phi}^{\pm}_{\ell m}/r\right) = (-1)^m\psi^{\pm}_{0,\ell,-m}\ ,$$

where

$$\mathcal{D}_l := \Delta^{-1}\left((r^2+a^2)\partial_v - ima\right).$$

- Angular relations don't have this nice feature, except for $a = 0$ where they become very simple (essentially $\phi^{\pm}_{,t} \sim \psi^{\pm}_0$). See Lousto & Whiting 2002.

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- The jumps across the worldline in ϕ and its v derivatives thus obey

$$\sum_{n=0}^{4} F_n(v)\left[\partial_v^n\phi_{\ell m}\right] = \left[\bar{\psi}_{0,\ell,-m}\right],$$

  where $\left[\bar{\psi}_{0,\ell,-m}\right]$ can be inferred directly from the source of the Teukolsky equation.

Junction conditions on the particle's worldline

- $\left[\partial_v^{n\geq 2}\phi\right]$ can be expressed in terms of $[\phi]$ and $[\phi_{,v}]$ and their τ derivatives along the orbit, via a repeated application of the vacuum Teukolsky equation.
- We obtain a coupled set of ODEs for $[\phi]$ and $[\phi_{,v}]$:

$$\sum_{n=0}^{3}\left(a_n(\tau)\frac{d^n[\phi]}{d\tau^n}+b_n(\tau)\frac{d^n[\phi_{,v}]}{d\tau^n}\right)+\mathcal{I}\text{ terms}=\left[\bar{\psi}_{2,\ell,-m}\right]$$

$$\sum_{n=0}^{4}\left(c_n(\tau)\frac{d^n[\phi]}{d\tau^n}+d_n(\tau)\frac{d^n[\phi_{,v}]}{d\tau^n}\right)+\mathcal{I}\text{ terms}=\left[(\bar{\psi}_{2,\ell,-m})_{,v}\right]$$

- These should be solved with suitable initial or periodicity conditions.
- Once $[\phi]$ and $[\phi_{,v}]$ are known, the jumps in all higher (u, v or mixed) derivatives are easily obtained using $[\dot{\phi}]=\dot{u}_{\rm p}[\phi_{,u}]+\dot{v}_{\rm p}[\phi_{,v}]$ and the Teukolsky equation.

Implementation for circular orbits in Schwarzschild

1. Field equation and junction conditions

- Specializing to Schwarzschild and IRG:

$$\boxed{\phi^{\ell}_{,uv} + U(r)\phi^{\ell}_{,u} + V(r)\phi^{\ell}_{,v} + W(r)\phi^{\ell} = 0,}$$

with

$$U(r) = \frac{2M}{r^2}, \qquad V(r) = -\frac{2f}{r}, \qquad W(r) = \frac{f}{4}\left(\frac{(\ell+2)(\ell-1)}{r^2} - \frac{2M}{r^3}\right).$$

- Specializing further to circular orbits, jump equations become algebraic:

$$\begin{aligned} a_\Sigma\,[\phi] + b_\Sigma\,[\phi_{,v}] &= \left[\bar{\psi}_{2,\ell,-m}\right], \\ c_\Sigma\,[\phi] + d_\Sigma\,[\phi_{,v}] &= \left[\partial_v\bar{\psi}_{2,\ell,-m}\right], \end{aligned}$$

so can be solved analytically.

Implementation for circular orbits in Schwarzschild

1. Field equation and junction conditions

- Even in this simple case, the jumps are complicated...

$$\begin{aligned}\left[\phi^{\mathrm{IRG}}_{\ell m}\right] &= \tilde{\Delta}^{-1}\left(d_\Sigma\left[\bar{\psi}_{2,\ell,-m}\right] - b_\Sigma\left[\partial_v\bar{\psi}_{2,\ell,-m}\right]\right),\\ \left[\partial_v\phi^{\mathrm{IRG}}_{\ell m}\right] &= \tilde{\Delta}^{-1}\left(a_\Sigma\left[\partial_v\bar{\psi}_{2,\ell,-m}\right] - c_\Sigma\left[\bar{\psi}_{2,\ell,-m}\right]\right),\end{aligned}$$

where

$$\tilde{\Delta} = \frac{1}{4} f_0^4 r_0^8\left[\lambda^2(\lambda+2)^2 + (12mM\Omega)^2\right],$$

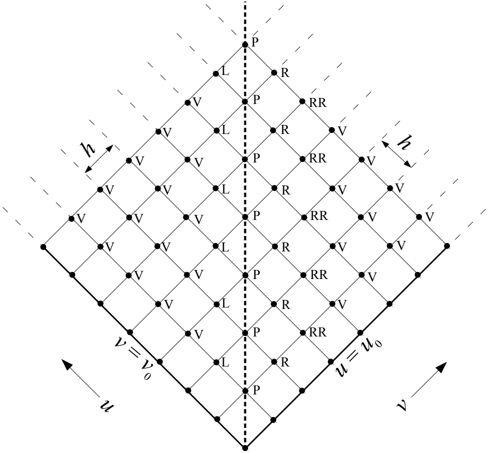
and, e.g., $c_\Sigma = \sum_{n=0}^4 (-im\Omega)^n c_n$ with

$$\begin{aligned} c_0 &= r_0^3 f_0^2(1-y)\lambda(\lambda+2),\\ c_1 &= f_0 r_0^4\left[\lambda(\lambda+5) - 2(\lambda^2+2\lambda-6)y - 2(4\lambda+17)y^2 + 12y^3\right],\\ c_2 &= 2r_0^5\left[3\lambda + (15-7\lambda)y + 2(\lambda-23)y^2 + 24y^3\right],\\ c_3 &= 2f_0 r_0^6(\lambda+22y),\\ c_4 &= 16Mr_0^6. \end{aligned}$$

Here $\lambda := (\ell+2)(\ell-1)$, $\quad f_0 := 1-2M/r_0$, $\quad y := M/r_0$.

Implementation for circular orbits in Schwarzschild

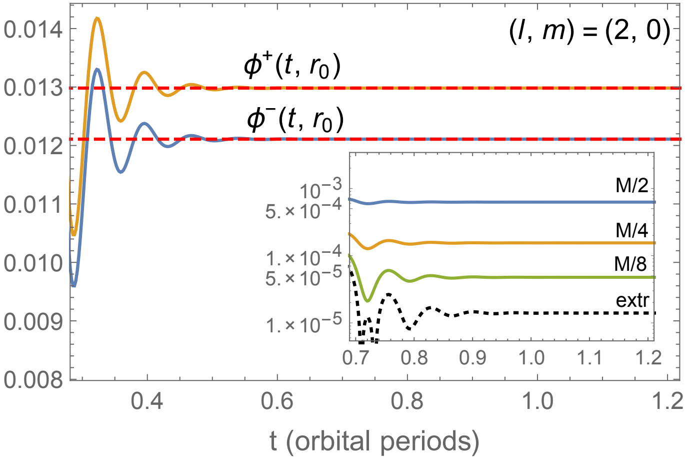
2. Numerical method



- Teukolsky equation discretized on a 1+1 mesh in *uv* coords
- Finite-difference scheme is 2nd-order convergent [local error is $O(h^4)$]
- FD scheme at near-particle points incorporates jump information
- Start with zero initial data on $u = u_0$ and $v = v_0$; wait for junk radiation to dissipate.

Implementation for circular orbits in Schwarzschild

3. Results and tests



Implementation for circular orbits in Schwarzschild

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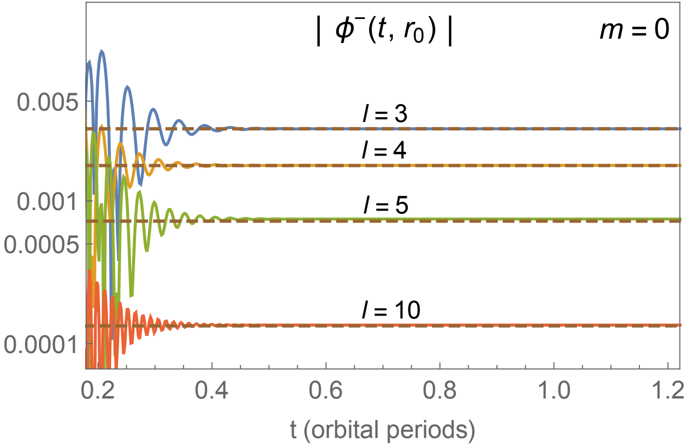

Implementation for circular orbits in Schwarzschild

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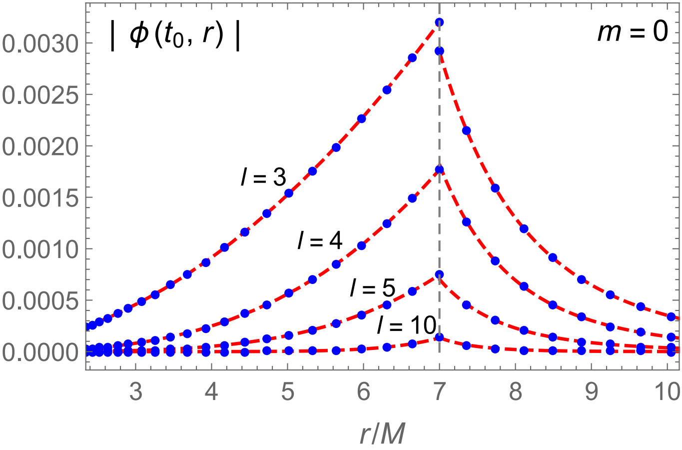

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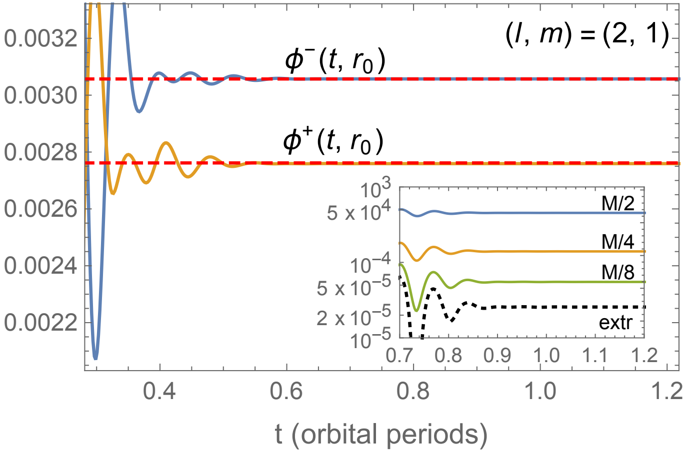

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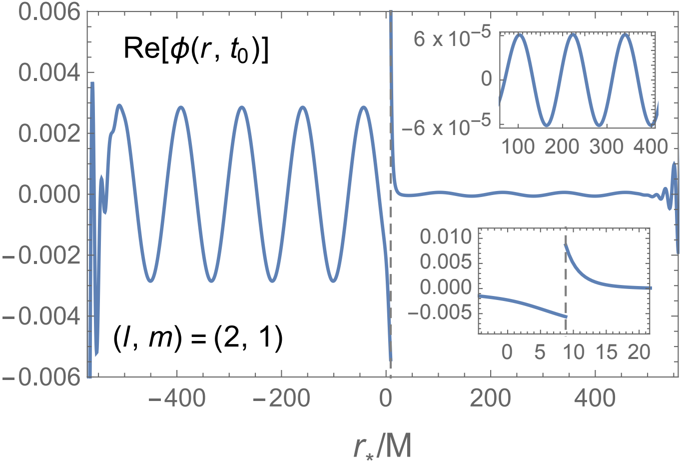

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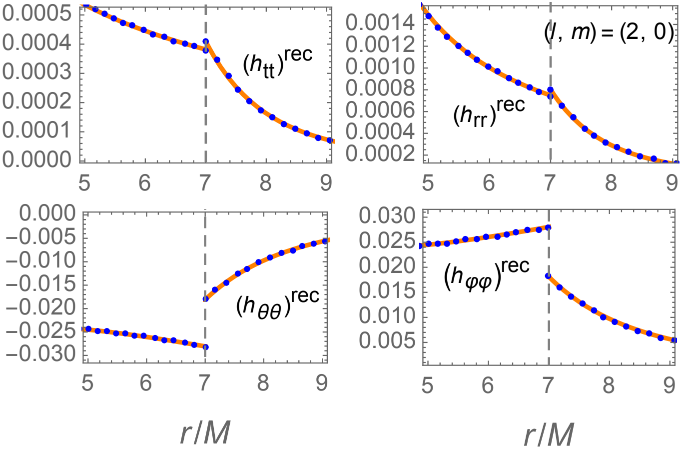

Going forward (1/4)

Extension to eccentric or unbound orbits

Main challenge is to solve 4th-order ODEs that determine the jumps of ϕ across the orbit.

- For periodic orbits: can decompose into harmonics of orbital frequencies
- For evolving bound orbits: can impose approximate periodic initial conditions at initial point
- For scatter orbits: can impose initial conditions at infinity; need to find a way to eliminate nonphysical unstable solutions.

Going forward (2/4)

Extension to Kerr

Need to incorporate mode coupling in both field & jump equations.

- Preliminary results show that's easily done in vacuum.
- In particle case, may need to deal with coupling at the level of the mode-sum regularization formula

Going forward (3/4)

Exploring the mixed-gauge approach

- Our IRG implementation seems to “automatically” select the physical (retarded) solution. Need to understand why and whether that’s always guaranteed.
- A mixed-gauge approach would have the advantage that a globally bounded solution is automatically the physical one.
- To set up a the mixed-gauge calculation, need (1) a formulation of the self-force from a mixed-gauge perturbation (easy), and (2) a method for determining the jump condition (hard!)

Going forward (4/4)

Improving the numerical method

Method could be directly implemented on existing platforms (e.g., Diener's or Thornburg's), to include

- Mesh refinement
- Hyperboloidal slicing
- Compactification
- Parallelization

Method is also amenable to finite-element discretization a la Canizares & Sopuerta.