Towards the self-consistent evolution of a scalar charge around a Schwarzschild black hole

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The problem.

We wish to determine the self-forced motion and field (e.g. energy and angular momentum fluxes) of a particle with scalar charge

$$\Box\psi^{\mathrm{ret}} = -4\pi q \int \delta^{(4)}(x - z(\tau))\, d\tau.$$

2 general approaches:

- Compute enough "geodesic"-based self-forces and then use these to drive the motion of the particle. (Post-processing, fast, accurate self-forces, relies on slow orbit evolution)
- Compute the "true" self-force <u>while</u> simultaneously driving the motion. (Slow and expensive, less accurate self-forces)

Effective source approach.

... is a general approach to self-force and self-consistent orbital evolution that doesn't use any delta functions.

Key ideas

- Compute a regular field, ψ^{R}, such that the self-force is

$$F_\alpha = \nabla_\alpha \psi^{\mathsf{R}}|_{x=z},$$

  where $\psi^{\mathsf{R}} = \psi^{\mathsf{ret}} - \psi^{\mathsf{S}}$, and ψ^{S} can be approximated via local expansions: $\psi^{\mathsf{S}} = \tilde{\psi}^{\mathsf{S}} + O(\epsilon^n)$.

- The effective source, S, for the field equation for ψ^{R} is regular at the particle location.

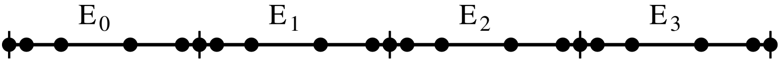
$$\Box\psi^{\mathsf{R}} = \Box\psi^{\mathsf{ret}} - \Box\tilde{\psi}^{\mathsf{S}} = S(x|z,u)$$

  where $\Box\,\tilde{\psi}^{\mathsf{S}} = -4\pi q \int \delta^{(4)}(x - z(\tau))\,d\tau - S$.

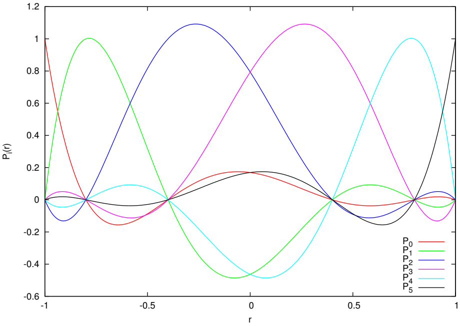
Self-consistent vs. geodesic evolutions.

- One main goal is to compare our self-consistent evolutions with Niels Warburton's geodesic evolutions.
- First attempt: 3+1 multi-patch finite difference code with a C^0 effective source.
- 3+1 accuracy limited by the non-smoothness of the source leading to high frequency noise with 2nd order convergent amplitude.
- Self-consistent evolutions agreed beautifully with geodesic evolutions within the errors (dominated by the noise).
- Next attempt: 3+1 multi-patch finite difference code with a C^2 effective source.
- Geodesic evolution agreed with the C^0 evolutions and the frequency domain result with the noise reduced by more than an order of magnitude.
- However, we found differences between C^2 and C^0 results as soon as the back-reaction was turned on.

Discontinuous Galerkin method.



- Split the domain into N nth order elements.
- Each element contains $n+1$ nodes.
- $u(t,x) \approx \sum_{i=0}^{n} \tilde{u}(t,x_i)P_i(x)$



- The numerical approximation is double valued at all element boundaries.
- Derivatives are approximated by multiplying the state vector in each element by a derivative matrix.
- Neighboring elements are glued together by numerical fluxes.

Discontinuous Galerkin method.

- Numerical fluxes can be constructed in many different ways in order to maintain numerical stability and to guarantee that the jumps in the solution at the element boundaries converge to zero.
- We use fluxes based on a characteristic decomposition of the wave equation.

The convergence properties of the DG method for smooth solutions are

- Exponential with the order n (with N kept fixed).
- polynomial with the element size $1/N$ (with n kept fixed).

As the DG scheme has discontinuities built in at the element boundaries, we retain these convergence properties even when the solution itself is non-smooth IF and only if, the non-smooth features can be placed at element boundaries.
(Hesthaven & Warburton, 2007)

Code description.

The code is 1+1 dimensional based on the spherical harmonic decomposition of the scalar wave equation in the Schwarzschild spacetime in tortoise coordinates $r_* = r + 2M\log(r/(2M) - 1)$ with a spherically harmonic decomposed effective source.

$$-\frac{\partial^2\psi_{\ell m}}{\partial t^2} + \frac{\partial^2\psi_{\ell m}}{\partial r_*^2} - V_\ell(r)\psi_{\ell m} = S^{\mathrm{eff}}_{\ell m}.$$

As $r_* \in [-\infty, \infty]$ we split the domain into three regions. In the inner ($r_* \in [-\infty, T_1]$) and outer ($r_* \in [T_2, \infty]$) regions we introduce new coordinates (τ, ρ) used in Bernuzzi, Nagar & Zenginoğlu (2011).

$$\begin{aligned} t &= \tau + h(\rho) \\ r_* &= \rho/\Omega(\rho) \end{aligned}$$

where $h(\rho)$ and $\Omega(\rho)$ are chosen suitably (hyperboloidal layers) in each region to make the inner boundary ($\rho_{\min}$) coincide with the horizon H and the outer boundary ($\rho_{\max}$) coincide with $\mathscr{I}^+$.

Code description.

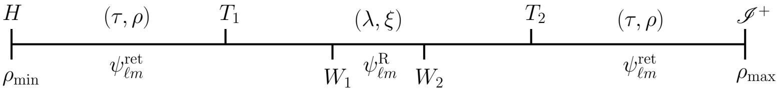
In the middle region ($r_* \in [T_1, T_2]$) we introduce a time dependent coordinate transformation (Field, Hesthaven & Lau, 2009)

$$
\begin{aligned}
t &= \lambda \\
r_* &= T_1 + \frac{r_*^p - T_1}{\xi^p - T_1}(\xi - T_1) + \\
&\quad \frac{(T_2 - r_*^p)(\xi^p - T_1) - (r_*^p - T_1)(T_2 - \xi^p)}{(\xi^p - T_1)(T_2 - \xi^p)(T_2 - T_1)}(\xi - T_1)(\xi - \xi^p)
\end{aligned}
$$

where r_*^p is the time-dependent particle location. This satisfies $r_*(\lambda, T_1) = T_1$, $r_*(\lambda, \xi^p) = r_*^p$, $r_*(\lambda, T_2) = T_2$.

In addition we use the world tube approach so that we evolve $\psi_{\ell m}^{\mathrm{R}} = \psi_{\ell m}^{\mathrm{ret}} - \psi_{\ell m}^{\mathrm{S}}$ in the region $r_* \in [W_1, W_2]$ (where typically $W_1 > T_1$ and $W_2 < T_2$), while elsewhere we evolve $\psi_{\ell m}^{\mathrm{ret}}$.

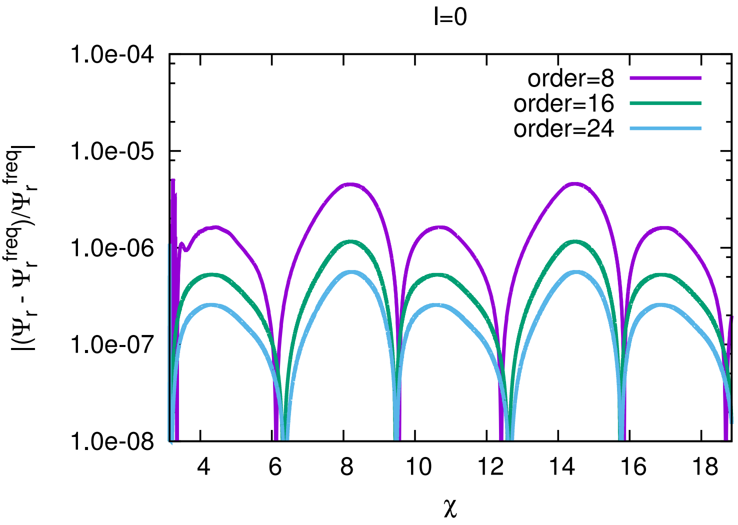
The values of T_1, W_1, W_2 and T_2 is of course chosen to coincide with element boundaries.



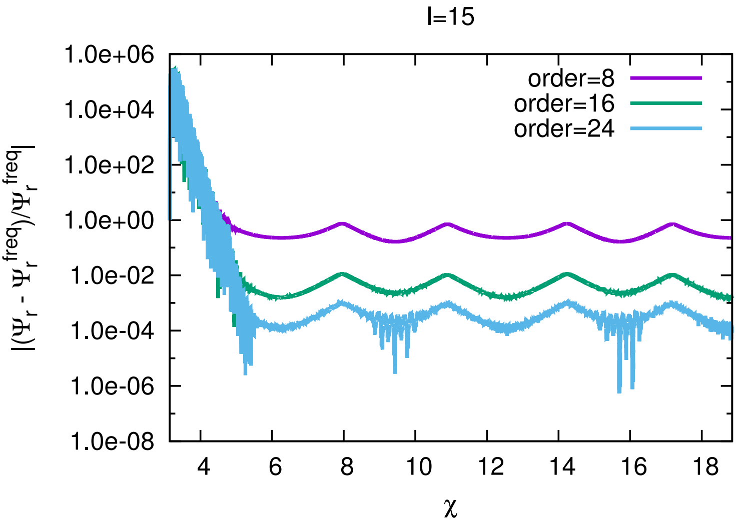
Code issues at last Capra meeting.

- Effective source with acceleration terms had been added. Code was working for constant accelerated circular orbit, but not for accelerated eccentric orbits.
- We appeared to lose mode sum convergence for non-constant accelerated circular orbits.
- Had done a few experiments with back reaction turned on but saw instabilities developing.
- Marathon debugging session took place in Dublin in January.
- Found that the effective source was actually implemented correctly. The error was in the expressions for the time-derivatives of the acceleration (in the case of eccentric orbits).
- Discontinuity of effective source during smooth turn on causes non-physical slowly decaying (related to Joost solutions?) modes.
- Implemented the ability to read in correct initial data from Niels' frequency domain code.

$p = 8,\ e = 0.1,\ dt/d\chi = 2(dt/d\chi)_{\rm geo}$



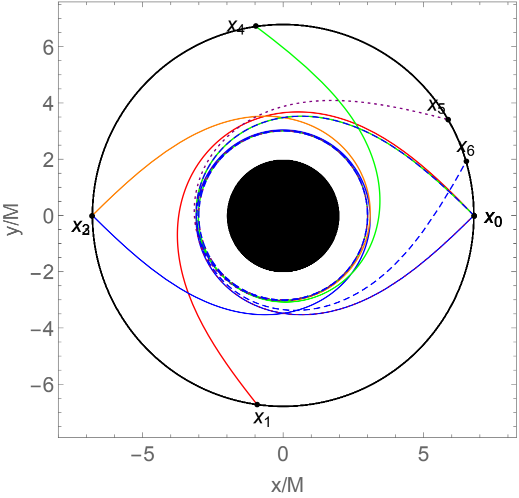
$p = 8,\ e = 0.1,\ dt/d\chi = 2(dt/d\chi)_{\rm geo}$



$p = 6.7862,\ e = 0.0,\ A = 0.05,\ \sigma = 1.8$

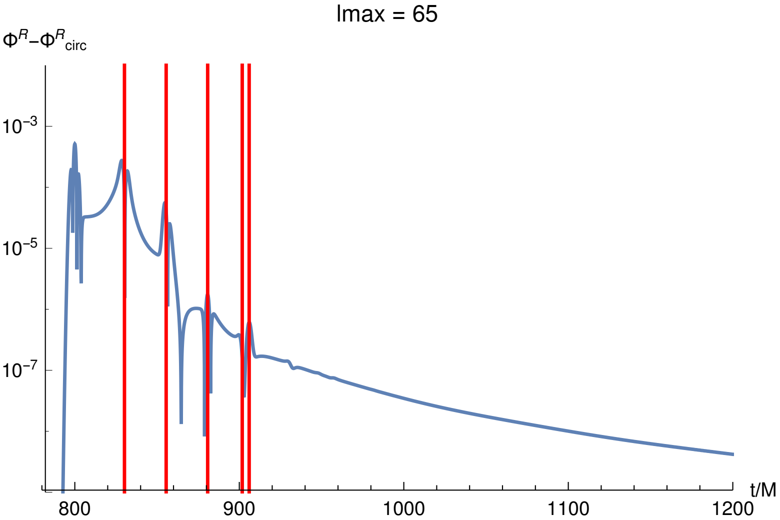
$t_1 = 30.1867$
$t_2 = 55.5379$
$t_3 = 55.5379$
$t_4 = 80.7708$
$t_5 = 101.779$
$t_6 = 106.003$

Evolve up to $\ell = 65$.
DG order up to 96.



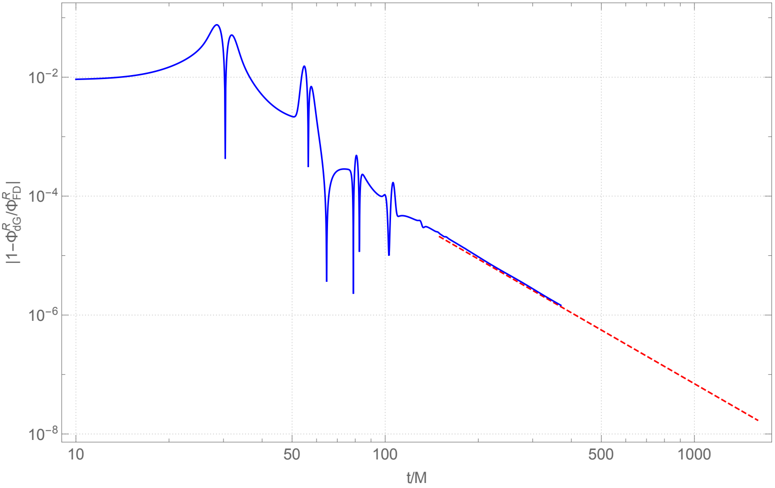
$$\Omega(t) = \Omega_0 + A e^{-\frac{(t-t_0)^2}{\sigma^2}}$$

$p = 6.7862,\ e = 0.0,\ A = 0.05,\ \sigma = 1.8$



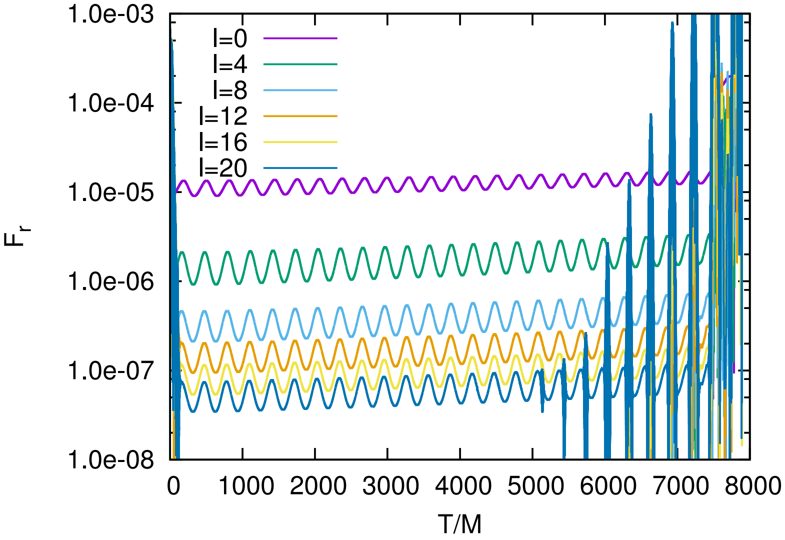
Movie

$p = 6.7862,\ e = 0.0,\ A = 0.05,\ \sigma = 1.8$



$p = 9.9$, $e = 0.1$, $q = 1/8$

Only four-acceleration passed in to the effective source!!!!



Conclusions and Outlook.

- The effective source for non-geodesic orbits is working and can reproduce frequency domain results for accelerated eccentric orbits with very good accuracy.
- For the case of a particle on a circular orbit experiencing a short acceleration phase, we maintain mode sum convergence and see a series of light crossings followed by a t^{-3} tail decay.
- Self-consistent evolutions goes unstable. Very fast if derivatives of the acceleration are passed into the effective source.
- Suspect it is a feedback instability triggered by noise in the acceleration and its derivatives but are still investigating other possible bugs in the code.
- May be able to reduce noise by using numerical derivatives of $\dot{E}$ and $\dot{L}$ instead of $\dot{a}$ and $\ddot{a}$
- May be able to use Anna's regularization parameters to improve mode sum convergence and decrease the needed number of ℓ-modes.

Conclusions and Outlook.

- The DG code seem to be a good fit for Leor's proposal of evolving the Hertz potential in the time domain. Will need to discuss more during Capra.
- Indeed it seems that a lot can be learned from Field, Hesthaven and Lau's approach to evolving RWZ with a δ-function source.