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Solving for binary inspiral dynamics using renormalization group methods

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For more details:
Galley & Rothstein, Phys. Rev. D 95, 104054 (2017) [arXiv:1609.08268]

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Motivation

- Solving equations of motion for compact binary inspirals is important and challenging
  - Must use numerical methods, which is a bottleneck for data analysis applications
  - Often can involve using high-order adaptive solvers to provide sufficiently accurate numerical solutions over a very large number of orbits
  - Important phase errors over many thousands of orbits (e.g., in LIGO's bandwidth) can be caused by inaccurately capturing the effects of very weak nonconservative forces
  - Perturbative solutions exhibit secular behavior making result invalid over short times

- Most analytical methods are based on orbit-averaging/adiabatic approximations
  - Advantages:
    - Simpler equations to solve
    - Often provides useful qualitative understanding of the system's physical tendencies
  - Disadvantages:
    - Ambiguity about timescale to use for averaging: Period is associated with mean, eccentric, or true anomalies? [see Pound & Poisson (2008)]
    - Not a systematic procedure
    - What are the errors of the resulting approximate solutions?
    - Lose real-time phase information
    - Tend to be less useful as a system becomes more complicated (e.g., precession) [see Chatziioannou et al (2016) for recent progress]

Dynamical Renormalization Group

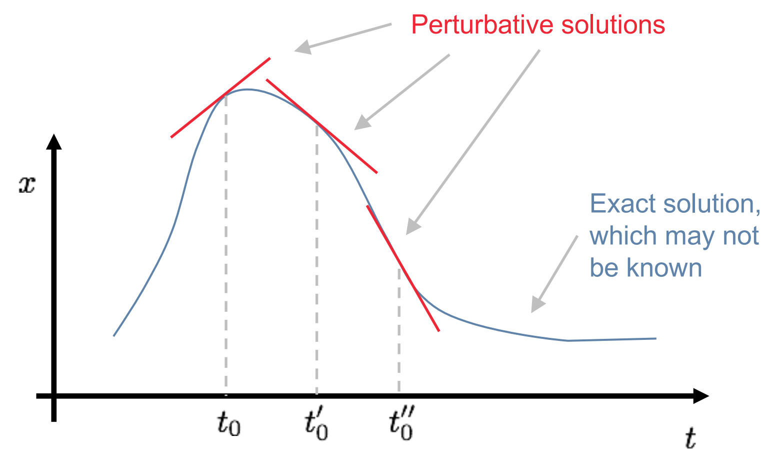
Overview

- Introduced as a method for solving ODE’s by Chen, Goldenfeld, and Oono (1996)
- Based on Renormalization Group Theory from high-energy and condensed matter physics
- Based on naive perturbation theory
- Encapsulates several other asymptotic methods of global analysis including:
  - Multiple-scale analysis
  - WKB theory
  - Boundary layer theory
- Systematic
  - Provides a turn-the-crank method of finding globally valid approximate solutions
  - Provides a formal error estimate on the perturbative solution
  - Contains strong self-consistency checks of the calculation

Dynamical Renormalization Group

Basic Idea

- Time at which to build a perturbative solution is arbitrary



- Perturbative solutions (at fixed order) at different times have the same form but different initial data parameters

$$x(t) = X_0 + V_0(t - t_0) + \mathcal{O}(t - t_0)^2$$
$$x(t) = X_0' + V_0'(t - t_0') + \mathcal{O}(t - t_0)^2$$

- These solutions are related to each other by "renormalization group flows" from one initial data set to another.

$$t_0' = t_0 + \delta t \implies X_0' \approx X_0 + V_0 \delta t \ , \quad V_0' \approx V_0$$

- What gets renormalized? Initial data parameters.

Dynamical Renormalization Group

The algorithm

- Write down the equations of motion
- Write down a background solution around which to perturb
  - This solution is written in terms of "bare" parameters (i.e., $R_B(t_0)$), which implicitly depend upon the initial time t_0, away from which we flow.
- Use this background to calculate perturbatively the solution to equations of motion.
  - The perturbation will in general have secular "divergences" (i.e., terms that grow as $(t-t_0)$).
- Take this solution and write the bare parameters as renormalized parameters (i.e., $R_R(\tau)$) plus "counter-terms".
  - Counter-terms will be proportional to $(\tau-t_0)^p$ and are chosen to eliminate the t_0 dependence of the aforementioned solution.
  - τ is known as the "subtraction point" or "renormalization scale."
  - This step yields the "renormalized" perturbative solution.
  - Renormalized solution must be independent of the choice of τ.
  - The solutions' explicit dependence on τ is cancelled by the implicit dependence of the renormalized parameters on τ.
  - Use this fact to derive a first-order differential equation (called the "renormalization group (RG) equation") for the renormalized parameter.
  - The right-hand side of the RG equation is the "beta (β) function."

Quantum Field Theory		Dynamical Renormalization Group
p_α	$\longrightarrow$	t
Λ	$\longrightarrow$	t_0
μ	$\longrightarrow$	τ

- Solve the RG equations and set $\tau = t$, the observation time.
  - All of the secularly growing terms are resummed at this order in perturbation theory.

Binary inspirals at leading post-Newtonian order

Equations of motion

- 0PN equations of motion in polar coordinates (motion occurs in a plane for all time)

$$\ddot{r} - r\omega^2 = -\frac{M}{r^2} + \frac{64M^3\nu}{15r^4}\dot{r} + \frac{16M^2\nu}{5r^3}\dot{r}^3 + \frac{16M^2\nu}{5r}\dot{r}\omega^2$$
$$r\dot{\omega} + 2\dot{r}\omega = -\frac{24M^3\nu}{5r^3}\omega - \frac{8M^2\nu}{5r^2}\dot{r}^2\omega - \frac{8M^2\nu}{5}\omega^3$$

- Radiation reaction from gravitational wave emission causes orbit to depart from a background orbit
  - For definiteness, consider a background circular orbit with a Keplerian angular frequency

$$\omega_B^2 = \frac{M}{r_B^3}$$

  - Perturbed orbit is described by:

$$r(t) = r_B + \delta r(t) \qquad \delta r / r_B = \mathcal{O}(v_B^5)$$
$$\omega(t) = \omega_B + \delta\omega(t) \qquad \delta\omega/\omega_B = \mathcal{O}(v_B^5) \qquad v_B \sim r_B\omega_B$$

- Expand equations of motion to first order in perturbations off of background orbit

$$\delta\ddot{r}(t) - 3\omega_B^2\delta r(t) - 2r_B\omega_B\delta\omega(t) = \mathcal{O}(r_B v_B^{10})$$
$$r_B\delta\dot{\omega}(t) + 2\omega_B\delta\dot{r}(t) = -\frac{32}{5}\nu r_B^6\omega_B^7 + \mathcal{O}(\omega_B v_B^{11})$$

General solution

- General solution is parameterized by four numbers (the bare parameters, “B”)

$$r(t) = r_B - \frac{64\nu}{5}\omega_B^6 r_B^6 (t-t_0) + \frac{64\nu}{5}\omega_B^5 r_B^6 \sin\omega_B(t-t_0) + A_B \sin\big(\omega_B(t-t_0)+\phi_B\big) + \mathcal{O}(v_B^{10} r_B \omega_B^2 T^2, v_B^{10} r_B \omega_B T)$$

$$\omega(t) = \omega_B + \frac{96\nu}{5} r_B^5 \omega_B^7 (t-t_0) - \frac{128\nu}{5} r_B^5 \omega_B^6 \sin\omega_B(t-t_0) - \frac{2\omega_B A_B}{r_B}\sin\big(\omega_B(t-t_0)+\phi_B\big) + \mathcal{O}(v_B^{10}\omega_B^3 T^2, v_B^{10}\omega_B^2 T)$$

- Can shift some bare parameters to remove non-secular sinusoids using trig identities

$$A_B \to A_B - \frac{64}{5}\nu r_B^6 \omega_B^5 \cos\phi_B \ , \quad \phi_B \to \phi_B + \frac{64}{5}\frac{\nu r_B^6 \omega_B^5}{A_B}\sin\phi_B$$

- This results in the following general perturbed solution:

$$r(t) = r_B - \frac{64\nu}{5} r_B^6 \omega_B^6 (t-t_0) + A_B \sin\big((t-t_0)\omega_B + \phi_B\big) + \mathcal{O}(v_B^{10} r_B \omega_B^2 T^2, v_B^{10} r_B \omega_B T)$$

$$\omega(t) = \omega_B + \frac{96\nu}{5} r_B^5 \omega_B^7 (t-t_0) - \frac{2\omega_B A_B}{r_B}\sin\big((t-t_0)\omega_B+\phi_B\big) + \mathcal{O}(v_B^{10}\omega_B^3 T^2, v_B^{10}\omega_B^2 T)$$

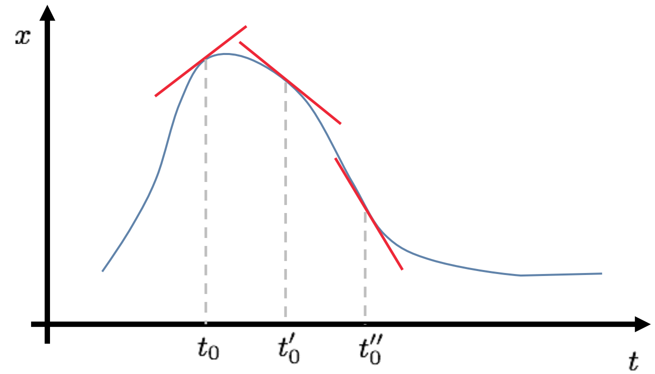
$$\phi(t) = \phi_B + (t-t_0)\omega_B + \frac{48\nu}{5} r_B^5 \omega_B^7 (t-t_0)^2 + \frac{2A_B}{r_B}\cos\big((t-t_0)\omega_B+\phi_B\big) + \mathcal{O}(v_B^{10}\omega_B^3 T^3, v_B^{10}\omega_B^2 T^2)$$

- Two types of perturbations off of background orbit
  - Non-secular terms (bounded in time)
  - Secular terms (grow linearly with time and eventually invalidate the perturbative solution)

$$T = t - t_0 \sim \frac{5}{32\nu r_B^5 \omega_B^6}$$

Renormalization

- Renormalize the initial data parameters
  - Parameters depend implicitly on initial time
  - Write a bare ("B") parameter as a renormalized ("R") parameter plus a "counter-term"
  - Use counter-terms to absorb secular divergences



$$
\begin{aligned}
r_B(t_0) &= r_R(\tau) + \delta_r(\tau, t_0) \\
\phi_B(t_0) &= \phi_R(\tau) + \delta_\phi(\tau, t_0) \\
\omega_B(t_0) &= \omega_R(\tau) + \delta_\omega(\tau, t_0) \\
A_B(t_0) &= A_R(\tau) + \delta_A(\tau, t_0)
\end{aligned}
$$

Counter-terms

$$
\begin{aligned}
\delta_\phi &= \mathcal{O}(1) \\
\delta_r &= \mathcal{O}(v_R^5 r_R \omega_R T) \\
\delta_\omega &= \mathcal{O}(v_R^5 \omega_R^2 T) \\
\delta_A &= \mathcal{O}(v_R^5 A_R \omega_R T)
\end{aligned}
$$

- Write perturbative solutions in terms of renormalized parameters
  - Drop higher order terms in $v_R^5(t-t_0)$ for perturbative consistency

$$
r(t) = r_R + \delta_r - \frac{64\nu}{5} r_R^6 \omega_R^6 (t - t_0) + A_R \sin\left((t - t_0)\omega_R + \phi_R + \delta_\phi\right) + \mathcal{O}(v_R^{10} r_R \omega_R^2 T^2, v_R^{10} r_R \omega_R T)
$$

$$
\omega(t) = \omega_R + \delta_\omega + \frac{96\nu}{5} r_R^5 \omega_R^7 (t - t_0) - \frac{2\omega_R A_R}{r_R} \sin\left((t - t_0)\omega_R + \phi_R + \delta_\phi\right) + \mathcal{O}(v_R^{10} \omega_R^3 T^2, v_R^{10} \omega_R^2 T)
$$

$$
\phi(t) = \phi_R + \delta_\phi + (t - t_0)\omega_R + \frac{48\nu}{5} r_R^5 \omega_R^7 (t - t_0)^2 + \frac{2A_R}{r_R} \cos\left((t - t_0)\omega_R + \phi_R + \delta_\phi\right) + \mathcal{O}(v_R^{10} \omega_R^3 T^3, v_R^{10} \omega_R^2 T^2)
$$

- Introduce the subtraction point/renormalization scale τ through $t\text{-}t_0 = (t\text{-}\tau)+(\tau\text{-}t_0)$
- Choose counter-terms to remove $(\tau\text{-}t_0)$ dependencies

$$
\begin{aligned}
r(t) = {} & r_R + \boxed{\delta_r} - \frac{64\nu}{5} r_R^6 \omega_R^6 (t-\tau) \boxed{- \frac{64\nu}{5} r_R^6 \omega_R^6 (\tau - t_0)} + A_R \sin\left((t-\tau)\omega_R + \boxed{(\tau - t_0)\omega_R} + \phi_R + \boxed{\delta_\phi}\right) \\
& + \mathcal{O}(v_R^{10} r_R \omega_R^2 T^2, v_R^{10} r_R \omega_R T) \\
\omega(t) = {} & \omega_R + \boxed{\delta_\omega} + \frac{96\nu}{5} r_R^5 \omega_R^7 (t-\tau) \boxed{+ \frac{96\nu}{5} r_R^5 \omega_R^7 (\tau - t_0)} \\
& - \frac{2\omega_R A_R}{r_R} \sin\left((t-\tau)\omega_R + \boxed{(\tau - t_0)\omega_R} + \phi_R + \boxed{\delta_\phi}\right) + \mathcal{O}(v_R^{10} \omega_R^3 T^2, v_R^{10} \omega_R^2 T) \\
\phi(t) = {} & \phi_R + \boxed{\delta_\phi} + (t-\tau)\omega_R + \boxed{(\tau - t_0)\omega_R} + \boxed{(t-\tau)\delta_\omega} + \boxed{(\tau - t_0)\delta_\omega} \\
& + \frac{48\nu}{5} r_R^5 \omega_R^7 (t-\tau) + \boxed{\frac{96\nu}{5} r_R^5 \omega_R^7 (t-\tau)(\tau - t_0)} + \boxed{\frac{48\nu}{5} r_R^5 \omega_R^7 (\tau - t_0)^2} \\
& + \frac{2A_R}{r_R} \cos\left((t-\tau)\omega_R + \boxed{(\tau - t_0)\omega_R} + \phi_R + \boxed{\delta_\phi}\right) + \mathcal{O}(v_R^{10} \omega_R^3 T^3, v_R^{10} \omega_R^2 T^2)
\end{aligned}
$$

- Counter-terms through first-order are:

$$
\begin{aligned}
\delta_r(\tau, t_0) &= \frac{64}{5} \nu r_R^6 \omega_R^6 (\tau - t_0) + \mathcal{O}(v_R^{10} r_R \omega_R^2 T^2, v_R^{10} r_R \omega_R T) \\
\delta_\omega(\tau, t_0) &= -\frac{96}{5} \nu r_R^5 \omega_R^7 (\tau - t_0) + \mathcal{O}(v_R^{10} \omega_R^3 T^2, v_R^{10} \omega_R^2 T) \\
\delta_\phi(\tau, t_0) &= -\omega_R (\tau - t_0) + \frac{48}{5} \nu r_R^5 \omega_R^7 (\tau - t_0)^2 + \mathcal{O}(v_R^{10} \omega_R^3 T^3, v_R^{10} \omega_R^2 T^2) \\
\delta_A(\tau, t_0) &= \mathcal{O}(v_R^5 A_R \omega_R T)
\end{aligned}
$$

Renormalization Group equations

- Recall: *bare parameter = renormalized parameter + counter-term*

$$r_B(t_0) = r_R(\tau) + \frac{64}{5}\nu r_R^6(\tau)\omega_R^6(\tau)(\tau - t_0) + \mathcal{O}(v_R^{10} r_R \omega_R^2 T^2, v_R^{10} r_R \omega_R T)$$
$$\omega_B(t_0) = \omega_R(\tau) - \frac{96}{5}\nu r_R^5(\tau)\omega_R^7(\tau)(\tau - t_0) + \mathcal{O}(v_R^{10}\omega_R^3 T^2, v_R^{10}\omega_R^2 T)$$
$$\phi_B(t_0) = \phi_R(\tau) - \omega_R(\tau)(\tau - t_0) + \frac{48}{5}\nu r_R^5(\tau)\omega_R^7(\tau)(\tau - t_0)^2 + \mathcal{O}(v_R^{10}\omega_R^3 T^3, v_R^{10}\omega_R^2 T^2)$$
$$A_B(t_0) = A_R(\tau) + \mathcal{O}(v_R^{10} A_R \omega_R T)$$

- Note that the bare parameters are independent of τ
  - Differentiate the bare parameters with respect to τ and set the result to zero.
  - Solve for the derivative of the renormalized parameter.

$$\frac{dr_R}{dt} = -\frac{64}{5}\nu r_R^6(\tau)\omega_R^6(\tau) + \mathcal{O}(\boxed{v_R^{10} r_R \omega_R^2 T,}\, v_R^{10} r_R \omega_R)$$
$$\frac{d\omega_R}{dt} = \frac{96}{5}\nu r_R^5(\tau)\omega_R^7(\tau) + \mathcal{O}(\boxed{v_R^{10}\omega_R^3 T,}\, v_R^{10}\omega_R^2)$$
$$\frac{d\phi_R}{d\tau} = \omega_R(\tau) + \boxed{\frac{d\omega_R}{d\tau}(\tau - t_0)}\boxed{-\frac{96}{5}\nu r_R^5(\tau)\omega_R^7(\tau)(\tau - t_0)} + \mathcal{O}(\boxed{v_R^{10}\omega_R^3 T^2, v_R^{10}\omega_R^2 T})$$
$$\frac{dA_R}{d\tau} = \mathcal{O}(v_R^5 A_R \omega_R)$$

  - Secular pieces at first-order automatically cancel (self-consistency check)

- The RG equations describe quantities that are finite, which cannot have secular divergences

- Solve the RG equations to describe the "flow" from $\tau = t_i$ to $\tau = t$
  - Analytically, if possible
  - Numerically, otherwise (coupled first-order differential equations)

$$
\begin{aligned}
r_R(t) &= r_R(t_i)\left(1 - \frac{256}{5}\nu r_R^5(t_i)\omega_R^6(t_i)(t - t_i)\right)^{1/4} + \mathcal{O}(v_R^{10} r_R \omega_R T) \\
\omega_R(t) &= \omega_R(t_i)\left(\frac{r_R(t_i)}{r_R(t)}\right)^{3/2} + \mathcal{O}(v_R^{10}\omega_R^2 T) \\
\phi_R(t) &= \phi_R(t_i) + \frac{1}{32\nu r_R^5(t_i)\omega_R^5(t_i)} - \frac{1}{32\nu r_R^5(t)\omega_R^5(t)} + \mathcal{O}(v_R^{10}) \\
A_R(t) &= A_R(t_i) + \mathcal{O}(v_R^5 A_R \omega_R)
\end{aligned}
$$

- Substitute the RG solutions into the perturbative solutions and evaluate at $\tau = t$

$$
\begin{aligned}
r(t) &= r_R(t) + A_R(t)\sin\phi_R(t) + \mathcal{O}(v_R^{10} r_R) \\
\omega(t) &= \omega_R(t) - \frac{2\omega_R(t)A_R(t)}{r_R(t)}\sin\phi_R(t) + \mathcal{O}(v_R^{10}\omega_R) \\
\phi(t) &= \phi_R(t) + \frac{2A_R(t)}{r_R(t)}\cos\phi_R(t) + \mathcal{O}(v_R^{10})
\end{aligned}
$$

Comments

- In analogy with quantum field theory calculations, the first-order perturbative calculation is sometimes referred to as a "1-loop" calculation
- Solutions to RG equations resum secular divergences order-by-order in powers of $v_R{}^5 T$

$$r_R(t) = r_R(t_i)\left(1 - \frac{256}{5}\nu r_R^5(t_i)\omega_R^6(t_i)(t-t_i)\right)^{1/4} + \mathcal{O}(v_R^{10} r_R \omega_R T)$$
$$= r_R(t_i)\left(1 - \frac{64}{5}\nu r_R^5(t_i)\omega_R^6(t_i)(t-t_i) - \frac{6144}{25}\left(\nu r_R^5(t_i)\omega_R^6(t_i)(t-t_i)\right)^2 + \mathcal{O}(v_R^5\omega_R T)^3\right) + \mathcal{O}(v_R^{10} r_R \omega_R T)$$

  - Third term is a secular divergence that appears at 2nd order but is already captured at 1st order

- Error estimates are naturally provided during the calculation
- Dynamical Renormalization Group identifies (1-loop) invariants along the RG trajectory

$$r_R^3(t)\omega_R^2(t) = \text{constant} = M \qquad \phi_R(t) + \frac{1}{32\nu r_R^5(t)\omega_R^5(t)} = \text{constant}$$
$$r_R^4(t)\left(1 + \frac{256}{5}\nu r_R^5(t)\omega_R^5(t)\, t\right) = \text{constant} \qquad A_R(t) = \text{constant}$$

- Terms involving *(t-τ)(τ-t_0)* must be cancelled by pieces generated from counter-terms
  - Provides another self-consistency check of the calculation
  - Removal of such cross terms is important for the renormalizability of the perturbative solution

DRG to second order in ϵ: The 2-loop calculation

- Use same equations of motion but expanded to 2nd order in the perturbations.
- Find general solution to the 2nd order equations
- Shift bare parameters (i.e., initial data) to absorb redundant, finite pieces
  - These shifts have some freedom parameterized by μ.
  - Easiest to choose a "renormalization scheme" so as to keep the resulting 2-loop RG equations as simple as possible, which is equivalent to choosing μ to remove all the finite, t-dependent pieces in the expression for the 2nd order angular frequency solution.
- Renormalize initial data parameters to remove secular divergences.
  - For example:

$$
\begin{aligned}
r_{2-\mathrm{loop}}(t) = {} & \frac{1}{2}\frac{A_R^2}{r_R} - \frac{29\,696}{75}\nu^2 r_R^{11}\omega_R^{10} \boxed{- \frac{6144}{25}\nu^2 r_R^{11}\omega_R^{12}}\left[(t-\tau)^2 \boxed{- (\tau - t_0)^2}\right] \\
& - \frac{656}{15}\nu A_R r_R^5 \omega_R^5 \cos\left(\phi_R + \omega_R(t-\tau)\right) + \frac{48}{5}\nu A_R r_R^5 \omega_R^7 (t-\tau)^2 \cos\left(\phi_R + \omega_R(t-\tau)\right) \\
& + \frac{1}{2}\frac{A_R^2}{r_R}\cos\left(2\phi_R + 2\omega_R(t-\tau)\right) \boxed{- \frac{496}{15}\nu A_R r_R^5 \omega_R^6}\left[(t-\tau) + \boxed{(\tau - t_0)}\right] \sin\left(\phi_R + \omega_R(t-\tau)\right) \\
& + \boxed{\delta_R^{v^{10}}} + \boxed{\delta_A^{v^{10}}} \sin\left(\phi_R + (t-\tau)\omega_R\right)
\end{aligned}
$$

  - Yields the counter-terms for r_B and A_B through 2-loops
  - Importantly, cross terms involving $(t-\tau)^p(\tau-t_0)^q$ automatically cancel with other terms containing lower-order counter-terms (self-consistency).

- At the end of the day, the counter-terms through 2-loops are

$$
\begin{aligned}
\delta_R &= \frac{64\nu}{5} r_R^6 \omega_R^6 (\tau - t_0) - \frac{6144}{25} \nu^2 r_R^{11} \omega_R^{12} (\tau - t_0)^2 + \mathcal{O}(v_R^{15} r_R \omega_R^2 T^2, v_R^{15} r_R \omega_R T) \\
\delta_\Omega &= -\frac{96\nu}{5} r_R^5 \omega_R^7 (\tau - t_0) + \frac{16896}{25} \nu^2 r_R^{10} \omega_R^{13} (\tau - t_0)^2 + \mathcal{O}(v_R^{15} \omega_R^3 T^2, v_R^{15} \omega_R^2 T) \\
\delta_\Phi &= -\omega_R (\tau - t_0) + \frac{48\nu}{5} r_R^5 \omega_R^7 (\tau - t_0)^2 - \frac{5632}{25} \nu^2 r_R^{10} \omega_R^{13} (\tau - t_0)^3 \\
&\quad + \frac{504}{5} \nu A_R r_R^4 \omega_R^5 \sin \Phi_B(t_0) - \frac{5}{4} \frac{A_R^2}{r_R^2} \sin 2\Phi_B(t_0) + \mathcal{O}(v_R^{15} \omega_R^3 T^3, v_R^{15} \omega_R^2 T^2) \\
\delta_A &= \frac{496}{15} A_R \nu r_R^5 \omega_R^6 (\tau - t_0) + \mathcal{O}(v_R^{10} A_R \omega_R T)
\end{aligned}
$$

- RG equations for initial data parameters are

$$
\begin{aligned}
\frac{dr_R}{d\tau} &= -\frac{64}{5} \nu r_R^6 \omega_R^6 + \mathcal{O}(v_R^{15} r_R \omega_R) \\
\frac{d\omega_R}{d\tau} &= \frac{96}{5} \nu r_R^5 \omega_R^7 + \mathcal{O}(v_R^{10} \omega_R^2) \\
\frac{d\phi_R}{d\tau} &= \omega_R + \mathcal{O}(v_R^{15}) \\
\frac{dA_R}{d\tau} &= -\frac{496}{15} A_R \nu r_R^5 \omega_R^6 + \mathcal{O}(v_R^{10} A_R \omega_R)
\end{aligned}
$$

  - A large number of cancellations happen to prevent secular terms from remaining in the RG equations (self-consistency)
  - RG equations and solutions for all renormalized quantities (except *A)* are same as at 1-loop

- Solution for A_R (= $e_R R_R$ where e_R is the orbit's small eccentricity) is

$$A_R(t) = A_R(t_i)\left(\frac{r_R(t)}{r_R(t_i)}\right)^{31/12} + \mathcal{O}(v_R^{10} A_R \omega_R T) \implies e_R(t) \equiv \frac{A_R(t)}{r_R(t)} = e_R(t_i)\left(\frac{r_R(t)}{r_R(t_i)}\right)^{19/12} + \mathcal{O}(v_R^{10} e_R r_R \omega_R T)$$

  - Power of 19/12 accounts for the circularization of a compact binary inspiral
  - Matches the well-known expression of Peters (1964) in the limit of small orbital eccentricity.

- RG invariants are same as at 1-loop except for a 2-loop modification to A_R invariant:

$$A_R(t) = \text{constant} \implies e_R^{12}(t) r_R^{19}(t) = \text{constant}$$

- Full, resummed perturbative solution through 2$^{\text{nd}}$ order is:

$$r(t) = r_R(t)\Big[1 + e_R(t)\sin\phi_R(t) + \frac{1}{2}e_R^2(t) - \frac{29\,696}{75}\nu^2 r_R^{10}(t)\omega_R^{10}(t) - \frac{656}{15}\nu e_R(t) r_R^5(t)\omega_R^5(t)\cos\phi_R(t) + \frac{1}{2}e_R^2(t)\cos 2\phi_R(t)\Big] + \mathcal{O}(v_R^{15} r_R)$$

$$\omega(t) = \omega_R(t)\Big[1 - 2e_R(t)\sin\phi_R(t) + \frac{904}{15}\nu e_R(t) r_R^5(t)\omega_R^5(t)\cos\phi_R(t) - \frac{5}{2}e_R^2(t)\cos 2\phi_R(t)\Big] + \mathcal{O}(v_R^{15}\omega_R)$$

$$\phi(t) = \phi_R(t) + 2e_R(t)\cos\phi_R(t) + \frac{504}{5}\nu e_R(t) r_R^5(t)\omega_R^5(t)\sin\phi_R(t) - \frac{5}{4}e_R^2(t)\sin 2\phi_R(t) + \mathcal{O}(v_R^{15})$$

Binary inspirals at first post-Newtonian order

- Include 1PN radiation reaction force but 0PN potential (for demonstration)
- Following the same steps as for 0PN order, the 1-loop RG equations are

$$
\begin{aligned}
\frac{dr_R}{d\tau} &= -\frac{64}{5}\nu r_R^6 \omega_R^6 - \frac{4\nu}{105}(336\nu - 3179) r_R^8 \omega_R^8 \\
\frac{d\omega_R}{d\tau} &= \frac{96\nu}{5} r_R^5 \omega_R^7 + \frac{2\nu}{35}(336\nu - 3179) r_R^7 \omega_R^9 \\
\frac{d\phi_R}{d\tau} &= \omega_R \,, \quad \frac{dA_R}{d\tau} = 0
\end{aligned}
$$

- Analytical solutions can be found when integrating these RG equations

$$
\begin{aligned}
-\frac{64\nu}{5} M^3 (t - t_i) = {} & \frac{1}{4}\left(r_R^4(t) - r_R^4(t_i)\right) + \frac{1}{3}\alpha M \left(r_R^3(t) - r_R^3(t_i)\right) + \frac{1}{2}\alpha^2 M^2 \left(r_R^2(t) - r_R^2(t_i)\right) \\
& + \alpha^3 M^3 \left(r_R(t) - r_R(t_i)\right) + \alpha^4 M^4 \log\left(\frac{r_R(t) - \alpha M}{r_R(t_i) - \alpha M}\right)
\end{aligned}
$$

$$
\omega_R(t) = \omega_R(t_i) \left(\frac{r_R(t)}{r_R(t_i)}\right)^{3/2} = \frac{M^{1/2}}{r_R^{3/2}(t)}
$$

(same as 0PN)

$$
\begin{aligned}
-\frac{32\nu}{5} M^{5/2} \left(\phi_R(t) - \phi_R(t_i)\right) = {} & \frac{1}{5}\left(r_R^{5/2}(t) - r_R^{5/2}(t_i)\right) + \frac{1}{3}\alpha M \left(r_R^{3/2}(t) - r_R^{3/2}(t_i)\right) + \alpha^2 M^2 \left(r_R^{1/2}(t) - r_R^{1/2}(t_i)\right) \\
& - \alpha^{5/2} M^{5/2} \left[\tanh^{-1}\sqrt{\frac{r_R(t)}{\alpha M}} - \tanh^{-1}\sqrt{\frac{r_R(t_i)}{\alpha M}}\right]
\end{aligned}
$$

$$
\alpha = \frac{3179}{336} - \nu
$$

Summary

- The Dynamical Renormalization Group method:
  - Is a systematic, turn-the-crank way to solve differential equations
  - Provides formal error estimates on the resulting globally valid approximate solutions
  - Generates perturbatively invariant quantities along a RG flow
  - Has built-in checks for self-consistency that can be used to verify correctness of the calculation
  - Subsumes other well-known global approximation methods including:
    - WKB
    - Multiple scale analysis
    - Boundary layer theory

- We've applied DRG to several problems, at varying levels of completion:
  - Damped harmonic oscillator (useful test ground for understanding the method in detail)
  - Nonspinning 0PN compact binary inspirals
  - Nonspinning 1PN compact binary inspirals (in progress)
  - Tidal dissipation of spinning, extended bodies in a binary (in progress)
  - Poynting-Robertson effect on motion of dust irradiated by a star (in progress)
  - Scalar self-force inspirals in a weak gravitational field

Future work (1)

- Apply DRG to precessing compact binary inspirals and other spinning systems
  - Can analytic solutions to the RG equations be found?
  - Provide a formal error estimate for the validity of the resummed perturbative solutions

- Other interesting applications include:
  - Exoplanet orbital evolutions
  - Binary inspirals/outspirals of not-so-compact bodies (e.g., mass-transferring stellar bodies)
  - Orbital mechanics of satellites and spacecraft

- Could DRG handle transient (orbital) resonances since averaging methods are not used? [e.g., see Flanagan & Hinderer (2012) for the breakdown of averaging]

- Can DRG be combined with numerical solutions of backgrounds?
  - If so, could be useful for resumming secular divergences encountered in numerical simulations of binary black holes for theories with corrections to general relativity [see Okounkova et al (2017)]
  - Could be useful for calculating gravitational self-force inspirals [see Gralla & Wald (2008), Warburton et al (2012), Osburn et al (2016)]

Future work (2)

- Do the RG invariants have symmetries associated with them?
  - Is there a “Noether’s Theorem” that relates continuous symmetry transformations to these quantities conserved throughout the RG flow (e.g., inspirals)?
  - Equal-mass and equal-spin-magnitude compact binary inspirals possess an inspiral-invariant quantity found empirically in Galley et al (2010):

$$\frac{2\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2 + (\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{L}})(\hat{\boldsymbol{S}}_2 \cdot \hat{\boldsymbol{L}})}{\sqrt{5}}$$

    Is it derivable using the Dynamical Renormalization Group approach?
    Is there a similar expression more generally applicable?

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