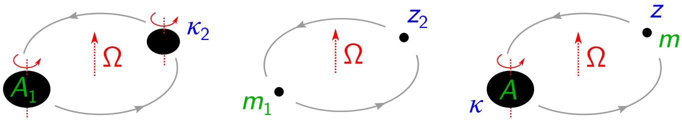
The laws of binary black hole mechanics: An update

Alexandre Le Tiec

Laboratoire Univers et Théories
Observatoire de Paris / CNRS

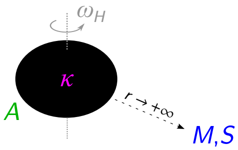


The laws of black hole mechanics

[Hawking 1972, Bardeen, Carter & Hawking 1973]

- Zeroth law of mechanics:

$$\kappa = \text{const.} \quad (\text{on } \mathcal{H})$$

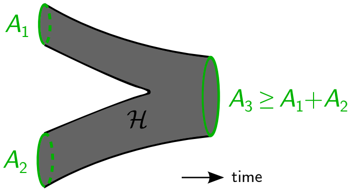


- First law of mechanics:

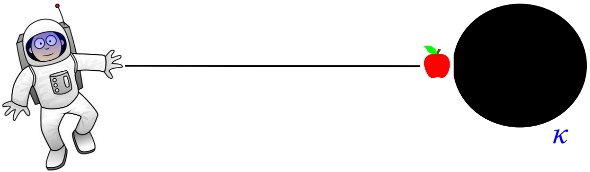
$$\delta M = \omega_H \, \delta S + \frac{\kappa}{8\pi} \, \delta A$$

- Second law of mechanics:

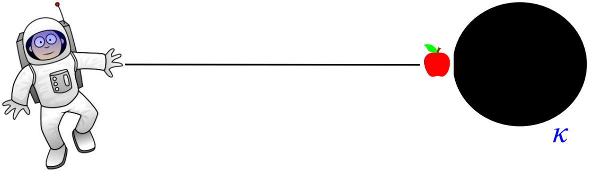
$$\delta A \geq 0$$



What is the horizon surface gravity?



What is the horizon surface gravity?



- For an event horizon $\mathcal{H}$ generated by a Killing field k^α, i.e. for a Killing horizon:

$$\kappa^2 \equiv \frac{1}{2} \left(\nabla^\alpha k^\beta \, \nabla_\beta k_\alpha \right)\Big|_{\mathcal{H}}$$

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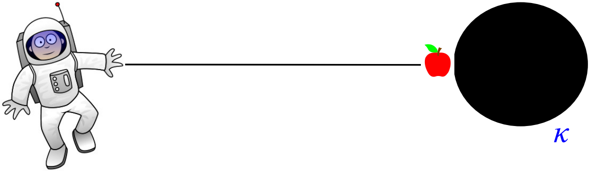

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- For a Schwarzschild black hole of mass M, this yields

$$\kappa = \frac{1}{4M} = \frac{GM}{R_S^2}$$

Outline

1. Laws of binary black hole mechanics
   - In general relativity
   - In post-Newtonian theory
   - First laws for generic bound orbits

2. Applications
   - Calibration of EOB models
   - Horizon surface gravity

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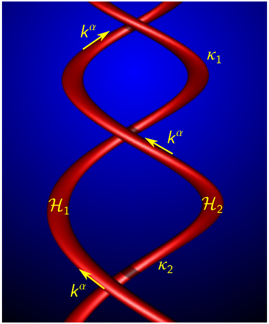
Generalized zeroth law of mechanics

[Friedman, Uryū & Shibata 2002]

- Black hole spacetimes with *helical* Killing vector field k^α
- On each component $\mathcal{H}_i$ of the event horizon, the expansion and shear of the generators vanish
- Generalized rigidity theorem: $\mathcal{H} = \bigcup_i \mathcal{H}_i$ is a Killing horizon
- *Constant* horizon surface gravity

$$\kappa_i^2 = \frac{1}{2}(\nabla^\alpha k^\beta \, \nabla_\beta k_\alpha)\big|_{\mathcal{H}_i}$$

- The binary black hole system is in a state of *corotation*

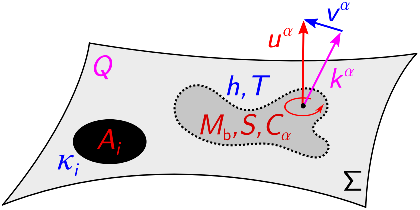


Generalized first law of mechanics

[Friedman, Uryū & Shibata 2002]

- Spacetimes with black holes + perfect fluid matter sources
- One-parameter family of solutions $\{g_{\alpha\beta}(\lambda), u^\alpha(\lambda), \rho(\lambda), s(\lambda)\}$
- *Globally* defined Killing field $k^\alpha \to$ conseved Noether charge Q

$$\delta Q = \sum_i \frac{\kappa_i}{8\pi}\,\delta A_i + \int_\Sigma \left[\, \bar{h}\,\delta(\mathrm{d}M_\mathrm{b}) + \bar{T}\delta(\mathrm{d}S) + v^\alpha\delta(\mathrm{d}C_\alpha)\right]$$



Issue of asymptotic flatness

[Friedman, Uryū & Shibata 2002]

- Binaries on circular orbits have a *helical* Killing symmetry k^α
- Helically symmetric spacetimes are *not* asymptotically flat
  [Gibbons & Stewart 1983, Detweiler 1989, Klein 2004]
- Asymptotic flatness can be recovered if radiation can be "turned off":
  - Conformal Flatness Condition
  - Post-Newtonian approximation
- For asymptotically flat spacetimes:

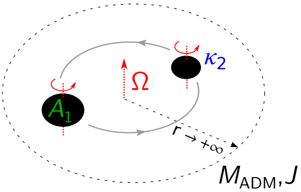
$$k^\alpha \to t^\alpha + \Omega\,\phi^\alpha \quad \text{and} \quad \delta Q = \delta M_{\mathrm{ADM}} - \Omega\,\delta J$$

Application to black hole binaries

[Friedman, Uryū & Shibata 2002]

- Rigidity theorem $\rightarrow$ black holes are in a state of corotation
- Conformal flatness condition $\rightarrow$ asymptotic flatness recovered
  ↳ preferred normalization of κ_i [Le Tiec & Grandclément 2017]
- For binary black holes the generalized first law reduces to

$$\delta M_{\mathrm{ADM}} = \Omega\,\delta J + \sum_i \frac{\kappa_i}{8\pi}\,\delta A_i$$



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First law for point-particle binaries

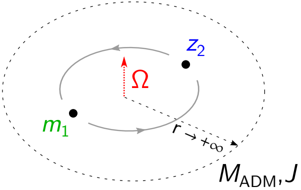
[Le Tiec, Blanchet & Whiting 2012]

- For balls of dust, the generalized first law reduces to

$$\delta Q = \int_\Sigma z\,\delta(\mathrm{d}M_\mathrm{b}) + \cdots\,, \quad \text{where} \quad z = -k^\alpha u_\alpha$$

- Conservative PN dynamics $\rightarrow$ asymptotic flatness recovered
- Two *spinless* compact objects modelled as point masses m_i and moving along circular orbits obey the first law

$$\delta M_\mathrm{ADM} = \Omega\,\delta J + \sum_i z_i\,\delta m_i$$



Extension to spinning binaries

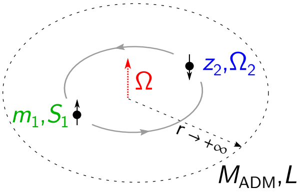
[Blanchet, Buonanno & Le Tiec 2013]

- Canonical ADM Hamiltonian $H(\mathbf{x}_i, \mathbf{p}_i, \mathbf{S}_i; m_i)$ of two point particles with masses m_i and spins $\mathbf{S}_i$ [Steinhoff *et al.* 2008]
- Redshift observables and spin precession frequencies:

$$\frac{\partial H}{\partial m_i} = z_i \quad \text{and} \quad \frac{\partial H}{\partial \mathbf{S}_i} = \mathbf{\Omega}_i$$

- First law for aligned spins ($J = L + \sum_i S_i$) and circular orbits:

$$\delta M = \Omega \, \delta L + \sum_i \left(z_i \, \delta m_i + \Omega_i \, \delta S_i \right)$$



Corotating point particles

[Blanchet, Buonanno & Le Tiec 2013]

- A point particle with rest mass m_i and spin S_i is given an *irreducible* mass μ_i and a proper rotation frequency ω_i *via*

$$\delta m_i = \omega_i \, \delta S_i + c_i \, \delta \mu_i \quad \text{and} \quad m_i^2 = \mu_i^2 + S_i^2/(4\mu_i^2)$$

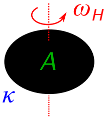
- The first law of binary point-particle mechanics becomes

$$\delta M = \Omega \, \delta J + \sum_i \left[z_i c_i \, \delta \mu_i + (z_i \, \omega_i + \Omega_i - \Omega) \, \delta S_i \right]$$

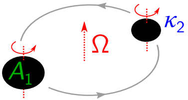
- Comparing with the first law for *corotating* black holes, $\delta M = \Omega \, \delta J + \sum_i (4\mu_i \kappa_i) \, \delta \mu_i$, the corotation condition is

$$z_i \, \omega_i = \Omega - \Omega_i \quad \longrightarrow \quad \omega_i(\Omega) \quad \longrightarrow \quad S_i(\Omega)$$

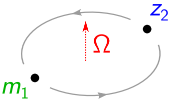
Black holes and point particles



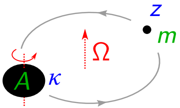
$$\delta M - \omega_H \, \delta S = \frac{\kappa}{8\pi} \, \delta A \qquad\qquad M - 2\omega_H S = \frac{\kappa A}{4\pi}$$



$$\delta M - \Omega \, \delta J = \sum_i \frac{\kappa_i}{8\pi} \, \delta A_i \qquad\qquad M - 2\Omega J = \sum_i \frac{\kappa_i A_i}{4\pi}$$



$$\delta M - \Omega \, \delta J = \sum_i z_i \, \delta m_i \qquad\qquad M - 2\Omega J = \sum_i z_i m_i$$



$$\delta M - \Omega \, \delta J = \frac{\kappa}{8\pi} \, \delta A + z \, \delta m \qquad\qquad M - 2\Omega J = \frac{\kappa A}{4\pi} + z m$$

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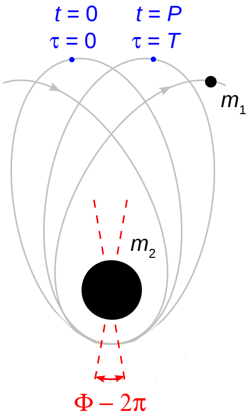
Averaged redshift for eccentric orbits

- Generic eccentric orbit parameterized by the two requencies

$$\Omega_r = \frac{2\pi}{P}\,, \quad \Omega_\phi = \frac{\Phi}{P}$$

- Time average of redshift $z = \mathrm{d}\tau/\mathrm{d}t$ over one radial period

$$\langle z \rangle \equiv \frac{1}{P}\int_0^P z(t)\,\mathrm{d}t = \frac{T}{P}$$



First law of mechanics for eccentric orbits

[Le Tiec 2015, Blanchet & Le Tiec 2017]

- Canonical ADM Hamiltonian $H(\mathbf{x}_i, \mathbf{p}_i; m_i)$ of two point particles with constant masses m_i

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- Variation δH + Hamilton's equation + orbital averaging:

$$\delta M = \Omega_\phi \, \delta L + \Omega_r \, \delta R + \sum_i \langle z_i \rangle \, \delta m_i$$

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- First integral associated with the variational first law:

$$M = 2 \left(\Omega_\phi L + \Omega_r R \right) + \sum_i \langle z_i \rangle m_i$$

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- These relationships are valid up to *at least* 4PN order, despite the tail-induced non-local-in-time dynamics

Particle Hamiltonian first law

- Geodesic motion of test mass m in Kerr geometry $\bar{g}_{\alpha\beta}$ derives from canonical Hamiltonian

$$\bar{H}(x^\mu, p_\mu) = \frac{1}{2m}\,\bar{g}^{\alpha\beta}(x)p_\alpha p_\beta$$

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- Canonical transformation $(x^\mu, p_\mu) \to (q_\alpha, J_\alpha)$ to *generalized* action-angle variables [Schmidt 2002, Hinderer & Flanagan 2008]

$$\frac{\mathrm{d}J_\alpha}{\mathrm{d}\tau} = -\frac{\partial \bar{H}}{\partial q_\alpha} = 0\,, \quad \frac{\mathrm{d}q_\alpha}{\mathrm{d}\tau} = \frac{\partial \bar{H}}{\partial J_\alpha} \equiv \omega_\alpha$$

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- Varying $\bar{H}(J_\alpha)$ yields a particle Hamiltonian first law valid for *generic* bound orbits [Le Tiec 2014]

$$\delta E = \Omega_\phi\,\delta L + \Omega_r\,\delta J_r + \Omega_\theta\,\delta J_\theta + \langle z\rangle\,\delta m$$

Inclusion of conservative GSF effects

[Isoyama *et al.* 2017]

- Geodesic motion of self-gravitating mass m in *time-symmetric* regular metric $\bar{g}_{\alpha\beta} + h^{\mathrm{R}}_{\alpha\beta}$ derives from canonical Hamiltonian

$$H(x^\mu, p_\mu; \gamma) = \bar{H}(x^\mu, p_\mu) + H_{\mathrm{int}}(x^\mu, p_\mu; \gamma)$$

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- In class of canonical gauges, one can define a *unique* effective Hamiltonian $\mathcal{H}(J) = \bar{H}(J) + \frac{1}{2}\langle H_{\mathrm{int}}\rangle(J)$ yielding a first law valid for *generic* bound orbits:

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- The actions $\mathcal{J}_\alpha$ and the averaged redshift $\langle z\rangle$, as functions of $(\Omega_r, \Omega_\theta, \Omega_\phi)$, include conservative self-force corrections from the *gauge-invariant* averaged interaction Hamiltonian $\langle H_{\text{int}}\rangle$

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Applications of the first law

- Fix 'ambiguity parameters' in 4PN 2-body equations of motion
  [Jaranowski & Schäfer 2012, Damour *et al.* 2014, Bernard *et al.* 2016]

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  [Zimmerman, Lewis & Pfeiffer 2016, Le Tiec & Grandclément 2017]

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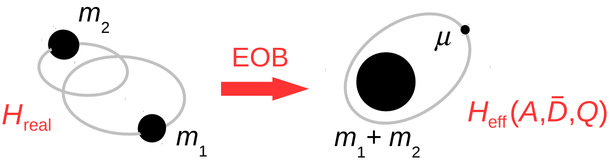
EOB dynamics beyond circular motion

[Le Tiec 2015]



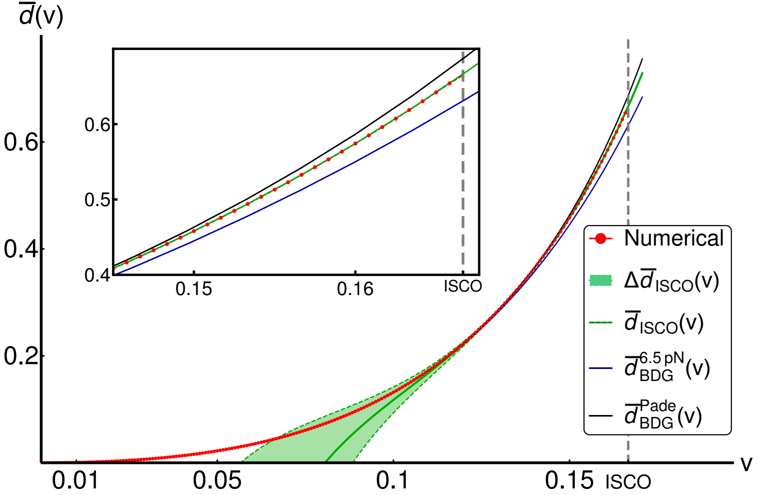
- Conservative EOB dynamics determined by "potentials"

$$A = 1 - 2M/r + \nu\, a(r) + \cdots$$
$$\bar{D} = 1 + \nu\, \bar{d}(r) + \cdots$$
$$Q = \nu\, q(r)\, p_r^4 + \cdots$$

- Functions $a(r)$, $\bar{d}(r)$ and $q(r)$ controlled by $\langle z \rangle_{\text{GSF}}(\Omega_r, \Omega_\phi)$

EOB dynamics beyond circular motion

[Akcay & van de Meent 2016]



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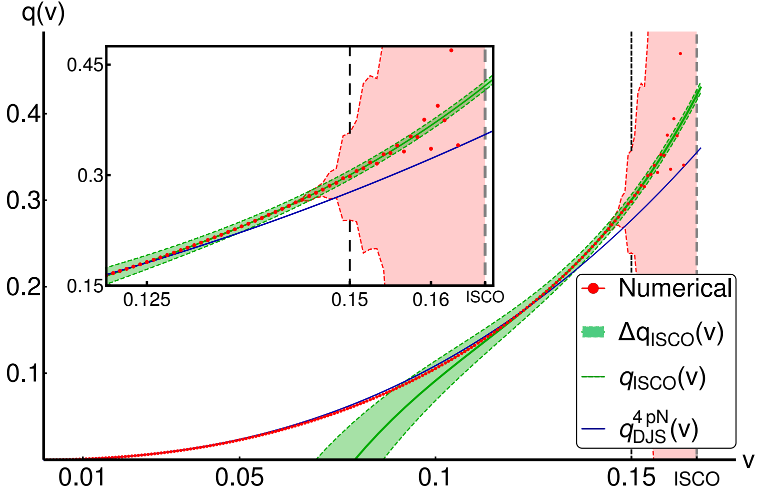
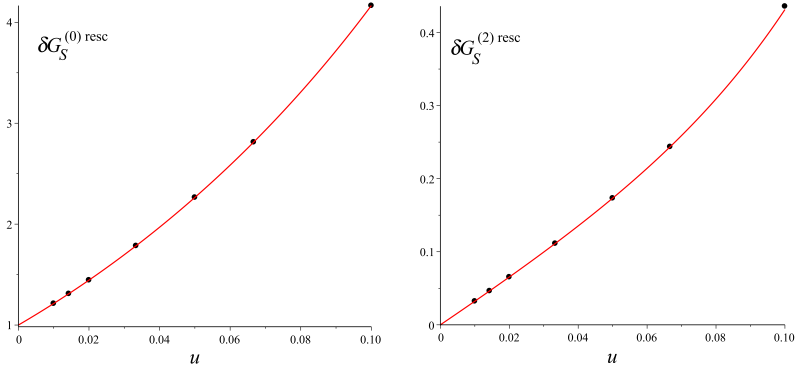


EOB dynamics for spinning bodies

[Bini, Damour & Geralico 2016]



First law for spinning bodies
GSF contribution to redshift

$\Bigg\} \Longrightarrow$ SO coupling function $\delta G_S(u, \hat{a})$

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Surface gravity and redshift

[Blanchet, Buonanno & Le Tiec 2013]

- First law for corotating black holes

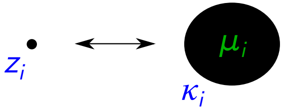
$$\delta M = \Omega\,\delta J + \sum_i (4\mu_i\kappa_i)\,\delta\mu_i$$

- First law for corotating point particles

$$\delta M = \Omega\,\delta J + \sum_i z_i c_i\,\delta\mu_i$$

- Analogy between BH surface gravity and particle redshift

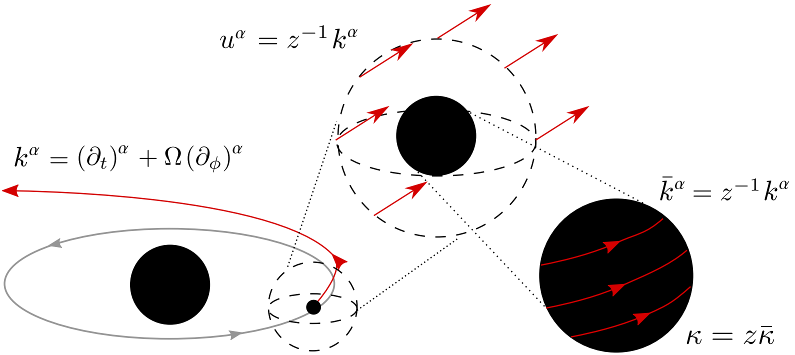
$$4\mu_i\kappa_i \longleftrightarrow z_i c_i$$



- New *invariant* relations for NR/BHP/PN comparison: $\kappa_i(\Omega)$

Surface gravity and redshift

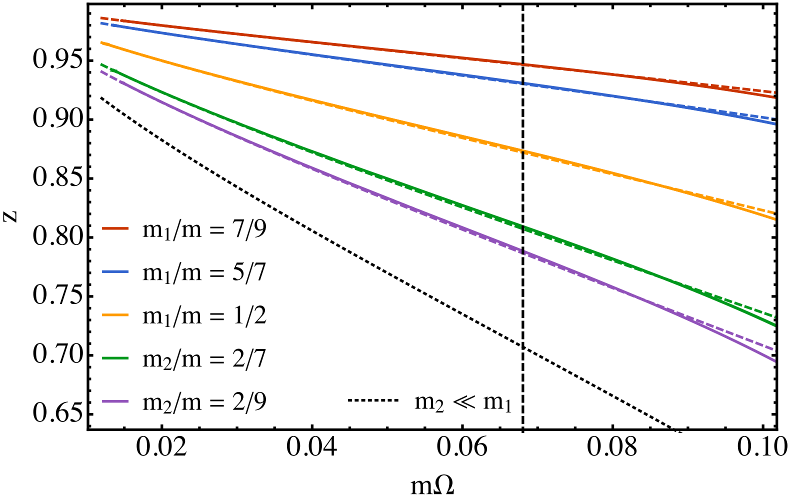
[Pound 2015 (unpublished)]



(Credit: Zimmerman, Lewis & Pfeiffer 2016)

Redshift *vs* orbital frequency

[Zimmerman, Lewis & Pfeiffer 2016]



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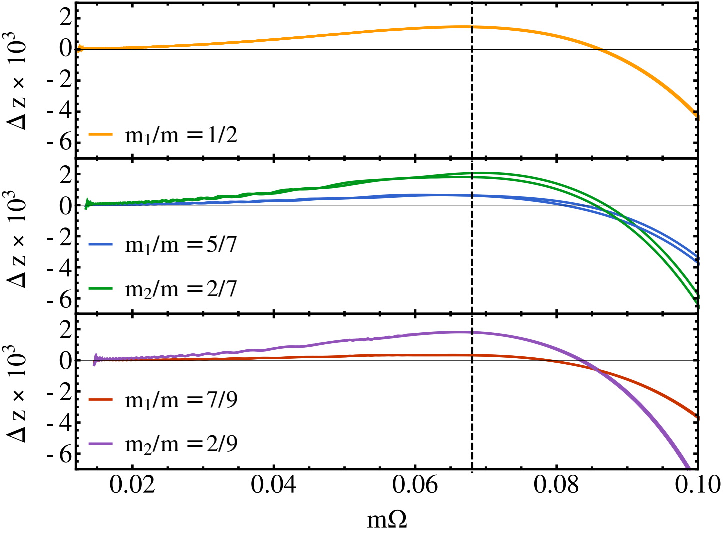

Quasi-equilibrium initial data

- 3+1 decomposition of the metric:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right)$$

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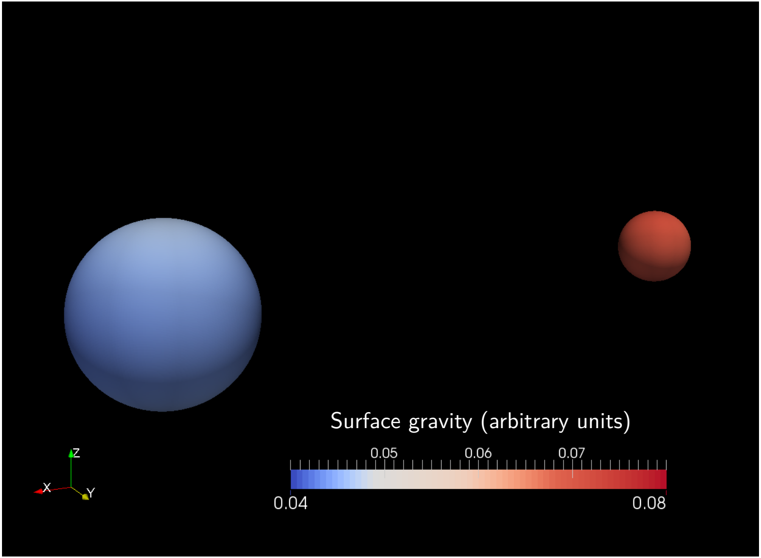
- Impose vanishing linear momentum to find rotation axis

Surface gravity for mass ratio 2 : 1



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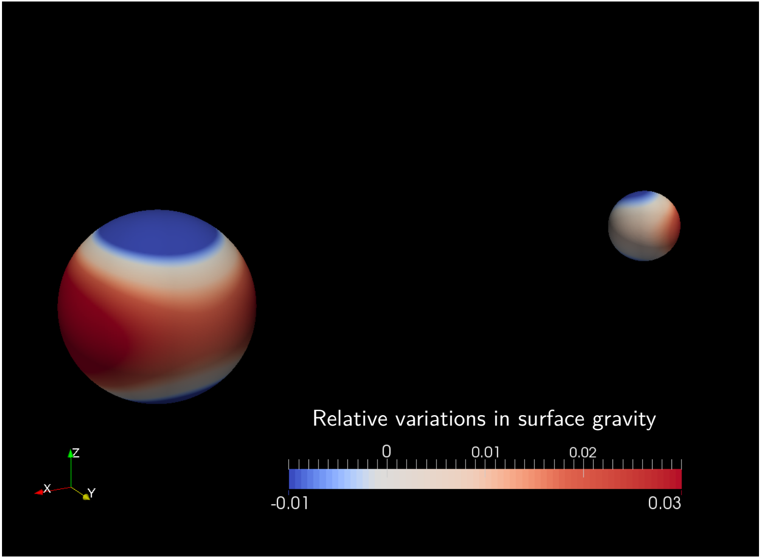
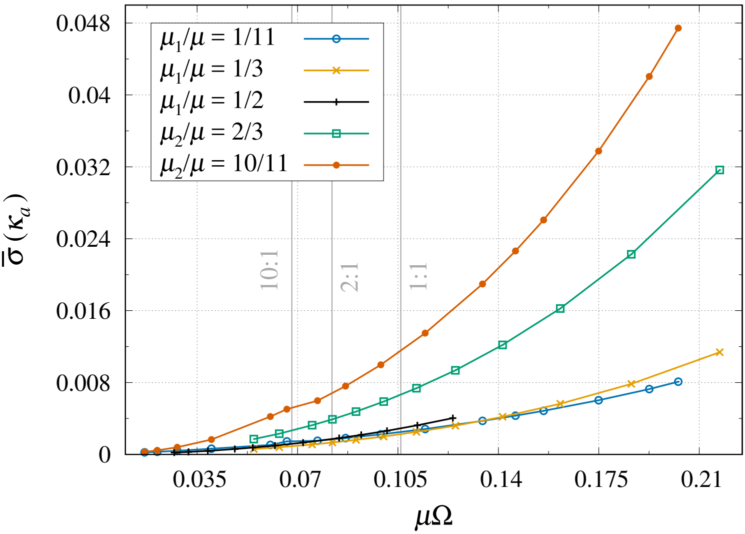
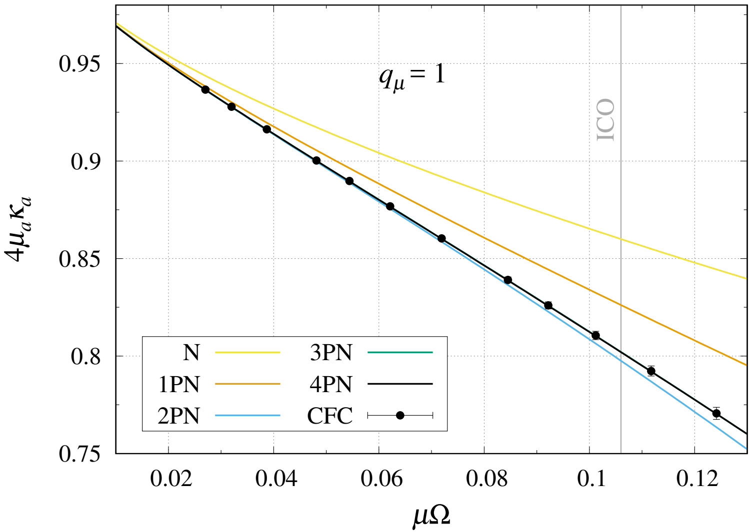


Variations in horizon surface gravity



Surface gravity *vs* orbital frequency



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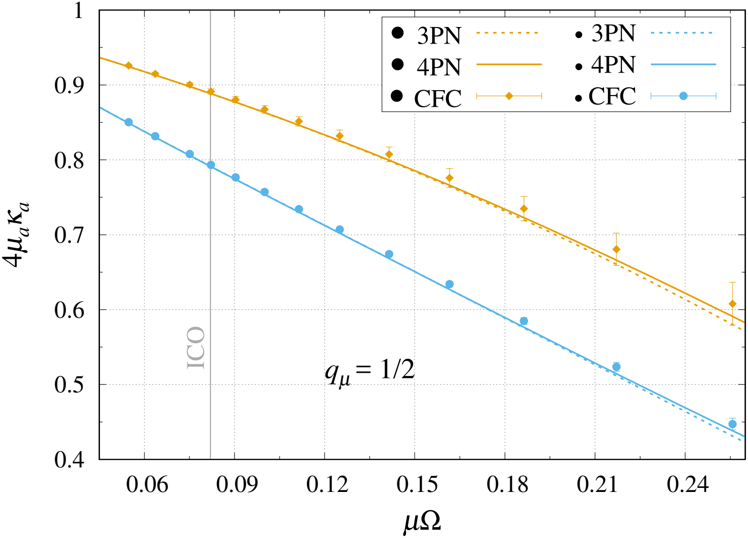

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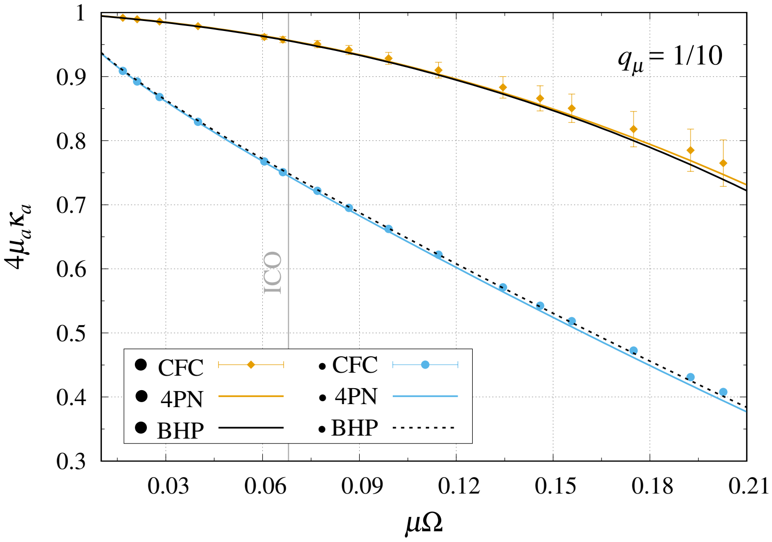

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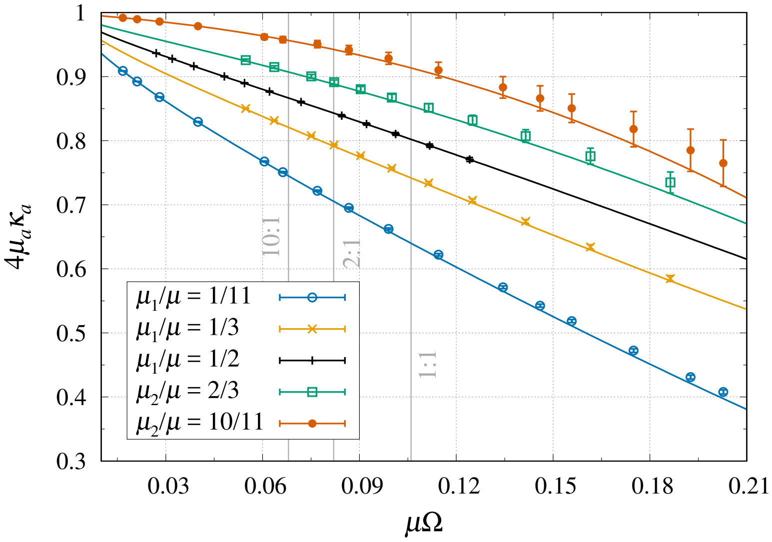

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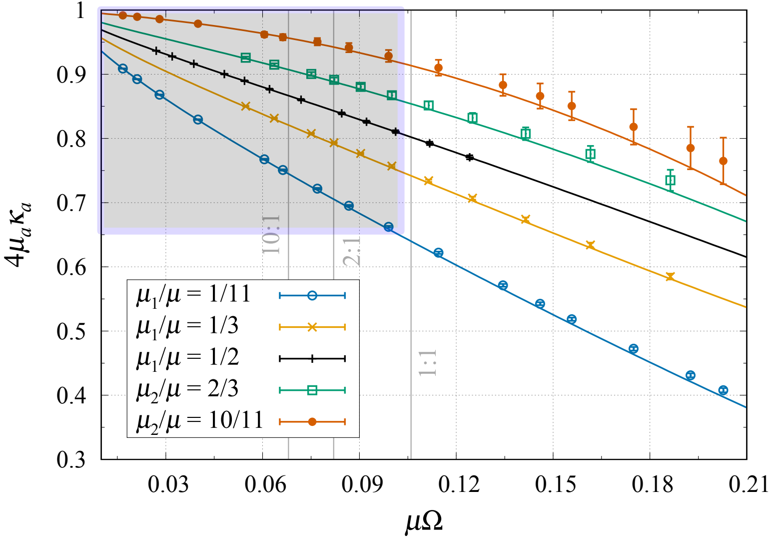
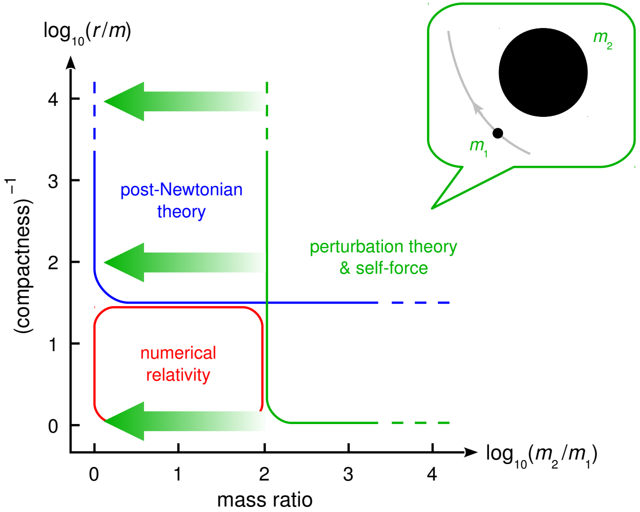


Perturbation theory for comparable masses



Summary

- The classical laws of BH mechanics can be extended to binary systems of compact objects
- First laws of mechanics come in a variety of different forms:
  - Context: exact GR, perturbation theory, PN theory
  - Objects: black holes, point particles
  - Orbits: circular, generic bound
  - Derivation: geometric, Hamiltonian
- Combined with the first law, the redshift $z(\Omega)$ provides crucial information about the binary dynamics:
  - Binding energy E and total angular momentum J
  - Innermost stable circular orbit frequency Ω_{ISCO}
  - EOB effective potentials A, $\bar{D}$, Q, G_S
  - Horizon surface gravity κ

Prospects

- Exploit the Hamiltonian first law for a particle in Kerr:
  - Innermost spherical orbit
  - Marginally bound orbits
- Extend PN Hamiltonian first law for two spinning particles:
  - Non-aligned spins and generic precessing orbits
  - Contribution from quadrupole moments
- Redshift at second order $\rightarrow \mathcal{O}(q^2)$ corrections in $E(\Omega), J(\Omega)$
- Perturbation theory may prove useful to build templates for IMRIs and even comparable-mass binaries

Additional Material

Why does BHPT perform so well?

- In perturbation theory, one traditionally expands as

$$f(\Omega; m_i) = \sum_{k=0}^{k_{\max}} a_k(m_2\Omega)\, q^k \quad \text{where} \quad q \equiv m_1/m_2 \in [0, 1]$$

- However, most physically interesting relationships $f(\Omega; m_i)$ are symmetric under exchange $m_1 \longleftrightarrow m_2$

- Hence, a better-motivated expansion is

$$f(\Omega; m_i) = \sum_{k=0}^{k_{\max}} b_k(m\Omega)\, \nu^k \quad \text{where} \quad \nu \equiv m_1 m_2/m^2 \in [0, 1/4]$$

- In a PN expansion, we have $b_n = \mathcal{O}\big(1/c^{2n}\big) = n\text{PN} + \cdots$

Why does BHPT perform so well?

- In perturbation theory, each surface gravity is expanded as

$$4\mu_1\kappa_1 = a(\mu_2\Omega) + q\, b(\mu_2\Omega) + \mathcal{O}(q^2)$$
$$4\mu_2\kappa_2 = c(\mu_2\Omega) + q\, d(\mu_2\Omega) + \mathcal{O}(q^2)$$

- From the first law we know that the general form is

$$4\mu_i\kappa_i = \sum_{k\geqslant 0} \nu^k f_k(\mu\Omega) \pm \sqrt{1-4\nu} \sum_{k\geqslant 0} \nu^k g_k(\mu\Omega)$$

- Each surface gravity can thus be rewritten as

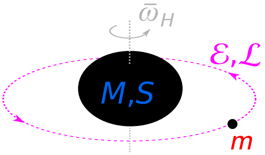
$$\begin{aligned} 4\mu_i\kappa_i = A(\mu\Omega) &\pm B(\mu\Omega)\sqrt{1-4\nu} + C(\mu\Omega)\,\nu \\ &\pm D(\mu\Omega)\,\nu\sqrt{1-4\nu} + \mathcal{O}(\nu^2) \end{aligned}$$

- Expand to linear order in q and match $\to A,\ B,\ C,\ D$

Rotating black hole + orbiting moon

- Kerr black hole of mass M and spin S perturbed by a moon of mass $m \ll M$:

$$g_{ab}(\varepsilon) = \bar{g}_{ab} + \varepsilon\, \mathcal{D}g_{ab} + \mathcal{O}(\varepsilon^2)$$



- Perturbation $\mathcal{D}g_{ab}$ obeys the linearized Einstein equation with point-particle source

$$\mathcal{D}G_{ab} = 8\pi\, \mathcal{D}T_{ab} = 8\pi\, m \int_\gamma \mathrm{d}\tau\, \delta_4(x,y)\, u_a u_b$$

- Particle has energy $\mathcal{E} = -m\, t^a u_a$ and ang. mom. $\mathcal{L} = m\, \phi^a u_a$
- Physical $\mathcal{D}g_{ab}$: retarded solution, no incoming radiation, perturbations $\mathcal{D}M_\mathrm{B} = \mathcal{E}$ and $\mathcal{D}J = \mathcal{L}$ [Keidl *et al.* 2010]

Rotating black hole + corotating moon

- We choose for the geodesic γ the unique equatorial, circular orbit with azimuthal frequency $\bar{\omega}_H$, i.e., the *corotating* orbit

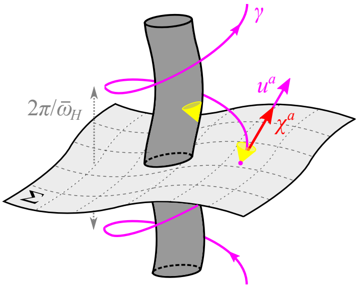
- Gravitational radiation-reaction is $\mathcal{O}(\varepsilon^2)$ and neglected
  The spacetime geometry has a helical symmetry

- In adapted coordinates, the helical Killing field reads

$$\chi^a = t^a + \bar{\omega}_H \, \phi^a$$

- Conserved orbital quantity associated with symmetry:

$$z \equiv -\chi^a u_a = m^{-1} \left(\mathcal{E} - \bar{\omega}_H \, \mathcal{L} \right)$$



Zeroth law for a black hole with moon

[Gralla & Le Tiec 2013]

- Because of helical symmetry and corotation, the expansion and shear of the *perturbed* future event horizon H vanish
- Rigidity theorems then imply that H is a Killing horizon
  [Hawking 1972, Chruściel 1997, Friedrich *et al.* 1999, etc]
- The horizon-generating Killing field must be of the form

$$k^a(\varepsilon) = t^a + \big(\underbrace{\bar{\omega}_H + \varepsilon\,\mathcal{D}\omega_H}_{\text{circular orbit frequency } \Omega}\big)\phi^a + \mathcal{O}(\varepsilon^2)$$

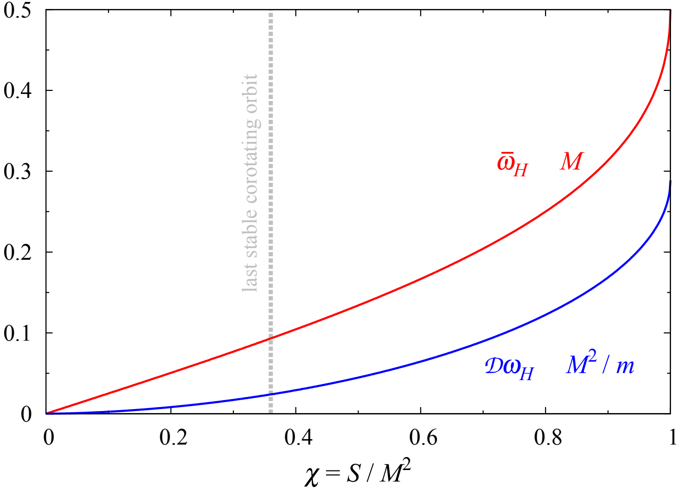
- The surface gravity κ is defined in the usual manner as

$$\kappa^2 = -\frac{1}{2}\,(\nabla^a k^b\,\nabla_a k_b)|_H$$

- Since $\kappa = \text{const.}$ over *any* Killing horizon [Bardeen *et al.* 1973], we have proven a zeroth law for the *perturbed* event horizon

Angular velocity *vs* black hole spin

[Gralla & Le Tiec 2013]



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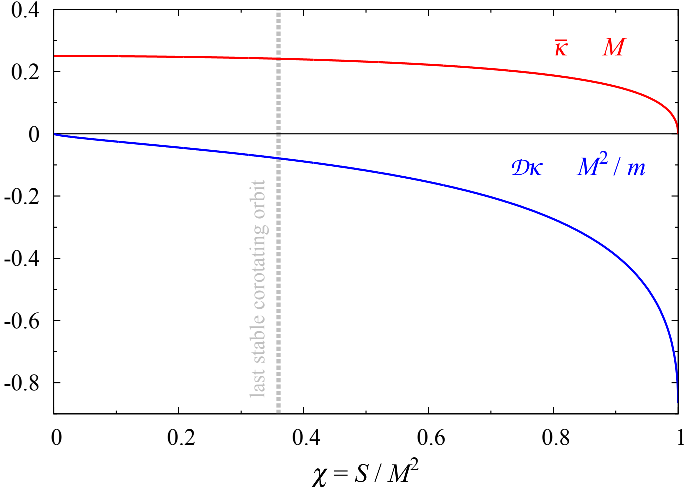

First law for a black hole with moon

[Gralla & Le Tiec 2013]

- Adapting [Iyer & Wald 1994] to non-vacuum perturbations of non-stationary spacetimes we find (with $Q_{ab} \equiv -\varepsilon_{abcd} \nabla^c k^d$)

$$\int_{\partial\Sigma} (\delta Q_{ab} - \Theta_{abc} k^c) = 2\,\delta \int_\Sigma \varepsilon_{abcd} G^{de} k_e - \int_\Sigma \varepsilon_{abcd} k^d G^{ef} \delta g_{ef}$$

- Applied to nearby BH with moon spacetimes, this gives the first law

$$\delta M_{\mathrm{B}} = \Omega\,\delta J + \frac{\kappa}{8\pi}\,\delta A + z\,\delta m$$

- Features variations of the Bondi mass and angular momentum