Progress Towards Multiscale EMRI Approximation: Zones and Scales

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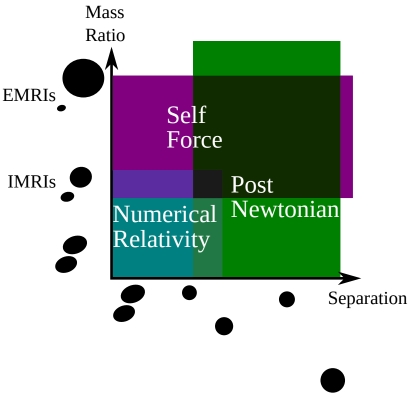
The multiscale expansion

- 'Multiscale' - a combination of approximations
  - Used to describe the use of two-timescale approximation where valid, combined with other methods:
    - Near the small companion
    - Far from the inspiral
    - Near the SMBH horizon
- Our (ambitious) goals
  - An algorithm built on existing SF tools for ensuring long scale ($t \sim M/\epsilon$) fidelity of:
    - Post-adiabatic waveform
    - Dynamical invariants of the inspiral for NR and PN comparison to second order

What we want from multiscale

- Post-adiabatic Waveform - important for parameter estimation for EMRIs, and possibly detection
  - Phase accuracy throughout waveform
  - Slowly varying memory effects

- Dynamical invariants - highly useful for comparisons and confirmations with NR and PN computations
  - Redshift z [Detweiler]
  - Surface gravity [Zimmerman]
  - Precession of Perihelion [Le Tiec]
  - Many of these are more demanding for a multiscale scheme than waveforms



Multiscale requirements

- Waveforms

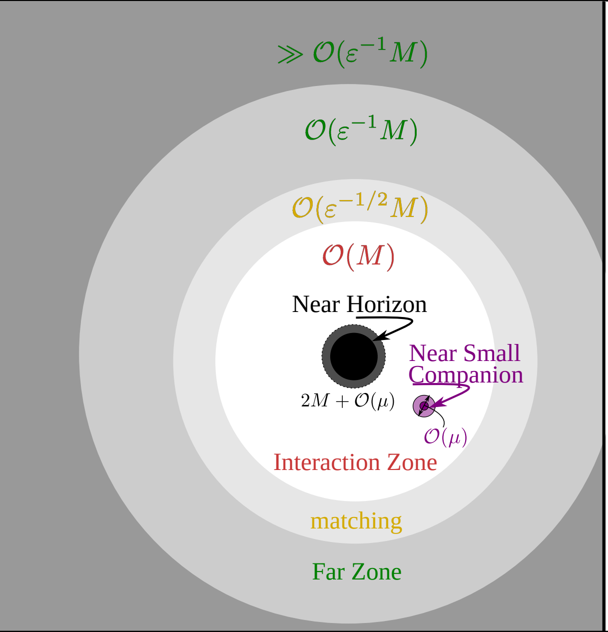
	Adiabatic	Post-adiabatic
Required Order of Self-Force	First Order Dissipative	Second Order Dissipative + First Order Conservative
Errors in Amplitude	$\mathcal{O}(\epsilon)$	$\mathcal{O}(\epsilon^2)$
Errors in Phase	$\mathcal{O}(1)$	$\mathcal{O}(\epsilon)$
Required oscillatory metric order	$\mathcal{O}(\epsilon)$	$\mathcal{O}(\epsilon^2)$
Required quasistatic metric order	$\mathcal{O}(1)$	$\mathcal{O}(\epsilon)$

- Dynamical Invariants (example: surface gravity)

	First order	Second order
Required Order of Self-Force	First Order Dissipative	Second Order Dissipative + First Order Conservative
Required oscillatory metric order	$\mathcal{O}(\epsilon)$	$\mathcal{O}(\epsilon^2)$
Required quasistatic metric order	$\mathcal{O}(\epsilon)$	$\mathcal{O}(\epsilon^2)$

  - Requires quasistatic matching from distant regions

Zones and scales



- Interaction zone: $-M/\epsilon \ll r_* \ll M/\epsilon$
- Near small companion: distance from small companion $\bar{r} \ll M$
- Far zone: $r_* \gg M$
- **Near-Horizon**: $r_* \ll -M$

Two-Timescale in interaction zone : $-M/\epsilon \ll r_* \ll M/\epsilon$

- Two-Timescale approximation promotes time dependence to multiple (temporarily) independent variables $t \to \{\tilde{t}, q^A\}$

$$\tilde{t} = \frac{\mu}{M} t \equiv \epsilon t \qquad\qquad \frac{dq^A}{dt} = \Omega(\tilde{t}, \epsilon)$$

- Action angle variables q^A coordinates on compact directions of the symplectic manifold
- Periodic behavior depends on q^A, secular depends on $\tilde{t}$
- Worldline can be expressed using action angle variables and geodesic parameters $P^M \equiv \{E, L_z, Q\}$:

$$\begin{aligned}\frac{dP^M}{dt} &= \epsilon G^{(1)M}(P^{(0)M}(\tilde{t}), q^A) + \mathcal{O}(\epsilon^2) \\ \frac{dq^A}{dt} &= \Omega^A(P^{(0)M}(\tilde{t})) + \epsilon g^{(1)A}(P^{(0)M}(\tilde{t}), q^A)\end{aligned}$$

Improved long time fidelity

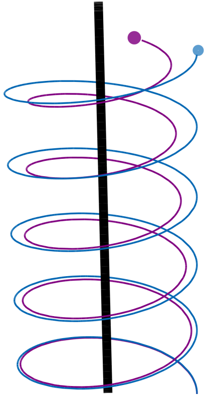
- Metric ansatz ($g^{(0)}_{\alpha\beta}$ taken to be Schwarzschild)

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta}(\bar{x}^i) + \epsilon h^{(1)}_{\alpha\beta}(\tilde{t}, q^A, \bar{x}^i) + \epsilon^2 h^{(1)}_{\alpha\beta}(\tilde{t}, q^A, \bar{x}^i) + \mathcal{O}(\epsilon^3)$$

- Worldline ansatz:

$$z^\mu(t) = z^{(0)}(\tilde{t}, q^A) + \epsilon z^{(1)}(\tilde{t}, q^A) + \mathcal{O}(\epsilon^2)$$

- Assume no resonances in the domain of interest
- Precision of approximation preserved: dephasing time is the entire inspiral $\sim M^2/\mu$, rather than the standard result for black hole perturbation theory - geometric mean $\sim \sqrt{\mu M}$
- Our method applies the Two-Timescale approximation to metric perturbations to preserve field precision for the full inspiral



Breakdown of Two-Timescale at long distances

- Two-Timescale approximation assumes radiation timescale longer than all other scales of the system
- At each order, we solve the wave equation

$$\Box_{q^A} h_{\mu\nu} + R_{\mu}{}^{\sigma}{}_{\nu}{}^{\rho} h_{\sigma\rho} = S,$$

  for some $\{\tilde{t}, q^A, x^i\}$-dependent source
- At long scales, inverting $\Box_{q^A}$ is solving for perturbations assuming an eternal source
- Leading second-order source scales as $\sim \Omega^2/r^2$
- For second order static Green's function, these contributions give divergent retarded field solution if integration domain $r' \in [a, \infty)$
- similar problems arise at $r_* \to -\infty$

Small companion puncture

Near Small Companion

- Two-timescale ansatz breaks down near small companion
  - Use either Self-Consistent evaluated at each fixed $\tilde{t}$, or an extended Self-Consistent
- Known puncture metric, derived by [Pound]
  - Independent of matching conditions, dependent only on small companion structure
- Non-exact worldline $z^\mu = z^{(0)\mu} + \epsilon z^{(1)\mu} + \ldots$ requires a re-expansion from Self-Consistent
  - Self-acceleration - direct re-expansion, up to slow time derivatives
  - Puncture dipole correction - $\mathcal{O}(\mu)$ displacement in worldline position
- Residual field derived in puncture region via relaxed EFE

$$E_{\mu\nu}[h^{(2)\mathcal{R}}_{\alpha\beta}] = -E_{\mu\nu}[h^{(2)\mathcal{P}}_{\alpha\beta}] + S_{\mu\nu}[h^{(1)}_{\alpha\beta}, h^{(1)}_{\alpha\beta}] + \delta T_{\mu\nu}$$

Breakdown of Self-Consistent at long times

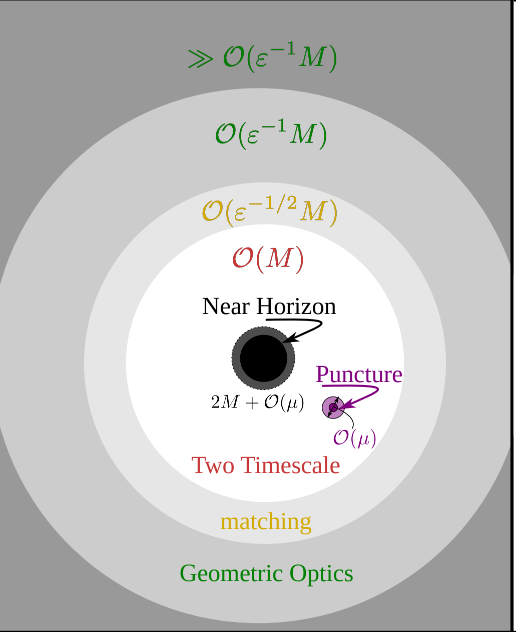
Near Small Companion

- Self-consistent formalism deals well with the slow evolution of the worldline by expanding the metric as a functional of the full worldline

$$g_{\mu\nu} = g^{(0)}_{\mu\nu}[x^\mu] + \epsilon h^{(1)}_{\mu\nu}[x^\mu; z^\mu] + \epsilon^2 h^{(2)}[x^\mu; z^\mu] + \mathcal{O}(\epsilon^3)$$

- Equations of motion are the Relaxed EFE and Lorenz gauge condition
- Slow evolution of background spacetime is incorrectly controlled
  - Mass and spin evolution enter at the order of energy flux $\sim \mathcal{O}(\epsilon^2)$
  - Entirely fixed by Lorenz gauge on initial data surface - no evolution during inspiral
  - Linearly growing mass and spin at second order invalidates the result at a radiation-reaction time
- Direct two-timescale extension does not solve these problems, but a more involved incorporation can recover long-time fidelity

Zones and scales of approximation methods



- Interaction zone: $|r_*| \ll M/\epsilon$
  Two-Timescale expansion, worldline Two-Time
  - Post-adiabatic evolution requires matching to adjacent regions
- Near small object : $\bar{r} \ll M$
  Puncture, Self-Consistent [Pound]
- Far zone: $r_* \gg M$
  Geometric optics, with some Post-Minkowski techniques -[Extending Pound 2015]
- **Near-Horizon**: $r_* \ll -M$
  Black hole perturbation theory

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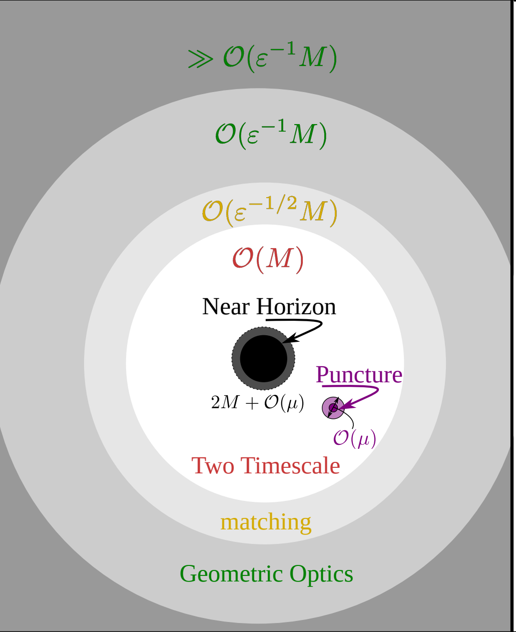

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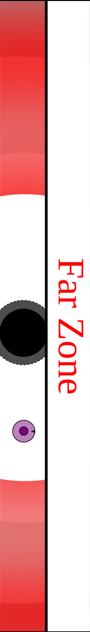
Geometric optics for the far zone



- Spatial scales vary with $\tilde{x}^i \sim \epsilon x^i$, on scale with slow inspiral
- Construct ansatz with single fast variation parameterized by scalar function $\Theta(x^\nu)/\epsilon$

$$
\begin{aligned}
g_{\mu\nu}(x^\nu, \varepsilon) =& \varepsilon^{-2}\Bigg(\eta_{\mu\nu} + \varepsilon h_{\mu\nu}\left[\tilde{x}^\nu\right] + \varepsilon^2 j_{\mu\nu}\left[\tilde{x}^\nu, \frac{\Theta}{\varepsilon}\right] \\
&+ \varepsilon^3 k_{\mu\nu}\left[\tilde{x}^\nu, \frac{\Theta}{\varepsilon}\right] + \mathcal{O}(\varepsilon^4)\Bigg)
\end{aligned}
$$

- The rescaling of the coordinates grants an additional order to the weak waves, as they depend on $1/r = \varepsilon/\tilde{r}$
- Define wave vector associated with the fast periodic dependence

$$k_\mu = \nabla_\mu \Theta$$

- Up to gauge, the leading dynamical equation enforces null wave vector

$$k^\mu k^\nu \eta_{\mu\nu} = 0$$

First order - direct wave solutions

Far Zone

- Define tetrad $\{k, l, e^A\}$ such that

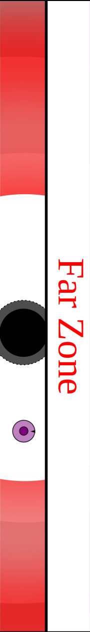
$$l^\mu l_\mu = k^\mu k_\mu = 0 \qquad k^\mu e_{A\mu} = l^\mu e_{A\mu} = 0$$
$$k^\mu l_\mu = -1 \qquad e^\alpha{}_A e^\beta{}_B \eta_{\alpha\beta} = \delta_{AB}$$

- Leading wave equation implies

$$\delta^{AB}\partial_\Theta^2 j_{AB} = \partial_\Theta^2 j_{ll} = 0$$
$$\partial_\Theta^2 j_{lA} = 0$$

- Lorenz gauge not imposed, but compatible with the results after EFE is calculated
- Compatible with [Blanchet and Damour] Post-Minkowski leading order in $1/r$ outgoing waves

Second order - propagation along null cones



- Null cone propagation at leading order gives simple $1/\tilde{r}$ radiation dependence

$$\frac{1}{\tilde{r}}\partial_\Theta j_{AB} + \partial_{\tilde{r}}\partial_\Theta j_{AB} = 0$$

- Subleading Lorenz gauge condition informs otherwise unconstrained parts (for instance, the $\ell = 0, 1$ parts not expressible as TT waves)

$$\nabla^\mu j_{\mu\nu} + k^\mu \partial_\Theta k_{\mu\nu} = 0$$

- Remaining components fix the now nontrivial non-TT components of $k_{\mu\nu}$:

$$\begin{aligned} -\frac{1}{2}\delta^{AB}\partial_\Theta^2 k_{AB} &= -G^{(1,1)}_{kk}[j] \\ -\frac{1}{2}\partial_\Theta^2 k_{lA} &= -G^{(1,1)}_{kA}[j] \\ \partial_\Theta^2 k_{ll} &= 0 \end{aligned}$$

Third order equations - quasistatic j_0

Far Zone

- Impose Lorenz gauge on the quasistatic part j_0
- Background correction + General wave equation

$$\Box j_{0\mu\nu}[\tilde{x}^{\nu}] + R_{\mu}{}^{\sigma}{}_{\nu}{}^{\rho} j_{0\sigma\rho} = -\left\langle G^{(2,2)}_{\mu\nu}[j,j]\right\rangle$$

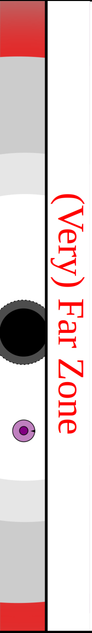
- Solvable via techniques first introduced by [Blanchet and Damour]
- General solution written as integral:

$$j_0 = \mathsf{FP}_{B\to 0}\left[\frac{1}{K(B)}\int_{\tilde{r}}^{\infty} d\tilde{z}\,\frac{S^{(k)}(\tilde{t}-\tilde{z})}{\tilde{r}^k}\,\hat{\tilde{\partial}}_L\left(\frac{(\tilde{z}-\tilde{r})^{B-k+l+2}-(\tilde{z}+\tilde{r})^{B-k+l+2}}{\tilde{r}}\right)\right]$$

- With some manipulation, we can re-write the retarded solution as a homogeneous + particular solution

$$j_{0,\ell} = \tilde{\partial}_L \frac{j^G_{\ell}(u)}{\tilde{r}} + j^H_{\ell}(u)$$

Third order quasistatic - asymptotic evaluation



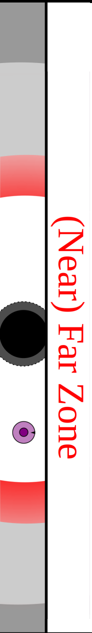
- Evaluate integral assuming large $\tilde{r}$. Geometric optics construction gives $G^{(2,2)} \sim \tilde{r}^{-2}$

$$j_\ell^H = \frac{\hat{n}_L}{\tilde{r}} \int_0^\infty d\tilde{z} \left(\frac{1}{2} \ln \frac{\tilde{z}}{2\tilde{r}} + \sum_{n=1}^{\ell} \frac{1}{n} \right) \left\langle G^{(2,2)}[j,j] \right\rangle + \mathcal{O}(\tilde{r}^{-2} \ln(\tilde{r}))$$

$$\tilde{\partial}_L \frac{j_\ell^G(\tilde{u})}{\tilde{r}} = \tilde{\partial}_L \frac{1}{\tilde{r} K_k} \int_{-\infty}^{\tilde{u}} d\tilde{s} \left\langle G^{(2,2)} \right\rangle (\tilde{s})(\tilde{u} - \tilde{s})^\ell$$

- Scales similarly with ε to outgoing waves - ‘memory’-like effect
- Scaled coordinates $\tilde{x}$ explicitly incorporate the long scale dependence of the system
  - Region of nonlinear source $r \sim M/\varepsilon \Rightarrow \tilde{r} \sim M$

Third order quasistatic - interaction region matching



- assume $\tilde{r} \ll M$
- near-cancellation within integral suppresses solution
  - for small $\tilde{r}$

$$(\tilde{z} - \tilde{r})^{B-k+\ell+2} = (\tilde{z} + \tilde{r})^{B-k+\ell+2} + \mathcal{O}(\tilde{r})$$

- Scaled coordinate solution proceeds as in [Pound 2015], resulting in

$$j_{0,\ell=0} = -S_\ell^{(2)}(\tilde{w}) + S_\ell^{(2)}(\tilde{w})\ln(2\tilde{r}/\epsilon^2) - \int_0^\infty d\tilde{z}\dot{S}_\ell^{(2)}(\tilde{w} - \tilde{z})\ln(\tilde{z})$$

$$j_{0,\ell\geq 0} = -\frac{\hat{n}_L}{\ell(\ell+1)}$$

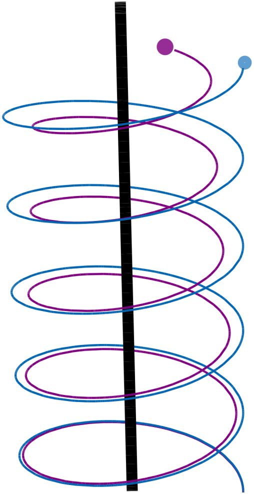
- Note that the constant-in-$\tilde{r}$ contributions remain second order as we take $\tilde{r} \ll M$, unlike the leading order wave solution
  - We recover the near-zone reasoning from PM that the nonlinearly sourced quasistatic should be $\mathcal{O}(\epsilon^2)$

Near-horizon (work in progress)

- Asymptotically plane wave solutions approaching horizon
- Only quasistatic, second-order perturbations for matching to interaction zone
- Two-Timescale assumption violated near the horizon
- Generally, we're exploring methods of using adjusted or scaled coordinate dependence to simplify the near-horizon expansion
  - Geometric optics approximation is confounded by exponentially small $\{r, t\}$ components $\sim e^{r_*/2M}$
  - No additional separation of scales - leading order solution is constant in r_*, compared to $1/r$ dependence in far zone

Multiscale status report

- Interaction zone computations fairly well-understood
- Several methods work in concert to form a globally valid approximation scheme
- Far zone well under control (largest update since last Capra)
- Open questions and future work for multiscale
  - **Near horizon** - similarities to far zone, but with confounding scaling details
  - resonances - generally introduce powers $\epsilon^{1/2}$
  - second order Kerr ...



Self-Consistent hybrid construction

- Consider the Two-Timescale form of the metric perturbation $h^{(n)}[\varphi^A, \tilde{t}, x^i]$
- Construct an equivalent functional of the worldline z^μ by evaluating the instantaneous 'osculating' geodesic action-angle variables
- Lorenz gauge condition at the heart of the difficulty - in Self-Consistent, it determines the motion of the small companion, and in Two-Timescale, it determines the $\delta\dot{M}$ and $\delta\dot{a}$.
- Separate equations of motion for the two distinct (but both valid) expressions of the worldline
  - Exact worldline obeys direct Self-Consistent equation of motion
  - Perturbatively expanded worldline obeys re-expanded Self-Consistent
  - Consistency is easy to show by summing the orders of the perturbative worldline
- Finally, the perturbative worldline may be used with the Lorenz gauge condition in the Two-Timescale expressions to derive the mass and spin evolution

Suggested Algorithm for Post-Adiabatic Computation