Enhanced determination of eccentric orbit PN expansions from perturbation theory: Finding structure in the LMN modes

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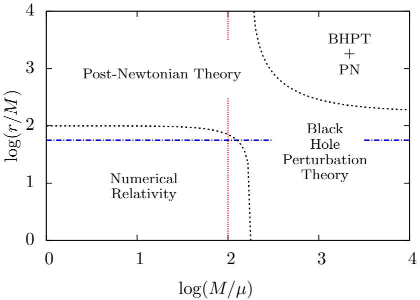
Aim: Use perturbation theory to push PN knowledge

- Continuing the work of Forseth, Evans, Hopper (2016)
- Calculate fluxes/flux components to 1st-order in mass ratio μ/M
- By choice focus on wide orbits $r \gg M$
  $\rightarrow$ overlap with PN theory

  Pluck off new higher-order PN terms
  $\rightarrow$ 3.5PN, 4PN,…, 7PN or higher
- Related work:



Poisson (1993); Poisson and Sasaki (1995); various by Sasaki, Tagoshi, Tanaka, Shibata, Takasugi, Mano (mid-1990s); Detweiler (2008); Blanchet, Detweiler, Le Tiec, and Whiting (2010,2011); Fujita (2012); Bini and Damour (2013, 2014, etc); Shah, Friedman, and Whiting (2014); Shah (2014); Fujita (2014); Johnson-McDaniel, Shah, and Whiting (2015); Sago and Fujita (2015); Kavanagh, Ottewill, and Wardell (2015a,b); Forseth, CRE, Hopper (2016)

Overview of the calculational method

- Analytic function expansions for $R^{\pm}_{lm\omega}$ using MST formalism (here $a = 0$)

$$\left[r^2 f \frac{d^2}{dr^2} - 2(r - M)\frac{d}{dr} + U_{l\omega}(r)\right] R^{\pm}_{lm\omega}(r) = 0,$$

- Convert $R^{\pm}_{lm\omega}$ to RWZ functions $X^{\pm}_{lm\omega}$

- Specify orbit $(p,\, e)$, typically using large p; solve source problem with SSI

- $\boxed{\text{MST} \to \text{RWZ} \to p{,}e \to \text{SSI} \to \textit{Mathematica} \to \text{Fluxes } |C^{\pm}_{lmn}|^2}$

- Then, on the PN side, we perform a "double fit" on these fluxes, first over p (y), then over e to obtain numeric coefficients.

  Use PSLQ to pluck off analytical forms.

Reminder: Eccentricity dependent PN energy flux

Flux at infinity depends on e_t and $y = (\omega M)^{2/3} \ll 1$

e_t is PN "time eccentricity"

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 y^5 \Big[\mathcal{I}_0(e_t) + y\,\mathcal{I}_1(e_t) + y^{3/2}\,\mathcal{K}_{3/2}(e_t) + y^2\,\mathcal{I}_2(e_t)$$
$$+ y^{5/2}\,\mathcal{K}_{5/2}(e_t) + y^3\,\mathcal{I}_3(e_t) + y^3 \log y\,\mathcal{I}_{3L}(e_t) + y^3\,\mathcal{K}_3(e_t)$$
$$+ y^{7/2}\,\mathcal{L}_{7/2}(e_t) + y^4\,\mathcal{L}_4(e_t) + y^4 \log y\,\mathcal{L}_{4L}(e_t) + \cdots\Big]$$

Enhancement functions: $\mathcal{I}_n(e_t)$ are instantaneous; $\mathcal{K}_n(e_t)$ are hereditary

$$\mathcal{I}_0(e_t) = \frac{1}{(1-e_t^2)^{7/2}}\left(1 + \frac{73}{24}\,e_t^2 + \frac{37}{96}\,e_t^4\right)$$

Peters-Mathews (1963)

$$\mathcal{I}_1(e_t) = \frac{1}{(1-e_t^2)^{9/2}}\left(-\frac{1247}{336} + \frac{10475}{672}e_t^2 + \frac{10043}{384}e_t^4 + \frac{2179}{1792}e_t^6\right)$$

See Arun et al. 2008a,b; 2009; Blanchet 2014 (LRR)

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Eccentricity dependent PN energy flux (cont.)

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- The relationship between time and Darwin eccentricities currently known to 3PN order

$$\frac{e_t^2}{e^2} = 1 - 6y - \frac{(15 - 19\sqrt{1-e^2}) + (-15 + 15\sqrt{1-e^2})e^2}{(1-e^2)^{3/2}}\,y^2$$
$$\frac{1}{(1-e^2)^{5/2}}\left[(30 - 38\sqrt{1-e^2}) + (59\sqrt{1-e^2} - 75)e^2 + (45 - 18\sqrt{1-e^2})e^4\right]y^3$$

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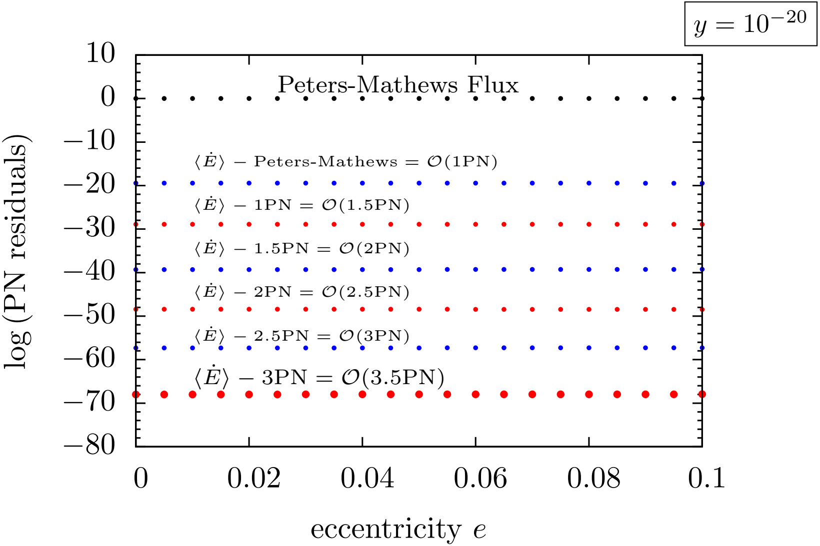
Confirming 3PN: Subtracting 3PN from $\ell \leq 5$ flux



Beyond 3PN

$$\langle \dot{E} \rangle = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 y^5 \Big[\mathcal{I}_0 + y\mathcal{I}_1 + y^{3/2}\mathcal{K}_{3/2} + y^2\mathcal{I}_2 + y^{5/2}\mathcal{K}_{5/2} + y^3\mathcal{I}_3 + y^3\mathcal{K}_3$$

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Beyond 3PN

Peters and Mathews, 1963

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Beyond 3PN

Wagoner and Will, 1976

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Blanchet and Schäfer; Wiseman; Poisson (1993)

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Gopakumar and Iyer, 1997

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Arun, et. al., 2008

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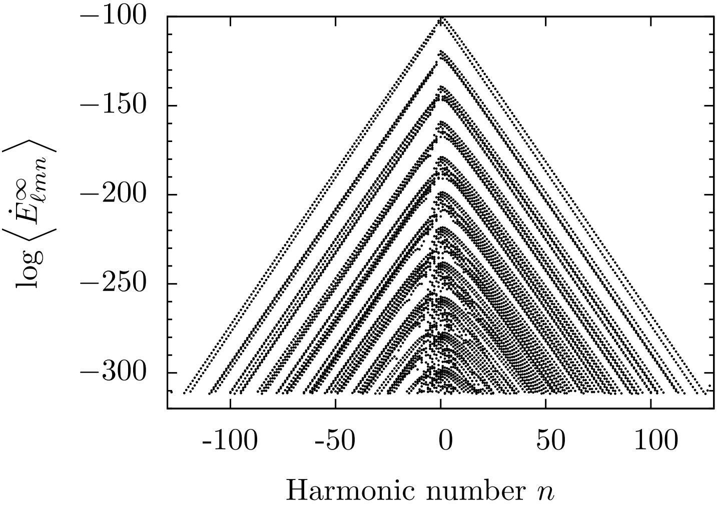
Forseth, CRE, Hopper (2016) + current work

(lowest order in mass ratio)

Fluxes $\dot{E}^{\infty}_{\ell mn}$ from one orbital model

Orbit: $y = 10^{-20}$, $e = 0.1$, accuracy: 200 digits, total modes > 7000



Previous methods/results (cont.)

- After adding up the ∼7500 lmn modes for the full energy flux of the ∼1700 orbits, we perform a double fit for numeric coefficients. Then, look for the corresponding exact numbers

- Example, for 3.5PN, we expect the form

$$\mathcal{L}_{7/2} = -\frac{16285\pi}{504(1-e^2)^7}\left(1 + a_2 e^2 + a_4 e^4 + \cdots\right)$$

- Fit gives

$$\begin{aligned} a_2 = 13.&7525630692866646197932657865111042881997748439259031\\ &8288813836864189949851601678435984223341134824177051198\\ &83034494121793 \end{aligned}$$

- Integer relation algorithm (PSLQ; see `FindIntegerNullVector`) finds

$$a_2 = \frac{21500207}{1563360}$$

  to 108 digits

- Likelihood of coincidence $\sim 10^{-93}$ (plus other checks)

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Example: 4PN Log term has an exact closed form

- Next consider 4PN Log term

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- The $\mathcal{L}_{4L}$ enhancement function is:

$$\mathcal{L}_{4L} = \frac{232597}{8820(1-e^2)^{15/2}} \left(1 + \frac{14770533}{465194} e^2 + \frac{142278179}{930388} e^4 + \frac{318425291}{1860776} e^6 + \frac{1256401651}{29772416} e^8 + \frac{64986219}{59544832} e^{10} \right)$$

- This is an exact, closed-form expression! (2016)

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New results: Extracting the most from flux data

- When performing the fit, a number of parameters have to be chosen that affect the yield of exact coefficients
- Examples: Where to truncate/round the numeric coefficient, the number of y terms to include in the radial fit, the number of e terms to include in the second fit
- Automation $\rightarrow$ greater sampling of parameter space $\rightarrow$ better coefficient yield
- Most importantly, affords the opportunity to look/test for patterns in the output

Example (new result): Finding the 5PN Log exact form

- 5PN Log term:

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- Recall that the 2PN enhancement function has the form:

$$\mathcal{I}_2 = \frac{1}{(1-e^2)^{11/2}}\left(-\frac{203471}{9072} - \frac{1430873e^2}{18144} + \frac{2161337e^4}{24192} + \frac{231899e^6}{2304} + \frac{499451e^8}{64512}\right) + \frac{1}{(1-e^2)^5}\left(\frac{35}{2} + \frac{1715e^2}{48} - \frac{2975e^4}{64} - \frac{1295e^6}{192}\right)$$

- Considering that partly, $\mathcal{L}_{5L} = \mathcal{I}_2 * \mathcal{L}_{3L}$, we might expect something similar.

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5 PN Log: Extending the series

- Improvements in parameter choice brought the yield all the way from this (2016):

$$\mathcal{L}_{5L} = \frac{1}{(1-e^2)^{17/2}}\Big(\frac{916628467}{15717240} + \frac{11627266729e^2}{31434480} - \frac{84010607399e^4}{10478160} - \frac{67781855563e^6}{1632960} - \frac{87324451928671e^8}{2011806720} - \frac{301503186907e^{10}}{29804544} - \frac{752883727e^{12}}{1290240} - \frac{22176713e^{14}}{129024} - \frac{198577769e^{16}}{2064384} - \frac{250595605e^{18}}{4128768} - \frac{195002899e^{20}}{4718592} - \frac{280151573e^{22}}{9437184} - \frac{1675599991e^{24}}{75497472} + \cdots\Big)$$

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- to this (2017):

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5 PN Log: exact form

- Just enough to solve an "overdetermined" system of equations for the expected form. Result:

$$\mathcal{L}_{5L} = \frac{1}{(1-e^2)^{17/2}}\Big(\frac{5080948627}{15717240} + \frac{117123377449e^2}{31434480} - \frac{4199642054e^4}{654885} - \frac{78989239933e^6}{1632960} - \frac{88593702010771e^8}{2011806720} - \frac{261925436695e^{10}}{29804544} - \frac{245975507e^{12}}{1290240}\Big) + \frac{1}{(1-e^2)^8}\Big(-\frac{5564}{21} - \frac{219778e^2}{63} - \frac{1136447e^4}{336} + \frac{132145e^6}{28} + \frac{5390125e^8}{2304} + \frac{137709e^{10}}{1792}\Big)$$

- Interestingly, a simpler form exists:

$$\mathcal{L}_{5L} = \frac{1}{(1-e^2)^{17/2}}\Big(\frac{5080948627}{15717240} + \frac{117123377449e^2}{31434480} - \frac{4199642054e^4}{654885} - \frac{78989239933e^6}{1632960} - \frac{88593702010771e^8}{2011806720} - \frac{261925436695e^{10}}{29804544} - \frac{245975507e^{12}}{1290240}\Big) - \frac{65}{2(1-e^2)^7}\Big(\frac{856}{105} + \frac{7276e^2}{63} + \frac{553297e^4}{2520} + \frac{187357e^6}{2520} + \frac{10593e^8}{4480}\Big)$$

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5 PN Log: simplifying 2 PN

- As it turns out, the 2PN function simplifies in the same way:

$$\mathcal{I}_2 = \frac{1}{(1-e^2)^{11/2}}\left(-\frac{203471}{9072} - \frac{1430873e^2}{18144} + \frac{2161337e^4}{24192} + \frac{231899e^6}{2304} + \frac{499451e^8}{64512}\right) + \frac{35}{2\,(1-e^2)^4}\left(1 + \frac{73e^2}{24} + \frac{37e^4}{96}\right)$$

- where we have used

$$\frac{35}{2\,(1-e^2)^4}\left(1 + \frac{73e^2}{24} + \frac{37e^4}{96}\right) = \frac{1}{(1-e^2)^5}\left(\frac{35}{2} + \frac{1715e^2}{48} - \frac{2975e^4}{64} - \frac{1295e^6}{192}\right)$$

6 PN Log^2

- Also confirmed the exact form of the 6PN log^2

$$\mathcal{L}_{6L2} = \frac{1}{(1-e^2)^{19/2}}\Big(\frac{366368}{11025} + \frac{189812971e^2}{132300} + \frac{1052380631e^4}{105840} + \frac{9707068997e^6}{529200} + \frac{8409851501e^8}{846720} + \frac{4574665481e^{10}}{3386880} + \frac{6308399e^{12}}{301056}\Big)$$

New method: Fitting by LMN

- Optimization of parameter choice gave small improvements across PN orders, but limited effect

- In the circular orbit case, PN expansions had very nice forms when calculated by LM mode

  $\rightarrow$ Allowed for tail factorizations (see: Johnson-McDaniel Phys. Rev. D 90, 024043)

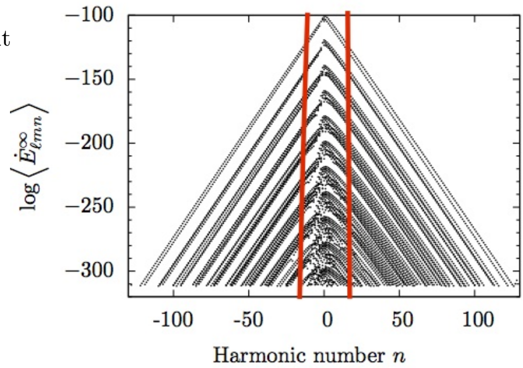
- Perhaps, eccentric orbit expansions would reveal exploitable patterns by LMN mode

- However, factorization not the only reason to fit by LMN. Fitting each mode and then summing the fits gives several additional advantages, esp. if focusing on analytic coefficients

Fitting by LMN: Advantages

1) Significantly reduces the number of necessary modes.

- Each e series of each y term in an N mode begins with e^{2N}
- Thus, an eccentricity goal of say e^{30} requires only 31 N modes for each LM
- Through 7PN, required LMN modes drops to about 1450 for this goal, a big improvement over the 7500 modes that went into the full flux!



Fitting by LMN: Advantages

1) Mode reduction (cont.)

- Modifying our eccentricity goals by PN (e.g. 6PN and 7PN would give low e powers at best, all else equal) lowers this number further

- Depending on numeric coefficient goals, could get down to $\sim$1000 LMN modes

- Can repurpose these CPU hours to improve other areas of the fit – e.g. increase number of orbits or decimals of precision

Fitting by LMN: Advantages

2) Allows for fine tuning of precision to meet e goals.

- Greatest obstacle in fitting = combinations of transcendentals. If we only had to deal with rationals, high powers of e would be easy!
- Combs. of transcendentals first appear at 3PN order, with the complexity greatly increasing at every $(3k)$PN for integer k
  Ex. $\{\pi^2, \gamma_E, \log 2, \cdots\}$
- At the LM level, this 3PN is not relative to the Peters-Mathews term, but to the lowest term of that mode, i.e. $L-2$ or $L-1$
- Thus, $\mathcal{L}_3$ will have transcendentals from modes 20 and 22, while $\mathcal{L}_4$ will have them from 20, 21, 22, 31, and 33
- Meaning: Higher L modes require much fewer decimals of precision. We can segregate LM modes into robust "classes" of needed precision based on the lowest order in the mode and our goals in y/e
- Higher N $\rightarrow$ fewer terms in series $\rightarrow$ precision goals drop with N as well

Fitting by LMN: Advantages

3) Gives the option of saving all y terms at the LM/LMN level

Fitting by LMN: Advantages

4) Patterns/Factorization

- Tail factorizations work for circular orbits on an LM basis (Damour and Nagar; Damour, Iyer and Nagar; Johnson-McDaniel).

- Many defined/derived through analysis of BHPT, but it is actually possible to find them on the PN side in a "bootstrap" manner
  Example: Dividing a circular LM flux by $(1 + \mathcal{L}_1^{LM} y)$ simplifies or eliminates terms at orders 5/2, 4, 11/2, and 7

- Unfortunately, does not appear to work so well by LMN. Moderate simplifications can be made from the BHPT side by modifying the circular factorizations above (work in progress)

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Fitting by LMN: Advantages

4) Patterns (cont.)

- However, even without factorizations, LMN modes characterized by nice patterns that drastically improve the output

- The eulerlog function is defined for LM circular orbits as
$$eulerlog_m(y) = \gamma_E + \log 2 + \log m + \tfrac{1}{2}\log y \qquad (m \neq 0)$$

- We have discovered a nice analog for LMN eccentric orbits. Define a modified eulerlog function as
$$eulerlog_{m,n}(y) = \gamma_E + \log 2 + \log|m+n| + \tfrac{1}{2}\log y \qquad (m \neq -n)$$

- Example: full 3, 4, and 5 PN terms in the 225 mode all have the form

$$\mathcal{L}^{225}_{3/4/5} + \mathcal{L}^{225}_{3L/4L/5L}\log y = \sum_{i=5}^{\infty} e^{2i}\left[a_i + b_i\pi^2 + c_i\left(\gamma_E + \log 2 + \log 7 + \frac{1}{2}\log y\right)\right]$$

Fitting by LMN: Advantages

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Exploiting patterns: 3, 4, and 5 PN (relative)

- Without eulerlog function, the integer 3, 4, and 5 PN terms in an LMN mode will have "components" $\{1, \pi^2, \gamma_E, \log 2, \log|m+n|\}$ – the "search vector" or "structure" of the term

- I.e., Mathematica assumes

$$\mathcal{L}^{LMN}_{3/4/5} = \sum_{i=1}^{\infty} e^{2i}\Big(a_i + b_i * \pi^2 + c_i * \gamma_E + d_i * \log 2 + e_i * \log|m+n|\Big)$$

and tries to find each $\{a_i, b_i, c_i, d_i, e_i\}$ all at once. Hard.

- With eulerlog function, there is a better way:
  - Fit the rational $\log y$ e-series first to find $\mathcal{L}^{LMN}_{3L/4L/5L}$.

$$\mathcal{L}^{LMN}_{3/4/5} = 2\mathcal{L}^{LMN}_{3L/4L/5L}\big(\gamma_E + \log 2 + \log|m+n|\big) + \sum_{i=1}^{\infty} e^{2i}\Big(a_i + b_i * \pi^2\Big)$$

  - Our remaining search vector is $\{1, \pi^2\}$. Much easier!

Example: 5 PN term, 221 mode

- With 200 decimals of accuracy, fit the 5 PN Log term to get

$$\mathcal{L}_{5L}^{221} = \Bigg(- \frac{3042117e^2}{896} + \frac{1337829453e^4}{125440} - \frac{249436511343e^6}{14049280} + \frac{55247886837e^8}{22478848} + \cdots + \frac{282046609802069394 1389e^{20}}{2887420613754880000} + \cdots \Bigg)$$

- Then, the 5 PN term will be

$$\mathcal{L}_{5}^{221} = 2\Big(\gamma_E + \log 2 + \log 3\Big)\Bigg(- \frac{3042117e^2}{896} + \frac{1337829453e^4}{125440} - \frac{249436511343e^6}{14049280} + \frac{55247886837e^8}{22478848} + \cdots + \frac{2820466098020693941389e^{20}}{2887420613754880000} \cdots \Bigg) + \sum_{i=1}^{\infty} e^{2i}\Big(a_i + b_i * \pi^2\Big)$$

Example: 5 PN term, 221 mode

- Given numerical results on LHS, we can find a_i and b_i to e^{12} as

$$\frac{2187\left(2196050489+182182000\pi^2\right)e^2}{179379200}-\frac{729\left(1296032113007+48070822800\pi^2\right)e^4}{5022617600}+\frac{2187\left(34135889421727+2987580195600\pi^2\right)e^6}{562533171200}-\frac{729\left(-7734258061849217+9925821906000\pi^2\right)e^8}{4500265369600}+\frac{243\left(3822096471553495061+59917530089617200\pi^2\right)e^{10}}{360021229568000}-\frac{243\left(-22356612501549133663+1278212748247234800\pi^2\right)e^{12}}{7200424591360000}$$

- Add back in the eulerlogs to get

$$\mathcal{L}_5=\left(\frac{2187e^2\left(2196050489-556956400\gamma+182182000\pi^2-556956400\log(2)-556956400\log(3)\right)}{179379200}+\cdots+large(e^{12})term\right)$$

- Similar to Johnson-McDaniel, Shah, Whiting (2015)

Exploiting patterns: 6, 7, and 8 PN (relative)

- Simplification at 6/7/8 order even more magical!
- Without eulerlog function, search vector now $\{1, \pi^2, \gamma_E, \log 2, \log|m+n|, \pi^2\gamma_E, \pi^2\log 2, \pi\log|m+n|, \gamma_E\log 2, \gamma_E\log|m+n|, \log 2\log|m+n|, \pi^4, \gamma_E^2, (\log 2)^2, (\log|m+n|)^2, \zeta(3)\}$
- Getting the correct answer from that almost comically difficult
- Instead, assume that various PN compositions will result in the following form:

$$\mathcal{L}_6^{LMN} + \mathcal{L}_{6L}^{LMN}\log y + \mathcal{L}_{6L^2}^{LMN}(\log y)^2 = (\text{zeta terms}) +$$

$$\sum_{i=N}^{\infty} e^{2i}\Big(a_i + b_i * \pi^2 + (c_i) * \text{eulog}(y)\Big)\Big(d_i + e_i * \pi^2 + (f_i) * \text{eulog}(y)\Big)$$

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Exploiting patterns: 6, 7, and 8 PN (relative)

- After multiplying out, collecting terms, and renaming, we find

$$\mathcal{L}_{6L^2}^{LMN} (\log y)^2 = \sum_{i=N}^{\infty} e^{2i} \left(\frac{C_i}{4} * (\log y)^2 \right)$$

$$\mathcal{L}_{6L}^{LMN} \log y = \sum_{i=N}^{\infty} e^{2i} \Big(A_i + B_i * \pi^2 + C_i * (\gamma_E + \log 2 + \log |m+n|) \Big) \log y$$

$$\mathcal{L}_{6}^{LMN} = \sum_{i=N}^{\infty} e^{2i} \Big(2 * A_i * \mathrm{eulog} + 2 * B_i * \pi^2 * \mathrm{eulog} + C_i * \mathrm{eulog}^2 + D_i + E_i * \pi^2 + F_i * \pi^4 + G_i * \zeta(k) \Big)$$

- Thus, by working from the top down, only have to solve for $\{D_i, E_i, F_i, G_i\}$ in the integer part

- This reduces our search vector from 16 elements to the 4 elements $\{1, \pi^2, \pi^4, \zeta(k)\}$!!

Procedure validation/example: 6 PN circular orbit limit

- Circular orbit limit at 6 PN gives a partial example describing and verifying this method

- Composed of the LMN modes $\{210, 220, 310, 320, 330, 410, \cdots, 820, 840, 860, 880\}$

- Only 220 provides terms with products of transcendentals in the integer part, terms with any transcendentals in the $(\log y)^1$ part, and any terms in the $(\log y)^2$ part – we expect those relationships to hold

$$\text{eulerlog}_{2,0}(y) = \gamma_E + 2 * \log 2 + \frac{1}{2} * \log y$$

- On the other hand, terms linear in transcendentals come from 12 different LMN modes with different eulerlog forms – don't expect those relationships to hold (besides γ_E)

6 PN circular orbit: results

- Indeed, we see

$$\mathcal{L}^{circ}_{6L^2}(\log y)^2 = \frac{366368}{11025} * (\log y)^2$$

$$\mathcal{L}^{circ}_{6L}\log y = \left(q - \frac{13696}{315} * \pi^2 + 4 * \frac{366368}{11025}\gamma_E + 8 * \frac{366368}{11025} * \log 2\right)\log y$$

$$\mathcal{L}^{circ}_{6} = \Big(2 * q * \gamma_E - 2 * \frac{13696}{315} * \pi^2 * (\gamma_E + 2\log 2) + 4 * \frac{366368}{11025} * (\gamma_E + 2\log 2)^2 + \left(\{1, \pi^2, \pi^4, \log 2, \log 3, \log 5, \zeta(3)\} \text{ terms}\right)\Big)$$

$$q = -\frac{246137536815857}{314659144800}$$

- Thus, confirms expectations and shows that even in the circular orbit limit, still better to fit each LMN mode individually (nonzero powers of e even worse!)

Extensions: Angular momentum at infinity, E and L at horizon

- Methods given all extend perfectly to radiated angular momentum at infinity
- Slightly different procedures at the horizon (work in progress)
  - Example: No γ_E or $\log|m+n|$ at 3PN
  - Example 2: 4PN and 5PN have full search vector, but more complicated ratios
- Currently have the lowest order (e.g. 0, 1, 2, 3, $4L$) horizon enhancement functions exactly

Conclusions

Use BH perturbations to probe PN limit in eccentric orbits

- Combined MST with SSI in *Mathematica*
- Calculated fluxes (and y) in terms of p and e

Enhanced calculation of PN coefficients

- Separate LMN data, set needed precision via "classes" of LM modes
- Use new LMN eulerlog function to simplify complicated combinations of transcendentals
- Found $\mathcal{L}_{5L}$, $\mathcal{L}_{6L2}$. May also provide $\mathcal{L}_{7L2}$, $\mathcal{L}_{8L2}$, and even $\mathcal{L}_{9L3}$

Applications: $\dot{E}^{\infty}(e)$, $\dot{J}^{\infty}(e)$, $\dot{E}^{H}(e)$, and $\dot{J}^{H}(e)$

- Obtain all 4 fluxes simultaneously, fit separately
- Same or similar eulerlog behavior can be exploited