Progress at second order

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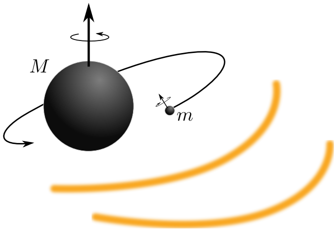
Outline

Outline

Modeling EMRIs



- treat m as source of perturbation of M's metric $g_{\mu\nu}$:

$$\mathsf{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^1_{\mu\nu} + \epsilon^2 h^2_{\mu\nu} + \ldots$$

  where $\epsilon \sim m/M$

- represent motion of m via worldline z^μ satisfying

$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon F^\mu_1 + \epsilon^2 F^\mu_2 + \ldots$$

- force is small; inspiral occurs very slowly, on time scale $\tau \sim 1/\epsilon$
- suppose we neglect F^μ_2; leads to error $\delta\left(\frac{D^2 z^\mu}{d\tau^2}\right) \sim \epsilon^2$

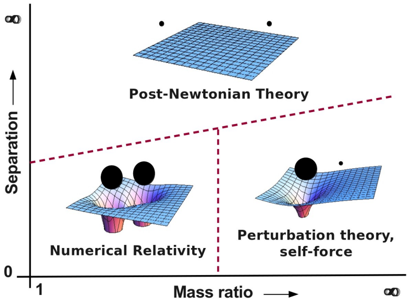
  $\Rightarrow$ error in position $\delta z^\mu \sim \epsilon^2 \tau^2$

  $\Rightarrow$ after inspiral time $\tau \sim 1/\epsilon$, error $\delta z^\mu \sim 1$
- so accurately describing orbital evolution requires second order

Improving models of IMRIs and similar-mass binaries

- at interface between models, SF data can fix high-order PN terms and calibrate EOB
- already done at first order
- second-order results will further improve these models
- also can use SF to *directly* model IMRIs

Binary parameter space



[Leor Barack]

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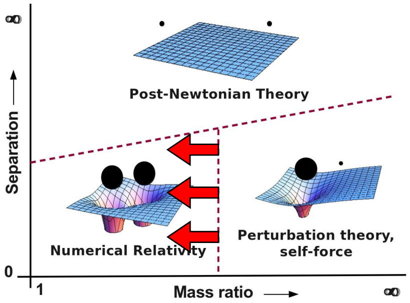
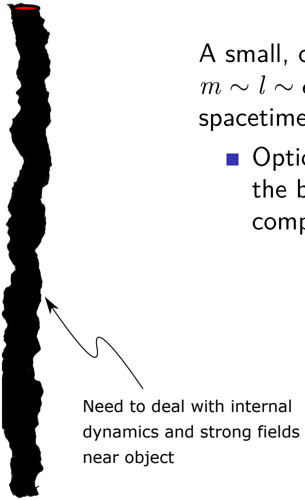


[Leor Barack]

Outline

1. Why second order?
2. The local problem and a new, highly regular gauge
3. The global problem and method of multiple scales
4. Application: quasicircular orbits in Schwarzschild
5. Conclusion

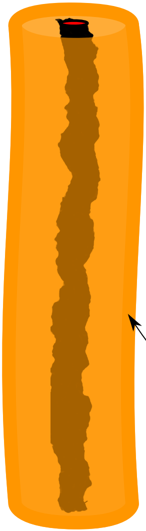
What is the problem we want to solve?



A small, compact object of mass and size $m \sim l \sim \epsilon$ moves through (and influences) spacetime

- Option 1: tackle the problem directly, treat the body as finite sized, deal with its internal composition

What is the problem we want to solve?



A small, compact object of mass and size $m \sim l \sim \epsilon$ moves through (and influences) spacetime

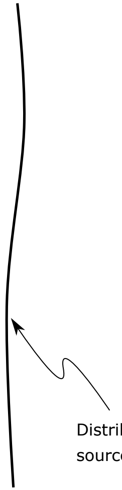
- Option 2: restrict the problem to distances $s \gg m$ from the object, treat m as source of perturbation of external background $g_{\mu\nu}$:

$$\mathrm{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^1_{\mu\nu} + \epsilon^2 h^2_{\mu\nu} + \dots$$

- This is a free boundary value problem

Metric here must agree with metric outside a small compact object; and "here" moves in response to field

What is the problem we want to solve?



A small, compact object of mass and size $m \sim l \sim \epsilon$ moves through (and influences) spacetime

- Option 3: treat the body as a point particle
  - takes behavior of fields outside object and extends it down to a fictitious worldline
  - so $h^1_{\mu\nu} \sim 1/s$ ($s =$distance from object)
  - second-order field equation $\delta G[h^2] \sim -\delta^2 G[h^1] \sim (\partial h^1)^2 \sim 1/s^4$ —no solution unless we restrict it to points off worldline, which is equivalent to FBVP

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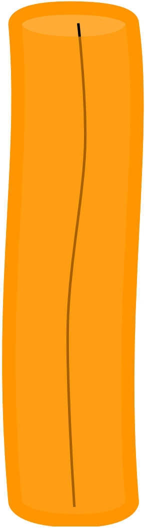
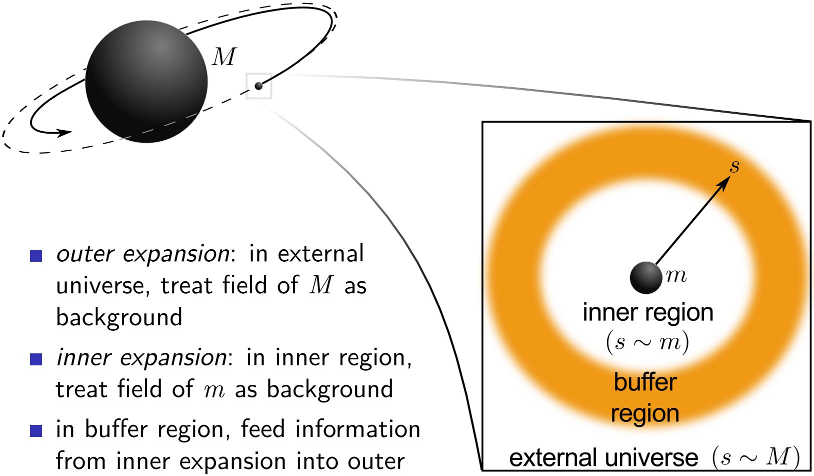


A small, compact object of mass and size $m \sim l \sim \epsilon$ moves through (and influences) spacetime

- Option 4: transform the FBVP into an *effective* problem using a *puncture*, a local approximation to the field outside the object
- This will be the method emphasized here
- But we'll see how the idea of a point particle can be sustained in a *highly regular gauge*

Matched asymptotic expansions

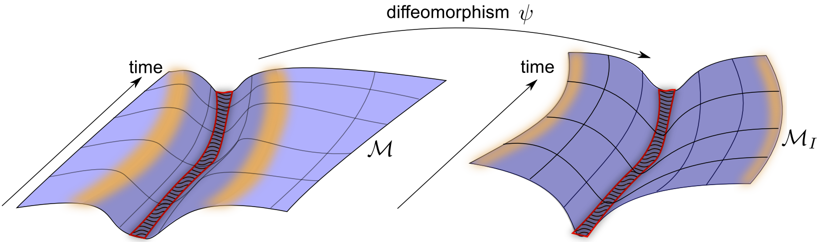


- *outer expansion*: in external universe, treat field of M as background
- *inner expansion*: in inner region, treat field of m as background
- in buffer region, feed information from inner expansion into outer expansion

The *inner expansion*

Zoom in on object

- use scaled distance $\tilde{s} \sim s/\epsilon$ to keep size of object fixed, send other distances to infinity as $\epsilon \to 0$
- unperturbed object defines background spacetime $g_{I\mu\nu}$ in inner expansion
- buffer region at asymptotic infinity $s \gg m$
  $\Rightarrow$ can define object's multipole moments as those of $g_{I\mu\nu}$



First and second order solutions

Given only the existence of inner expansion, local solution in Lorenz gauge found to be:

First order

- $h^1_{\mu\nu} = h^{\mathrm{S}1}_{\mu\nu} + h^{\mathrm{R}1}_{\mu\nu}$
- $h^{\mathrm{S}1}_{\mu\nu} \sim 1/s + O(s^0)$ defined by mass monopole m
- $h^{\mathrm{R}1}_{\mu\nu}$ is undetermined homogenous solution regular at $s = 0$

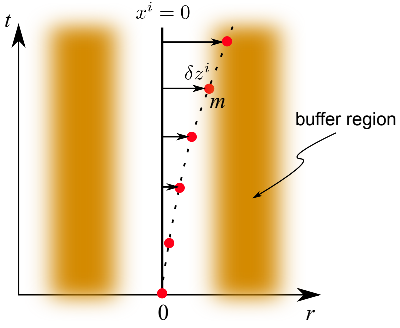
Second order

- $h^2_{\mu\nu} = h^{\mathrm{S}2}_{\mu\nu} + h^{\mathrm{R}2}_{\mu\nu}$
- $h^{\mathrm{S}2}_{\mu\nu} \sim 1/s^2 + O(1/s)$ defined by
  1. monopole correction δm
  2. mass dipole M^μ
  3. spin dipole S^μ

Perturbed position at first order [Mino et al, Gralla-Wald, Pound]

Reminder: mass dipole moment M^i:

- small displacement of center of mass from origin of coordinates
- e.g., Newtonian field $\dfrac{m}{|x^i - \delta z^i|} \approx \dfrac{m}{|x^i|} + \dfrac{m\delta z^j n_j}{|x^i|^2} \;\Rightarrow\; M^i = m\delta z^i$



Definition of object's worldline:

- work in coordinates (t, x^i) centered on a curve γ
- mass dipole is integral over small sphere:
  $$M^i = \tfrac{3}{8\pi} \lim_{s\to 0} \oint h^2_{\mu\nu} u^\mu u^\nu n^i dS$$
- equation of motion of z^μ: whatever ensures $M^\mu \equiv 0$

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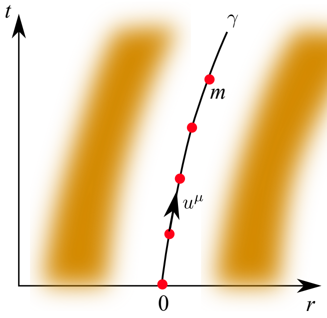

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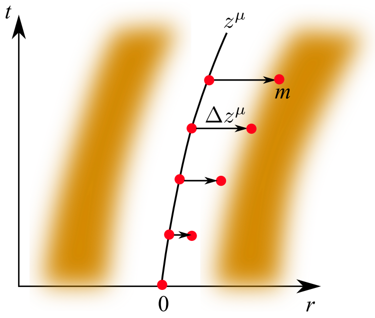
Perturbed position at second order [Pound]

Problem:

- mass dipole moment defined for asymptotically flat spacetimes
- beyond zeroth order, inner expansion is not asymptotically flat



Solution:

- start in "rest gauge" centered on z^μ
- demand that transformation to practical (e.g., Lorenz) gauge does not move z^μ
- i.e., insist $\Delta z^\mu = 0$
- ensures worldline in the two gauges is the same

0th-, 1st-, and 2nd-order equations of motion

0th order, arbitrary object: $\dfrac{D^2 z^\mu}{d\tau^2} = O(m)$ (geodesic motion in $g_{\mu\nu}$)

1st order, arbitrary object [MiSaTaQuWa]:

$$\frac{D^2 z^\mu}{d\tau^2} = -\tfrac{1}{2}\left(g^{\alpha\delta} + u^\alpha u^\delta\right)\left(2h^{\mathrm{R1}}_{\delta\beta;\gamma} - h^{\mathrm{R1}}_{\beta\gamma;\delta}\right) u^\beta u^\gamma + \tfrac{1}{2m} R^\alpha{}_{\beta\gamma\delta} u^\beta S^{\gamma\delta} + O(m^2)$$

(motion of spinning test body in $g_{\mu\nu} + h^{\mathrm{R1}}_{\mu\nu}$)

2nd-order, nonspinning, spherical object [Pound]:

$$\frac{D^2 z^\mu}{d\tau^2} = -\tfrac{1}{2}\left(g^{\mu\nu} + u^\mu u^\nu\right)\left(g_\nu{}^\rho - h^{\mathrm{R}\rho}_\nu\right)\left(2h^{\mathrm{R}}_{\rho\sigma;\lambda} - h^{\mathrm{R}}_{\sigma\lambda;\rho}\right) u^\sigma u^\lambda + O(m^3)$$

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Point particles and punctures

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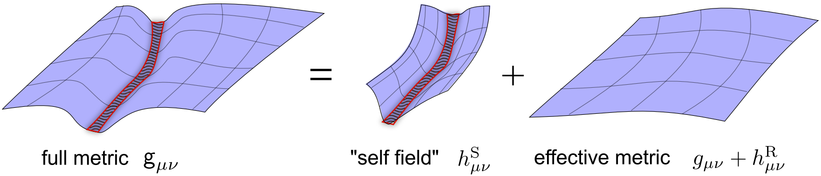

- at 1st order, can use this to *replace object with a point particle*:

$$T^1_{\mu\nu} := \frac{1}{8\pi}\delta G_{\mu\nu}[h^1]$$

- beyond 1st order, point particles not well defined—but can replace object with a *puncture*, a local singularity in the field, moving on γ, equipped with the object's multipole moments

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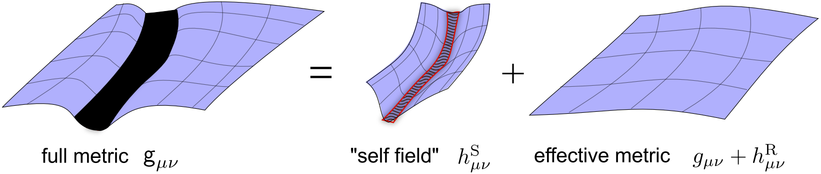

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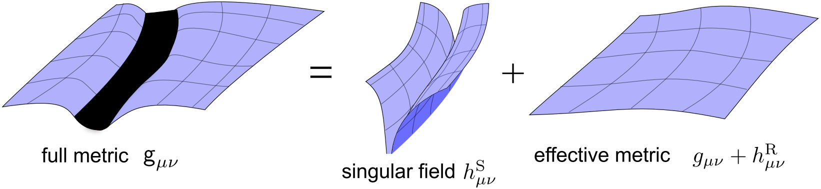

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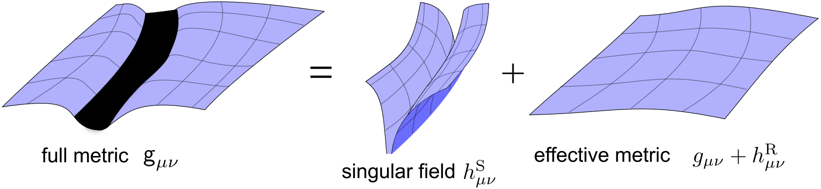

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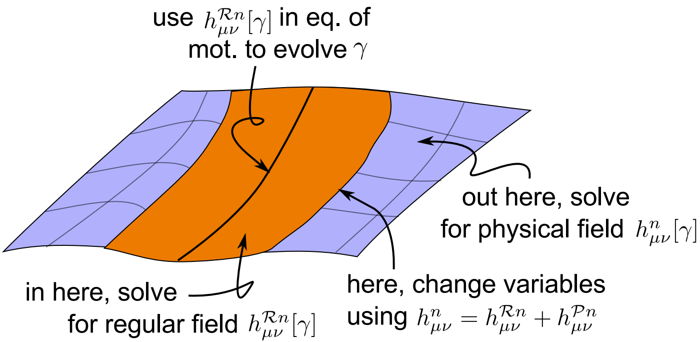
Puncture scheme

- use a local expansion of $h^{\mathrm{S}n}_{\mu\nu}$ as a "puncture" $h^{\mathcal{P}n}_{\mu\nu}$ that moves on γ
- solve field equations for $h^{n}_{\mu\nu}$ and $h^{\mathcal{R}n}_{\mu\nu} := h^{n}_{\mu\nu} - h^{\mathcal{P}n}_{\mu\nu} \approx h^{\mathrm{R}n}_{\mu\nu}$
- move the puncture with eqn of motion (using $\partial h^{\mathcal{R}n}_{\mu\nu}|_\gamma = \partial h^{\mathrm{R}n}_{\mu\nu}|_\gamma$)



Metric in a lightcone gauge

- The “rest gauge” metric is derived from an inner expansion —specifically, the metric of a tidally perturbed BH in Eddington-Finkelstein coordinates + lightcone gauge
- Exact Schwarzschild metric in EF coords *is linear in* m
- Translates to a highly regular metric perturbation in outer expansion: $h^2_{\mu\nu} \sim m^2 s^0 \mathcal{E} + ms\delta\mathcal{E}$
- Can define singular and regular fields

$$
\begin{aligned}
h^{\rm S1}_{\mu\nu} &\sim \frac{m}{s} \\
h^{\rm S2}_{\mu\nu} &\sim m^2 s^0 + ms\delta\mathcal{E} \\
h^{\rm R1}_{\mu\nu} &\sim s^2\delta\mathcal{E} \\
h^{\rm R2}_{\mu\nu} &\sim s^2\delta^2\mathcal{E}
\end{aligned}
$$

- $g_{\mu\nu} + h^{\rm R}_{\mu\nu}$ is a vacuum metric, and motion is geodesic in it

Motion in a highly regular gauge [Pound 2017]

- Transforming to a generic gauge such as Lorenz, we end up with the strong, $1/s^2$ divergences
- But we can keep the gauge highly regular by performing a *smooth* transformation (subject to worldline-preserving condition $\xi^\mu|_\gamma = 0$):

$$\Delta h^{\rm R1}_{\alpha\beta} = \mathcal{L}_{\xi_1} g_{\alpha\beta}$$
$$\Delta h^{\rm S1}_{\alpha\beta} = 0$$
$$\Delta h^{\rm R2}_{\alpha\beta} = \mathcal{L}_{\xi_2} g_{\alpha\beta} + \mathcal{L}_{\xi_1} h^{\rm R1}_{\alpha\beta} + \frac{1}{2}\mathcal{L}^2_{\xi_1} g_{\alpha\beta}$$
$$\Delta h^{\rm S2}_{\alpha\beta} = \mathcal{L}_{\xi_1} h^{\rm S1}_{\alpha\beta}$$

- This ensures $g_{\mu\nu} + h^{\rm R}_{\mu\nu}$ remains a vacuum metric and that motion remains geodesic in it

Point particles and punctures in a highly regular gauge

- In the highly regular gauge, $h^{\mathrm{S}2}_{\mu\nu} \sim \frac{m h^{\mathrm{R}1}}{s} + m^2 s^0 + \dots$
- The second-order field equation becomes

$$\delta G_{\mu\nu}[h^{\mathcal{R}2}] = -\delta^2 G_{\mu\nu}[h^1] - \delta G_{\mu\nu}[h^{\mathcal{P}2}]$$

  where $h^{\mathcal{R}2}_{\mu\nu}$ can be in any desired gauge

- But $\delta^2 G_{\mu\nu}$ now well defined as a distribution, so we can write a distributional equation for the retarded field:

$$\delta G_{\mu\nu}[h^2] = -\delta^2 G_{\mu\nu}[h^1] + 8\pi T^2_{\mu\nu}$$

- Here $T^2_{\mu\nu}$ defined from $\delta G_{\mu\nu}[h^2]$, not from expansion of exact point-particle stress-energy
  —I haven't yet examined its form, but it can *define* what we mean by the second-order stress-energy of a point particle

Benefits of working in highly regular gauge

- Eliminates most singular part of source and associated computational expense
- Can solve for retarded field directly, without a puncture
- Can use mode-sum regularization?
- Can eliminate troublesome singularities in EFT approach?
- Can express singular and regular fields in terms of Green's functions?
- Since the nondistributional singularity can be eliminated with a gauge transformation, invariant field equations (e.g., second-order Teukolsky) should be free of it

Outline

Solving the perturbed Einstein globally

- solving the local problem told us how to replace the small object with a moving puncture in the field equations:

$$E_{\mu\nu}[h^{\mathcal{R}1}] = -E_{\mu\nu}[h^{\mathcal{P}1}] \quad \text{inside } \Gamma$$
$$E_{\mu\nu}[h^{1}] = 0 \quad \text{outside } \Gamma$$

$$E_{\mu\nu}[h^{\mathcal{R}2}] = \delta^2 R_{\mu\nu}[h^1, h^1] - E_{\mu\nu}[h^{\mathcal{P}2}] \quad \text{inside } \Gamma$$
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where Γ is a tube around z^μ, $E_{\mu\nu}[h] \sim \Box h_{\mu\nu}$, $h^{\mathcal{P}n}_{\mu\nu} \approx h^{Sn}_{\mu\nu}$, $h^{\mathcal{R}n}_{\mu\nu} = h^n_{\mu\nu} - h^{\mathcal{P}n}_{\mu\nu}$

- the global problem: how do we solve these equations in practice?

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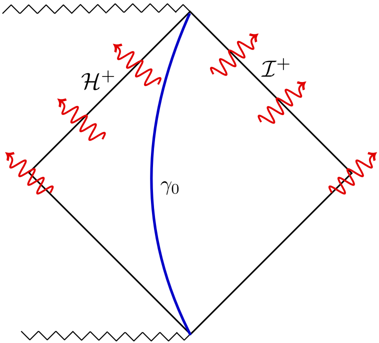

- approximate the source orbit as a bound geodesic
- impose outgoing-wave BCs at $\mathcal{I}^+$ and $\mathcal{H}^+$
- solve field equation numerically, compute self-force from solution
- system radiates forever; at any given time, BH has already absorbed infinite energy
- but on short sections of time the approximation is accurate
- breaks down on *dephasing time* $\sim 1/\sqrt{\epsilon}$, when $|z^\mu - z_0^\mu| \sim M$

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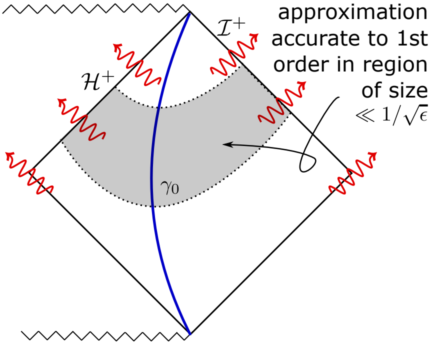

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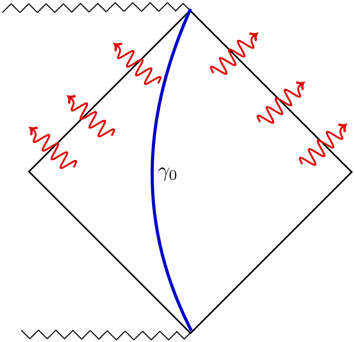
- breaks down on *dephasing time* $\sim 1/\sqrt{\epsilon}$, when $|z^\mu - z_0^\mu| \sim M$

Infrared problems at second order



- suppose we try to use "typical" $h^1_{\mu\nu}$ to construct source for $h^2_{\mu\nu}$
- because $|z^\mu - z^\mu_0|$ blows up with time, $h^2_{\mu\nu}$ does likewise
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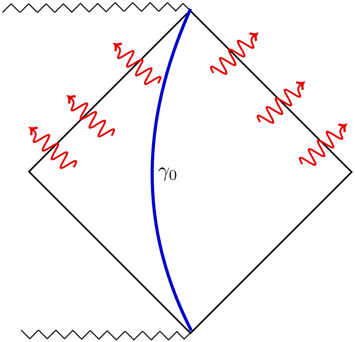

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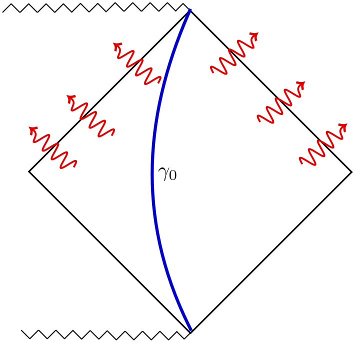

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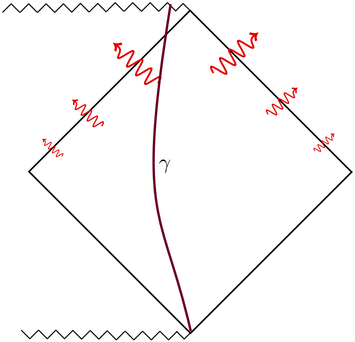

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Resolutions of the infrared problem

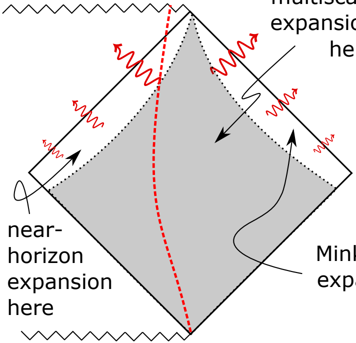
Option 1

- solve field equations and equation of motion self-consistently in the time domain
- problems:
  - limited accuracy
  - gauge instabilities
  - difficult to accurately account for growing BH mass (see Moxon's talk)

Option 2:

- again use matched expansions: different expansions in different regions
- advantages:
  - allows calculations in frequency domain; high accuracy
  - no instabilities
  - better control over behavior in each region, easier to impose correct initial data

Matched expansions [Pound, Moxon, Flanagan, Hinderer, Yamada, Isoyama, Tanaka]



Multiscale expansion

- multiscale expansion: expand orbital parameters and fields as

$$J = J_0(\tilde{t}) + \epsilon J_1(\tilde{t}) + \ldots$$

$$h^n_{\mu\nu} \sim \sum_{kk'} h^n_{kk'}(\tilde{t}) e^{-ikq_r(\tilde{t}) - ik' q_\phi(\tilde{t})}$$

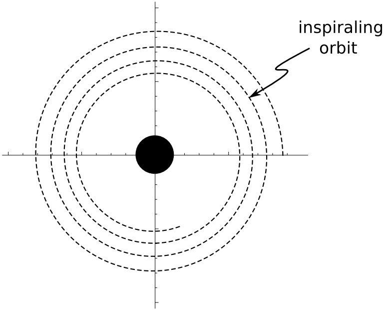
  where (J, q) are action-angle variables for z^μ, and $\tilde{t} \sim \epsilon t$ is a "slow time"
- solve for $h^n_{kk'}$ at fixed $\tilde{t}$ with standard frequency-domain techniques

Get boundary conditions from

- post-Minkowski expansion: expand $h^n_{\mu\nu}$ in powers of M
- near-horizon expansion: expand $h^n_{\mu\nu}$ in powers of gravitational potential near horizon

Outline

Quasicircular orbits in Schwarzschild [Pound, Wardell, Warburton, Miller, Barack]



- Multiscale expansion of the worldline:
  - radius $r_p = r_0(\tilde{t}) + \epsilon r_1(\tilde{t}) + \dots$
  - frequency $\Omega = \Omega_0(\tilde{t}) + \epsilon\Omega_1(\tilde{t}) + \dots$
  - orbital phase $\phi_p = \frac{1}{\epsilon}\int \Omega d\tilde{t}$

- Multiscale expansion of the field:

$$h^n_{\mu\nu} = \sum_{ilm} h^n_{ilm}(\tilde{t}, r)\, e^{-im\phi_p(\tilde{t})}\, Y^{ilm}_{\mu\nu}$$

- we take a "snapshot", doing our calculations at some $\tilde{t} = \tilde{t}_0$

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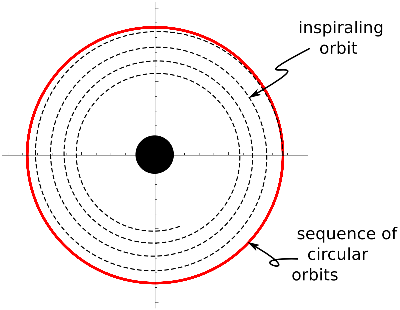

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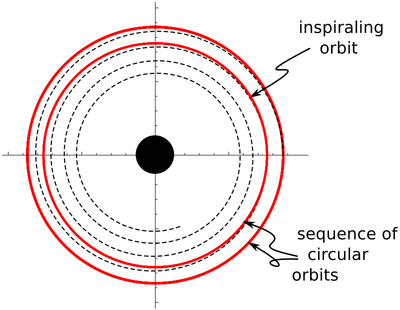

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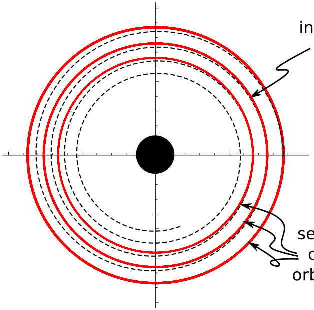

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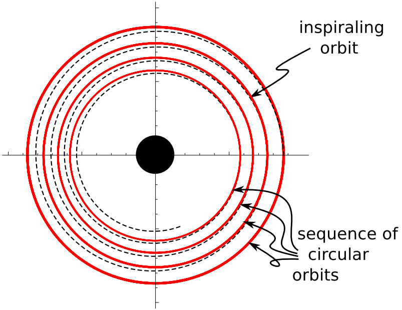

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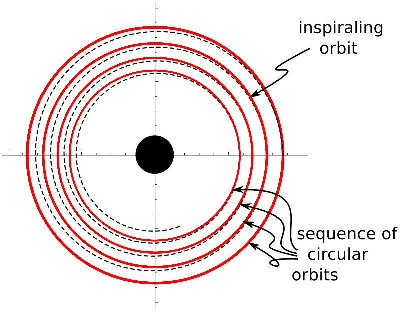

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Field equations

$$E_{\mu\nu}[h^{\mathcal{R}1}] = -E_{\mu\nu}[h^{\mathcal{P}1}] \quad \text{inside } \Gamma$$
$$E_{\mu\nu}[h^{1}] = 0 \quad \text{outside } \Gamma$$

$$E_{\mu\nu}[h^{\mathcal{R}2}] = \delta^2 R_{\mu\nu}[h^1, h^1] - E_{\mu\nu}[h^{\mathcal{P}2}] \quad \text{inside } \Gamma$$
$$E_{\mu\nu}[h^{2}] = \delta^2 R_{\mu\nu}[h^1, h^1] \quad \text{outside } \Gamma$$

$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon F_1^\mu + \epsilon^2 F_2^\mu$$

$$h^{\mathcal{P}1}_{\mu\nu} \sim \frac{m}{|x^\alpha - z^\alpha|}$$
$$h^{\mathcal{P}2}_{\mu\nu} \sim \frac{m^2}{|x^\alpha - z^\alpha|^2} + \frac{\delta m + m h^{\mathcal{R}1}}{|x^\alpha - z^\alpha|}$$

Field equations

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$$
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E^0_{ilm}[h^2_{jlm}] &= \delta^2 R^0_{ilm}[h^1_{jl'm'}, h^1_{kl''m''}] - E^1_{ilm}[h^1_{jlm}] \quad \text{outside } \Gamma
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$$h^{\mathcal{P}1}_{ilm} \sim m|r - r_0|$$
$$h^{\mathcal{P}2}_{ilm} \sim m^2 \log|r - r_0| + (\delta m + m h^{\mathcal{R}1} + r_1 + \dot{r}_0)|r - r_0|$$

Boundary conditions from PM/Near-Horizon expansions

At large r, adapt Blanchet-Damour PM methods

- The source behaves as $\delta^2 R^0_{il0} \sim \frac{S_{il0}}{r^2}$
- For $l = 0, 2$, hereditary terms arise:

$$h^2_{il0} \sim \ln(r/\epsilon) S_{il0} + \int_{-\infty}^{0} \frac{d}{d\tilde{t}} S_{il0}(\tilde{t} - \epsilon r + \tilde{z}) \ln \tilde{z} \, d\tilde{z}$$

At $r \approx 2M$, similar iteration using near-horizon retarded Green's function

- (Semi)hereditary terms arise:

$$h^2_{il0} \sim (r - 2M)\delta^2 R_{il0} + \int_{-\infty}^{0} \delta^2 R_{il0}(\tilde{t} + \epsilon r + \tilde{z}) d\tilde{z}$$

We use these asymptotic approximations as punctures $h^{\infty\mathcal{P}}_{il0}$ and $h^{\mathcal{HP}}_{il0}$ at infinity/horizon (actually, we currently use ad hoc $h^{\mathcal{HP}}_{il0}$!)

Specialization to $\ell = 0$

Advantages?

- Clean(?) split into dissipative and conservative sectors
  - Dissipative sector: h^2_{tr}, $\partial_{\tilde{t}} h^1_{tt}$, $\partial_{\tilde{t}} h^1_{rr}$, $g^{\mu\nu}\partial_{\tilde{t}} h^1_{\mu\nu}$, $\dot{r}_0$
  - Conservative sector: h^2_{tt}, h^2_{rr}, $g^{\mu\nu} h^2_{\mu\nu}$, r_1
- First-order solution known analytically

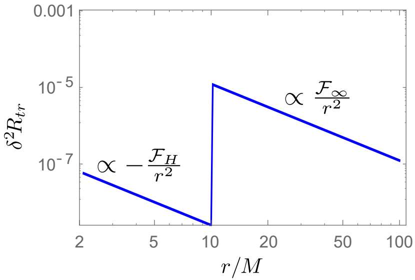
Things to mind:

- First-order perturbation must include slowly varying correction to BH mass
- We absorb $\delta M_{BH}(\tilde{t}_0)$ (and hereditary integrals) into background mass M
- We take our "snapshot" at the preferred time when $\Omega(\tilde{t}_0) = \Omega_0(\tilde{t}_0)$

$\ell = 0$, dissipative sector

- field equation:

$$\begin{aligned}\partial_r^2 h_{tr}^{\mathcal{R}2} \sim \delta^2 R_{tr}[h^1, h^1] \\ - \partial_r^2 h_{tr}^{S2} \\ - \partial_{\tilde{t}} h_{tt}^1\end{aligned}$$



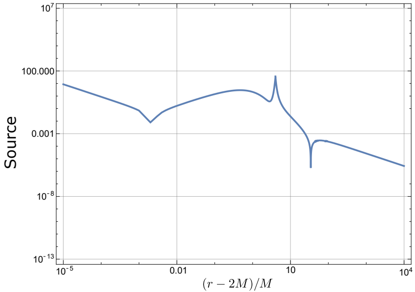
What comes out of the solution?

- balance law: $\dot{E}_0 + \delta \dot{M}_{BH} = \mathcal{F}_\infty$
- first major result/consistency check of our framework

$\ell = 0$, dissipative sector

- field equation:

$$\begin{aligned}\partial_r^2 h_{tt}^{\mathcal{R}2} \sim \delta^2 R_{tt}[h^1, h^1] \\ - \partial_r^2 h_{tt}^{S2} \\ - \partial_r^2 (h_{tt}^{\infty} + h_{tt}^{\mathcal{H}})\end{aligned}$$



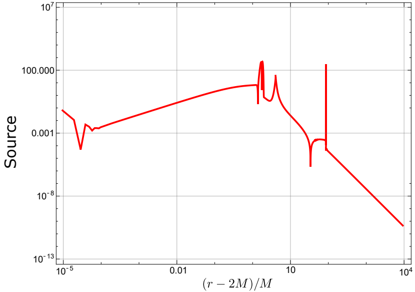
What comes out of the solution?

- binding energy $E_{bind} = (M_{\text{Bondi}} - m - M_{BH})/\mu$

$\ell = 0$, dissipative sector

- field equation:

$$\partial_r^2 h_{tt}^{\mathcal{R}2} \sim \delta^2 R_{tt}[h^1, h^1] - \partial_r^2 h_{tt}^{S2} - \partial_r^2 (h_{tt}^{\infty} + h_{tt}^{\mathcal{H}})$$



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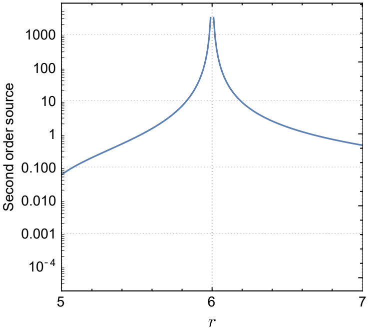

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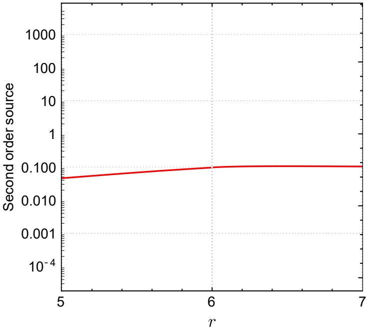

What comes out of the solution?

- binding energy $E_{bind} = (M_{\rm Bondi} - m - M_{BH})/\mu$

Computing the binding energy

$$E_{bind} = (M_{\rm Bondi} - m - M_{BH})/\mu$$

- $M_{\rm Bondi}$ obtained from matching numerical field to PM expansion
- How to identify M_{BH}?
  - A particular solution can always include a global mass perturbation $h^{\delta M}_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial M}\delta M$, implicitly determined by BCs
  - This unkown mass contributes to M_{BH}
  - We use $M_{BH} = M_{irr} = \sqrt{\frac{\text{Area of Apparent Horizon}}{16\pi}}$
  - Lengthy calculation leads to explicit formula for $M_{irr} = M + \epsilon M_1(\tilde{v}) + \epsilon^2 M_2(\tilde{v}) + O(\epsilon^3)$
  - Because it depends only on slow time $\tilde{v} = \epsilon v$, it should be gauge invariant through order ϵ^2
- Results for E_{bind}? — see Barry's talk

Outline

Status of formalism

The "local problem"

- in a highly regular gauge, nondistributional singularities are eliminated $\Rightarrow$ we can maintain the idea of a point particle
- this makes direct computation of gauge-invariant quantities (e.g., second-order Teukolsky) look very promising
- still missing spin and quadrupole effects at second order

The "global problem"

- multiscale formulation is under development, with a few remaining issues
- need to deal with resonances
- numerical tools are working well in Schwarzschild, but we need extensions to eccentric orbits and Kerr

Status of implementation in Schwarzschild

What I said last year:

"snapshot calculation" essentially complete for $\ell = 0$ field
—see talk by Wardell

What I say this year:

- "snapshot calculation" is complete for $\ell = 0$ field, many consistency checks passed
  — see talk by Wardell
- *but* niggling concerns about near-horizon BCs, possible error in expression for M_{irr} — see talk by Wardell
- dissipative higher-ℓ modes should be easier

Hierarchy of self-force models [Hinderer and Flanagan]

- on an inspiral timescale $t \sim 1/\epsilon$, the phase of the gravitational wave has an expansion

$$\phi = \frac{1}{\epsilon}\left[\phi_0 + \epsilon\phi_1 + O\big(\epsilon^2\big)\right]$$

- a model that gets ϕ_0 right is probably enough for signal detection in many cases
- a model that gets both ϕ_0 and ϕ_1 is enough for parameter extraction

Hierarchy of self-force models [Hinderer and Flanagan]

Adiabatic order

determined by

- averaged dissipative piece of F_1^μ

- on an inspiral timescale $t \sim 1/\epsilon$, the phase of the gravitational wave has an expansion

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Hierarchy of self-force models [H

Adiabatic order

determined by

- averaged dissipative piece of F_1^μ

Post-adiabatic order

determined by

- averaged dissipative piece of F_2^μ
- conservative piece of F_1^μ
- oscillatory dissipative piece of F_1^μ

- on an inspiral timescale $t \sim 1/\epsilon$, the phase of the gravitational wave has an expansion

$$\phi = \frac{1}{\epsilon}\left[\boxed{\phi_0} + \epsilon\boxed{\phi_1} + O(\epsilon^2)\right]$$

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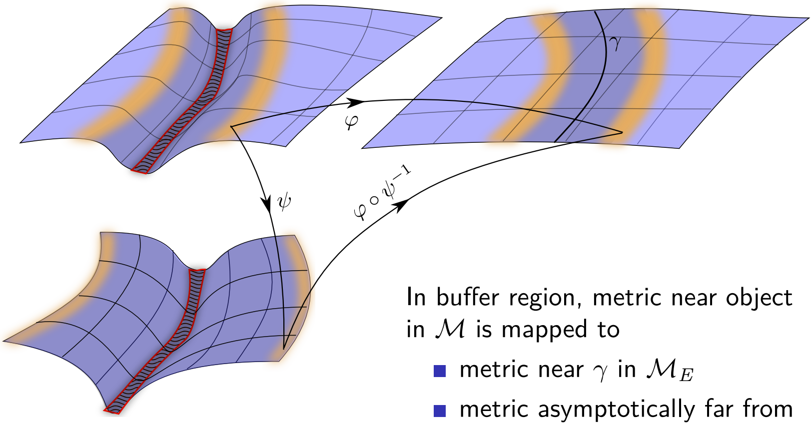
Relating the expansions

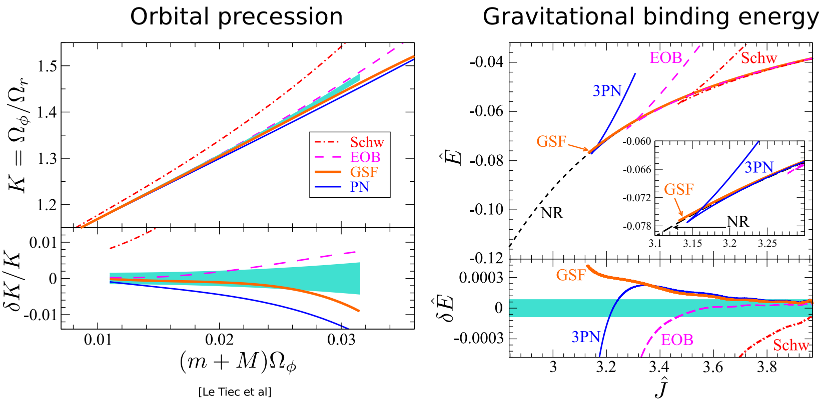


In buffer region, metric near object in $\mathcal{M}$ is mapped to

- metric near γ in $\mathcal{M}_E$
- metric asymptotically far from object in $\mathcal{M}_I$

Using SF to *directly* model IMRIs and similar-mass binaries

Comparisons for **equal-mass** binaries



[Le Tiec et al]

- SF results use "mass symmetrized" model: $\frac{m}{M} \to \frac{mM}{(m+M)^2}$
- with mass-symmetrization, second-order self-force might be able to directly model even comparable-mass binaries