Merger Simulation Using the Parker Sochacki Method and Finite Element Analysis, in a Model Explicitly Consistent with Quantum Mechanics.

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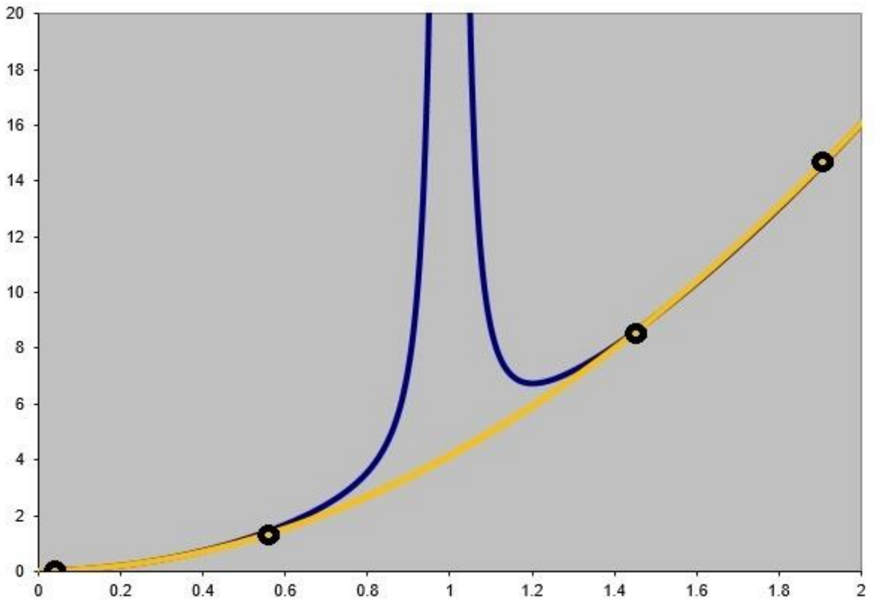
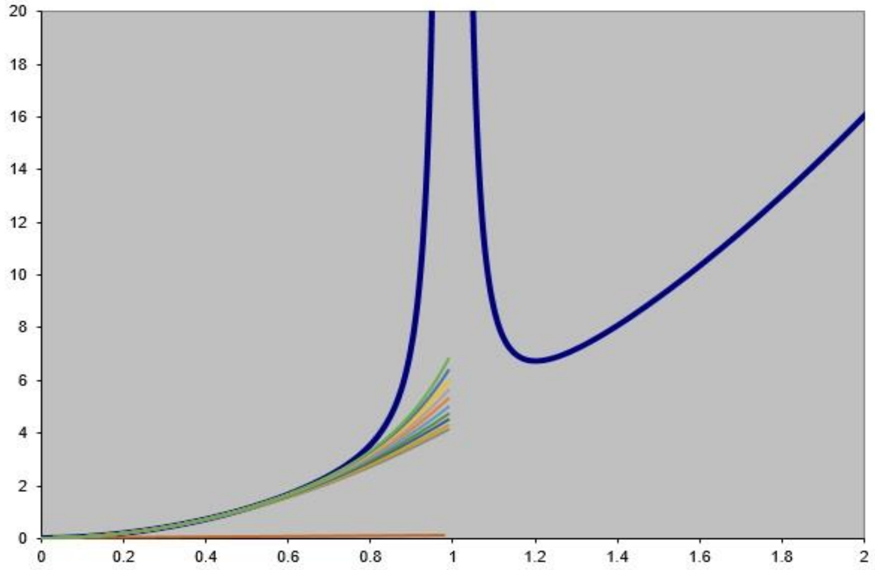
http://educ.jmu.edu/~sochacjs/PolynomialODEs-Sochacki.pdf

Outline of Talk.

I. General Relativistic Model: Isotropic Metric.
II. The Power Series Expansion Method.
III. Compound Pendulum Example.
IV. Identification of Terms that Advance the Series.
V. Calculation of an A-Priori Error Limit.
VI. History of the Method: Close Calls.





Abstract: The motion of two or more sources of a gravitational field is modelled using the Parker Sochacki Method in adaptive finite element analysis. In rest frames, the metric is isotropic but not conformally flat. A metric equation for the conjugate mass-energy-momentum equation provides explicit consistency with quantum mechanics: Unitarity is preserved because Planck's Constant is invariant with metric scaling. While a metric is invariant under a local lorentz transformation, it is not invariant in under a lorentz transformation at an observer with a different metric scaling. The lorentz-transformed metric provides the affine connection for the equations of motion, which gives the velocities of the rest frames of the metrics at each point in space as seen by an observer at an arbitrary location. The equivalence principle applied to the continuity equation (or Bianchi identities) for the Einstein Tensor as seen by any observer provides the equation which advances the Taylor series for the metric scaling: $G^{jk}(g^{-2})_{,k}=0$, where the metric scaling g appears in the metric equations in rest frames as $d\tau^2=g^{-2}dt^2-g^2\mathbf{r}^2$ and $dm_0^2=g^2E^2-g^{-2}\mathbf{p}^2$. This method is inherently symplectic because it uses the Parker Sochacki Method. It is inherently retarded and parallelizable because time evolution depends only on local conditions: Each processor can independently track its finite element.

General Relativistic Model: Isotropic Metric.

$d\tau^2 = dt'^2/g^2 - g^2\mathbf{r}'^2 \qquad d\tau^2 = g_{ij}d\xi^i d\xi^j \qquad (1)$

$dm_0^2 = g^2E'^2 - \mathbf{p}'^2/g^2 \qquad dm_0^2 = g^{ij}d\pi_i d\pi_j \qquad (2)$

$$\frac{d^2\xi^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{d\xi^\rho}{d\tau}\frac{d\xi^\sigma}{d\tau} = 0 \qquad (3) \qquad\qquad \frac{d\tau}{dt'} = \frac{c}{g\gamma} \qquad (4)$$

$$\frac{d^2\xi^\mu}{dt'^2} + \frac{1}{g\gamma}\frac{d(g\gamma)}{dt'}\frac{d\xi^\mu}{dt'} + \Gamma^\mu_{\rho\sigma}\frac{d\xi^\rho}{dt'}\frac{d\xi^\sigma}{dt'} = 0 \qquad (5)$$

$$G^{44} = -g^2(2\nabla^2 V_G + (\nabla V_G)^2)) \qquad (6)$$

$G^{ij}{}_{;j} = 0 \qquad (7) \qquad G^{ij\,\prime} = g^2G^{ij} \qquad (8) \qquad G^{ij}(g^2)_{,j} = 0 \qquad (9)$

The Power Series Expansion Method

1. Write dif. eqs. as sums of products.
2. Expand each function as a power series.
3. Treat each sum, product, or derivative as a separate step.
4. Equate powers of the independent variables.
5. Advance series until error reaches goal.

$$\left\|\mathrm{x}(t) - \sum_{j=0}^{k} \mathrm{x}_j t^j\right\| \leq ||c|| \left((1 - M|t|)^{-1/(m-1)} - \sum_{j=0}^{k} z_j |t|^j\right) \qquad (1)$$

where c are rescaled coefficients, $||c|| = c_{max}$, $m \geq 2$, $|t| < 1/M$
$M = (m-1)||\mathrm{B}_N||$, and $||\mathrm{B}_N||$ is the largest row sum
of the absolute value of coefficients of transformation B.

Taylor Series Expansion and Cauchy Product

$$f(t) = f_0 + f_1 t + f_2 t^2 + \ldots = \sum_{i=0} f_i t^i. \tag{2}$$

$$f' \equiv df/dt = f_1 + 2 f_2 t + 3 f_3 t^2 + \ldots = \sum_{i=0} (i+1) f_{i+1} t^i. \tag{3}$$

$$fg = f_0 g_0 + (f_0 g_1 + f_1 g_0) t + (f_0 g_2 + f_1 g_1 + f_2 g_0) t^2 + \ldots \tag{4}$$

$$(fg)_i = \sum_{j=0}^{i} f_j g_{i-j}. \tag{5}$$

$$g(t) = 1/f(t) : f_0 g_i = -\sum_{j=1}^{i} f_j g_{i-j}. \tag{6}$$

$$1/(1-t) = 1 + t + t^2 + t^3 + \ldots = \sum_{i=0} t^i. \tag{7}$$

$\$\, f(t) = 1/(1-t)$. Then $f' = f^2$.

Expansions for Functions of Functions

- $g(t) = f(t)^n$. Then $g' = nf'f^{n-1} = nf'g/f$.
- $g(t) = e^{f(t)}$. Then $g' = e^f f' = gf'$ by chain rule. Then

$$(i+1)g_{i+1} = \sum_{j=0}^{i-1} (j+1) f_{j+1} g_{i-j-1}. \tag{8}$$

- $g_t = \sin(f(t))$ and $h_t = \cos(f(t))$. Then

$$g' = \cos(f)f' = hf' \text{ and } h' = -\sin(f)f' = -gf'. \tag{9}$$

Compound Pendulum Example: Equations of Motion.

1) $x = a \sin\theta + b \sin\phi.$

2) $y = -a \cos\theta - b \cos\phi.$

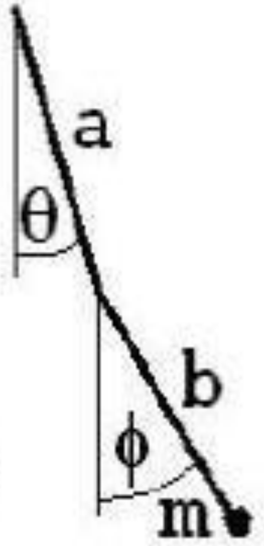
3) $L = T - V = \frac{m}{2}(a^2\theta'^2 + b^2\phi'^2 + 2ab\theta'\phi' \cos(\theta-\phi)) - mg(a\cos\theta + b\cos\phi).$

4) $0 = a\theta'' + b\phi'' \cos(\theta-\phi) + b\phi'^2 \sin(\theta-\phi) + g\sin\theta.$

5) $0 = b\phi'' + a\theta'' \cos(\theta-\phi) - a\theta'^2 \sin(\theta-\phi) + g\sin\phi.$

6) $\theta'' = \frac{b\cos(\theta-\phi)(a\theta'^2 \sin(\theta-\phi) - g\sin\theta + b^2\phi'^2 \sin(\theta-\phi) + gb\sin\theta}{ab - ab(\cos(\theta-\phi))^2}.$

7) $\phi'' = \frac{a\cos(\theta-\phi)(-b\phi'^2 \sin(\theta-\phi) - g\sin\phi - a^2\theta'^2 \sin(\theta-\phi) + ga\sin\phi}{ab - ab(\cos(\theta-\phi))^2}.$



Compound Pendulum Ex.: Polynomial Form.

$$6)\theta'' = \frac{b\cos(\theta-\phi)(a\theta'^2\sin(\theta-\phi)-g\sin\theta+b^2\phi'^2\sin(\theta-\phi)+gb\sin\theta}{ab-ab(\cos(\theta-\phi))^2}.$$

$$7)\phi'' = \frac{a\cos(\theta-\phi)(-b\phi'^2\sin(\theta-\phi)-g\sin\phi-a^2\theta'^2\sin(\theta-\phi)+ga\sin\phi}{ab-ab(\cos(\theta-\phi))^2}.$$

$8) u_1 \equiv \theta.\ \ u_3 \equiv \theta'.$	$u_1' = u_3.$
$9) u_2 \equiv \phi.\ \ u_4 \equiv \phi'.$	$u_2' = u_4.$
$10) u_5 \equiv \theta - \phi.$	$u_5' = u_3 - u_4.$
$11) u_6 \equiv \sin(\theta - \phi).$	$u_6' = u_7 u_5.$
$12) u_7 \equiv \cos(\theta - \phi).$	$u_7' = -u_6 u_5.$
$13) u_8 \equiv ab - ab(\cos(\theta - \phi))^2 = ab - abu_7^2.$	$u_8' = 2abu_7 u_7' = -2abu_5u_6u_7.$
$14) u_9 \equiv 1/u_8.$	$u_9' = -u_8'/u_8^2 = 2abu_5u_6u_7u_9^2.$
$15) u_{10} \equiv \sin\theta.$	$u_{10}' = u_{11}u_3.$
$16) u_{11} \equiv \cos\theta.$	$u_{11}' = -u_{10}u_3.$
$17) u_{12} \equiv \sin\phi.$	$u_{12}' = u_{13}u_4.$
$18) u_{13} \equiv \cos\phi.$	$u_{13}' = -u_{12}u_4.$

$19) u_3' = (bu_7(au_3^2u_6 - gu_{12}) + b^2u_4^2u_6u_{10})u_9.$

$20) u_4' = (au_7(-bu_4^2u_6 - gu_{10}) - a^2u_3^2u_6u_{12})u_9.$

Identification of Terms that Advance Series.

- $f' = f$ and $f(t = 0) = f_0$. When solved by series solution,
$f = f_0 + f_1 t + f_2 t^2 + f_3 t^3 + ... + f_i t^i + ...$
$f' = f_1 + 2 f_2 t + 3 f_3 t^2 ... + (i + 1) f_{i+1} t^i + ...$
When equating powers of t, $(i + 1) f_{i+1} = f_i$. $f = f_0 e^t$
- $t g' + g = t g$ and $g(t = 0) = g_0$. When solved by series solution,
$g = \sum_i g_i t^i$, $tg = \sum_i g_{i-1} t^i$, and $tg' = \sum_i i g_i t^i$.
When equating powers of t, $i g_i + g_i = g_{i-1}$.
Solving analytically for comparison, $(tg)' = (tg)$. $g = g_0 (e^t - 1)/t$.
- $a t h'' + b h' + h = 0$.

Calculation of Error Limit: Rescaling.

1.(Theorem 1)If $p = \prod_k u_k(t) = \sum_i p_i t^i$, $q = \prod_l v_l(t) = \sum_i q_i t^i$, and $k_{max} \le l_{max}$ and $u_k = \sum_j u_{kj} t^j$ and $v_l = \sum_l v_{lj} t^j$ and $\forall k$, $|u_{kl}| \le v_{kl}$, and $v_{k0} = 1$, then by induction, $p_{ik} \le q_{ik}$.

2. Suppose $\exists$ a system of N differential equations $\mathbf{x}' = \mathbf{f}(\mathbf{x})$.
Then rescale $\mathbf{x}$ to put initial conditions between 1 and -1:
If $x_i(0) = a_i$, then scaling $c_i \equiv a_i$ if $|a_i| > 1$, else $c_i \equiv 1$.
If $x_i \equiv c_i y_i$, then $y_i(0) = b_i$ with $|b_i| \le 1$ and $\mathbf{y}' = \mathbf{f}(\mathbf{x})$.

3. For the power series method, $\mathbf{y}$ is written as

$$y_i' = \sum_{j_1=0}^{N} \sum_{j_2=0}^{N} \cdots \sum_{j_n=0}^{N} B_{i,j_1,j_2,\ldots,j_n} y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n} \tag{10}$$

Calculation of Error Limit: Warne Transformation.

4. For Warne transformation $\mathrm{B}_N[\mathrm{y}] \equiv \mathrm{f} : \Re^n \to \Re^n$,

$$||\mathrm{B}_N|| \equiv \text{sup}\{||\mathrm{B}_N[\mathrm{u}]|| : \mathrm{u} \in \Re^n \text{where} ||\mathrm{u}|| = 1\} \tag{11}$$

$$||\mathrm{B}_N|| = \max_{1 \le i \le n} \left\{ \sum_{j_1=0}^{N} \sum_{j_2=0}^{N} \cdots \sum_{j_n=0}^{N} |B_{i,j_1,j_2,\ldots,j_n}| \right\} \tag{12}$$

5. Then on $||y - b|| < L$, $||\mathrm{f}|| \le ||\mathrm{B}_N||(L+1)^N$ and $||\partial f/\partial y_k|| \le m||\mathrm{B}_N||(L+1)^m$ where m is the largest degree of any nonzero term in f.
Then $\exists$ a unique y in Eq. (2) for $t \in (-L/(||\mathrm{B}_N||(L+1)^m), L/((\mathrm{B}_N||(L+1)^m))$.
For $m > 1$, the interval is a maximum for $L = m - 1$, yielding

$$t \in \left(\frac{-1}{M} \left(\frac{m-1}{m} \right)^m, \frac{1}{M} \left(\frac{m-1}{m} \right)^m \right) \tag{13}$$

where $M = (m-1)||\mathrm{B}_N||$.

Calculation of Error Limit: Majorizing Limit z on y.

6. Since $\mathrm{y}' = \mathrm{B}_N[\mathrm{y}]$, and $\mathrm{y}(0) = \mathrm{b}$, and $\mathrm{y} = \sum_j \mathrm{y}_j t^j$, $\mathrm{f} = \sum_j \mathrm{f}_j t^j$,
we can iteratively generate terms:
$\mathrm{y}_0 = \mathrm{b}$, $\mathrm{f}_0 = \mathrm{f}(\mathrm{y}_0)$, $\mathrm{y}_{k+1} = \mathrm{f}_k/(k+1)$. So $||\mathrm{y}_{k+1}|| = ||\mathrm{f}||/(k+1)$.
7. Likewise, if $z' = ||\mathrm{B}_N|| z^m$, and $z(0) = 1$, and $z = \sum_j z_j t^j$,
then iteratively, $z_0 = 1$, $z_{k+1} = (z^m)_k ||\mathrm{B}_N||/(k+1)$, yielding

$$z(t) = \begin{cases} e^{||\mathrm{B}_N|| t} & \text{for } m = 1, \\ (1 - Mt)^{-1/(m-1)} & \text{for } m > 1. \end{cases} \tag{14}$$

8. We can use Theorem 1 on the limits of products:
Since $|y_{ij}| \leq ||\mathrm{y}_j|| \leq z_j$, then $|[y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n}]_k| \leq (z^m)_k$.
9. Then $|f_{ik}| \leq \sum_{j_1=0}^N \sum_{j_2=0}^N \cdots \sum_{j_n=0}^N |B_{i,j_1,j_2,\ldots,j_n}| (z^m)_k \leq ||\mathrm{B}_N|| (z^m)_k$.
$||\mathrm{f}_k|| \leq ||\mathrm{B}_N|| (z^m)_k$ and $||\mathrm{y}_{k+1}|| \leq ||\mathrm{B}_N|| (z^m)_k/(k+1) = z_{k+1}$.

Calculation of Error Limit: Transfer of Limit to x.

10. Since $||\mathrm{y}_{k+1}|| \le z_{k+1}$ and

$$z(t) = \begin{cases} e^{||\mathbf{B}_N||t} & \text{for } m = 1, \\ (1 - Mt)^{-1/(m-1)} & \text{for } m > 1. \end{cases}$$

then the limit on y is

$$\left| y_i(t) - \sum_{j=0}^{k} y_{ij} t^j \right| \le \begin{cases} e^{||\mathbf{B}_N||\ |t|} - \sum_{j=0}^{k} z_j |t|^j & \text{for } m = 1, \\ (1 - Mt)^{-1/(m-1)} - \sum_{j=0}^{k} z_j |t|^j & \text{for } m > 1. \end{cases} \tag{15}$$

Since $x_i = c_i y_i(t)$, then the limit on x is

$$\left\| \mathrm{x}(t) - \sum_{j=0}^{k} \mathrm{x}_j t^j \right\| \le \begin{cases} ||c|| \left(e^{||\mathbf{B}_N||\ |t|} - \sum_{j=0}^{k} z_j |t|^j \right) & \text{for } m = 1, \\ ||c|| \left((1 - Mt)^{-1/(m-1)} - \sum_{j=0}^{k} z_j |t|^j \right) & \text{for } m > 1. \end{cases} \tag{16}$$

The coefficients z_j are easily calculated iteratively from $z_0 = 1$:

$$z_{k+1} = \left(\frac{(m-1)k + 1}{k+1} \right) \frac{M}{m-1} z_k \tag{17}$$

Calculation of Error Limit: A Simpler Bound.

Full error limit:

$$\left\|\mathrm{x}(t)-\sum_{j=0}^{k}\mathrm{x}_j t^j\right\| \leq \begin{cases} ||c||\left(e^{||\mathrm{B}_N||\ |t|}-\sum_{j=0}^{k} z_j|t|^j\right) & \text{for } m=1 \\ ||c||\left((1-Mt)^{-1/(m-1)}-\sum_{j=0}^{k} z_j|t|^j\right) & \text{for } m>1 \end{cases}$$

Since $((m-1)k+1)(k+1) \leq m-1$, a simpler, less tight bound:

$$\left\|\mathrm{x}(t)-\sum_{j=0}^{k}\mathrm{x}_j t^j\right\| \leq \frac{||c||\ |Mt|^{k+1}}{1-|Mt|} \text{ for } m>1. \qquad (18)$$

This is an example of a Cauchy-Kovalevsky error limit.

Peter Dirichlet

Leopold Kronecker

Karl Weierstrass

Augustin-Louis Cauchy

Henri Poincaré

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Lorentz Transformations.

$$\begin{pmatrix} 1+\alpha v_x^2 & \alpha v_x v_y & \alpha v_x v_z & \gamma v_x \\ \alpha v_x v_y & 1+\alpha v_y^2 & \alpha v_y v_z & \gamma v_y \\ \alpha v_x v_z & \alpha v_y v_z & 1+\alpha v_z^2 & \gamma v_z \\ \gamma v_x & \gamma v_y & \gamma v_z & \gamma \end{pmatrix} \begin{pmatrix} g_x & 0 & 0 & 0 \\ 0 & g_y & 0 & 0 \\ 0 & 0 & g_z & 0 \\ 0 & 0 & 0 & g_t \end{pmatrix} \gamma_\nu^k =$$

$$\begin{pmatrix} (1+\alpha v_x^2)^2 g_x + v_x^2(\alpha^2 v_y^2 g_y + \alpha^2 v_z^2 g_z + \gamma^2 g_t) & . & . & . \\ v_x v_y((1+\alpha v_x^2)\alpha g_x + (1+\alpha v_y^2)\alpha g_y + \alpha^2 v_z^2 g_z + \gamma^2 g_t) & . & . & . \\ v_x v_z((1+\alpha v_x^2)\alpha g_x + (1+\alpha v_z^2)\alpha g_z + \alpha^2 v_y^2 g_y + \gamma^2 g_t) & . & . & . \\ \gamma v_x((1+\alpha v_x^2) g_x + \alpha v_y^2 g_y + \alpha v_z^2 g_z + \gamma g_t) & . & . & . \end{pmatrix} = \begin{pmatrix} -g^2 + \gamma^2 v_x^2 (g^{-2} - g^2) & . & . & \gamma^2 v_x (g^{-2} - g^2) \\ \gamma^2 v_x v_y (g^{-2} - g^2) & . & . & \gamma^2 v_y (g^{-2} - g^2) \\ \gamma^2 v_x v_z (g^{-2} - g^2) & . & . & \gamma^2 v_z (g^{-2} - g^2) \\ \gamma^2 v_x (g^{-2} - g^2) & . & . & \gamma^2 (g^{-2} - v^2 g^2) \end{pmatrix}$$

$$\begin{pmatrix} 1+\alpha v_x^2 & \alpha v_x v_y & \alpha v_x v_z & \gamma v_x \\ \alpha v_x v_y & 1+\alpha v_y^2 & \alpha v_y v_z & \gamma v_y \\ \alpha v_x v_z & \alpha v_y v_z & 1+\alpha v_z^2 & \gamma v_z \\ \gamma v_x & \gamma v_y & \gamma v_z & \gamma \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & G_x \\ 0 & 0 & 0 & G_y \\ 0 & 0 & 0 & G_z \\ G_x & G_y & G_z & 0 \end{pmatrix} \gamma_\nu^k =$$

$$\begin{pmatrix} 2(1+\alpha v_x^2)\gamma v_x G_x + 2\alpha\gamma v_x^2 v_y G_y + 2\alpha\gamma v_x^2 v_z G_z & . & . & . \\ 2(1+\alpha v_y^2)\gamma v_y G_y + 2\alpha\gamma v_x v_y^2 G_x + 2\alpha\gamma v_y^2 v_z G_z & . & . & . \\ 2(1+\alpha v_z^2)\gamma v_z G_z + 2\alpha\gamma v_x v_z^2 G_x + 2\alpha\gamma v_y v_z^2 G_y & . & . & . \\ (1+\alpha v_x^2 + \gamma v_x^2)\gamma G_x + (\alpha+\gamma)\gamma v_x(v_y G_y + v_z G_z) & . & . & G_t' \end{pmatrix}$$

where $G_t' = 2\gamma^2(v_x G_x + v_y G_y + v_z G_z)$.