# Effective Source Formulations in the Regge-Wheeler Gauge

Jonathan Thompson

Department of Physics
University of Florida

June 21, 2017

# Regge-Wheeler Formalism

In perturbation theory, one expands the physical metric $g_{ab}$ to first-order,

$$g_{ab} = g^0_{ab} + h_{ab}.$$

Both the physical metric and the background metric are solutions to the Einstein Field Equations (EFEs),

$$G_{ab}(g) = 8\pi T_{ab}, \tag{1}$$

and we may expand the field equations in powers of the metric perturbation $h_{ab}$,

$$G_{ab}(g^0 + h) = G_{ab}(g^0) - \frac{1}{2} E_{ab}(h) + O(h^2), \tag{2}$$

with,

$$\begin{aligned} E_{ab}(h) = \nabla^c \nabla_c h_{ab} + \nabla_a \nabla_b h - 2\nabla_{(a} \nabla^c h_{b)c} \\ + 2R_a{}^c{}_b{}^d h_{cd} + g^0_{ab}(\nabla^c \nabla^d h_{cd} - \nabla^c \nabla_c h). \end{aligned}$$

# Regge-Wheeler Formalism

We take advantage of the spherical symmetry of our background spacetime and project all tensor fields and equations onto an orthogonal, pure-spin tensor harmonic basis[1],

$$Y^{\ell m},\ Y_A^{\ell m},\ X_A^{\ell m},\ Y_{AB}^{\ell m},\ X_{AB}^{\ell m},$$

with $Y$ and $X$ splitting the decomposition into even- and odd-parity modes, respectively.

When decomposing the EFEs, we recover 10 PDEs to solve (7 of even-parity and 3 of odd-parity).

Using the Bianchi identites, we may reduce the number of equations to 4 of even-parity and 2 of odd-parity.

[1] K. Martel and E. Poisson, Phys. Rev. D **71**, 104003 (2005).

# Regge-Wheeler Formalism

From the remaining free equations we recover two "master equations"[2]

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V^{\ell m}_{e/o}\right] \Psi^{\ell m}_{e/o} = \mathcal{S}^{\ell m}_{e/o},$$

which govern the evolution of the even- and odd-parity "master functions" $\Psi_{e/o}$, gauge-invariant scalar fields constructed from components of the metric perturbation,

$$\Psi^{\ell m}_o = \frac{r}{\lambda}\left[\partial_r h^{\ell m}_t - \partial_t h^{\ell m}_r - \frac{2}{r} h^{\ell m}_t\right],$$

$$\Psi^{\ell m}_e = \frac{2r}{\ell(\ell+1)}\left[K^{\ell m} + \frac{1}{\Lambda}(f^2 h^{\ell m}_{rr} - rf\partial_r K^{\ell m})\right].$$

[2] S. Hopper and C. Evans, Phys. Rev. D **82**, 084010 (2010).

# Gauge Invariance in RW Formalism

The master functions are true gauge-invariant quantities (per $(\ell, m)$ mode) for any gauge vector $\xi^a$.

The (rather simple) way to see this fact is to calculate the changes to the metric perturbation under such a gauge transformation, e.g., given a gauge vector $\xi_A^{\ell m} = \xi_{\text{odd}}^{\ell m} X_A^{\ell m}$,

$$h_r^{\ell m} \to h_r^{\ell m} + \left( \frac{\partial}{\partial r} - \frac{2}{r} \right) \xi_{\text{odd}}^{\ell m},$$

$$h_t^{\ell m} \to h_t^{\ell m} + \frac{\partial}{\partial t} \xi_{\text{odd}}^{\ell m},$$

and then expand the metric components in $\Psi_{e/o}^{\ell m}$.

In fact, several gauge-invariant scalar fields have been constructed throughout the years (see Moncrief, Sachs, Gerlach and Sengupta, etc.).

# Gauge-Invariant Local Singular Information

The idea is to construct gauge-invariant punctures to be used in regularizing the master functions.

- Begin with a puncture field (for convenience, choose Lorenz gauge as starting gauge)[3].
- Decompose in harmonic basis.
- Construct singular master functions through gauge-invariant combinations of these metric components.
- Use these singular master functions to construct an effective source.

[3] C. O. Lousto and H. Nakano, Class. Quant. Grav. **25**, 145018 (2008).

# Puncture Field

A few puncture field formulations have been introduced, but let's use the Lorenz gauge puncture from Wardell and Warburton[4].

[4] B. Wardell and N. Warburton, Phys. Rev. D **92**, 084019 (2015).

# Puncture Field

A few puncture field formulations have been introduced, but let's use the Lorenz gauge puncture from Wardell and Warburton[4].



$$\mathcal{M}^{(7)}{}_{(j)}\bar{h}^{(j)} = -\frac{f}{2r^2}(\bar{h}^{(7)} + \lambda \bar{h}^{(5)}), \tag{B7}$$

$$\mathcal{M}^{(8)}{}_{(j)}\bar{h}^{(j)} = \frac{1}{4}f'[i\omega(\bar{h}^{(9)} - \bar{h}^{(8)}) + \bar{h}^{(8)}_{,r_*} - \bar{h}^{(9)}_{,r_*}] - \frac{ff'}{4r}(3\bar{h}^{(8)} + 2\bar{h}^{(9)} - \bar{h}^{(10)}), \tag{B8}$$

$$\mathcal{M}^{(9)}{}_{(j)}\bar{h}^{(j)} = \frac{f}{r^2}\left(1 - \frac{9M}{2r}\right)\bar{h}^{(9)} - \frac{f}{2r^2}\left(1 - \frac{3M}{r}\right)\bar{h}^{(10)}, \tag{B9}$$

$$\mathcal{M}^{(10)}{}_{(j)}\bar{h}^{(j)} = -\frac{f}{2r^2}(\bar{h}^{(10)} + \lambda \bar{h}^{(9)}), \tag{B10}$$

where $\lambda = (\ell - 1)(\ell + 2)$.

The sources to the field equation (2.10) take the form

$$\mathcal{J}(r) = -\frac{16\pi E}{f_0^2}\alpha^{(i)}\delta(r - r_0)\begin{cases} Y^{\ell m *}(\pi/2, \Omega_\varphi t), & i = 1, \ldots, 7, \\ Y^{\ell m *}_{\theta}(\pi/2, \Omega_\varphi t), & i = 8, 9, 10, \end{cases} \tag{B11}$$

where

$$\alpha^{(1)} = f_0^2/r_0, \quad \alpha^{(2)} = 0, \quad \alpha^{(3)} = f_0/r_0, \quad \alpha^{(4)} = 2if_0 m\Omega_\varphi, \quad \alpha^{(5)} = 0,$$
$$\alpha^{(6)} = r_0\Omega_\varphi^2, \quad \alpha^{(7)} = r_0\Omega_\varphi^2[\ell(\ell+1) - 2m^2], \quad \alpha^{(8)} = 2f_0\Omega_\varphi, \quad \alpha^{(9)} = 0, \quad \alpha^{(10)} = 2imr_0\Omega_\varphi^2. \tag{B12}$$

### APPENDIX C: PUNCTURE FUNCTIONS FOR CIRCULAR ORBITS IN LORENZ GAUGE

In this appendix we give our explicit expressions for the Lorenz-gauge puncture fields $\bar{h}^{(i)\mathrm{P}}_{\ell m}$ for the case of a circular geodesic orbit in Schwarzschild spacetime. These punctures contain all pieces of the Detweiler-Whiting singular field necessary to compute the regularized components of the metric and its first derivatives. Written as tensor-harmonic modes in the $(\theta, \varphi)$ coordinate system, the punctures are given by

$$\bar{h}^{(1)\mathrm{P}}_{\ell m} = rD^{\ell}_{m,0}\sqrt{\frac{4\pi}{2\ell+1}}\left[\frac{8(r_0-2M)^{3/2}\mathcal{K}}{\pi r_0^2(r_0-3M)^{1/2}} - (2\ell+1)|\Delta r|\frac{2(r_0-2M)}{r_0^{5/2}(r_0-3M)^{1/2}} + \Delta r\frac{4(r_0-2M)^{1/2}[(r_0-2M)\mathcal{E} - 2(r_0-4M)\mathcal{K}]}{\pi r_0^3(r_0-3M)^{1/2}}\right], \tag{C1}$$

$$\bar{h}^{(2)\mathrm{P}}_{\ell m} = rf(r)[D^{\ell}_{m,1} - D^{\ell}_{m,-1}]\sqrt{\frac{4\pi}{2\ell+1}}\sqrt{\frac{1}{\ell(\ell+1)}}\left[\frac{64(r_0-2M)^{1/2}[(r_0-2M)\mathcal{E} - (r_0-3M)\mathcal{K}]}{\pi r_0^{3/2}M^{1/2}(r_0-3M)^{1/2}}\Lambda_1\right], \tag{C2}$$

$$\begin{aligned}\bar{h}^{(3)\mathrm{P}}_{\ell m} = \frac{r}{f(r)}D^{\ell}_{m,0}\sqrt{\frac{4\pi}{2\ell+1}}&\left[\frac{8(r_0-2M)^{3/2}\mathcal{K}}{\pi r_0^2(r_0-3M)^{1/2}} - (2\ell+1)|\Delta r|\frac{2(r_0-2M)}{r_0^{5/2}(r_0-3M)^{1/2}}\right. \\ &\left. + \Delta r\frac{4(r_0-2M)^{1/2}[(r_0-2M)\mathcal{E} - 2(r_0-4M)\mathcal{K}]}{\pi r_0^3(r_0-3M)^{1/2}}\right],\end{aligned} \tag{C3}$$

$$\begin{aligned}\bar{h}^{(4)\mathrm{P}}_{\ell m} = \ell(\ell+1)[D^{\ell}_{m,1} - D^{\ell}_{m,-1}]\sqrt{\frac{4\pi}{2\ell+1}}\sqrt{\frac{1}{\ell(\ell+1)}}&\left[-\frac{64(r_0-2M)^{3/2}(\mathcal{E}-\mathcal{K})}{\pi M^{1/2}r_0^{1/2}(r_0-3M)^{1/2}}\Lambda_1\right. \\ &+ \frac{48(r_0-2M)^{1/2}[2(r_0-2M)\mathcal{E} - (2r_0-5M)\mathcal{K}]}{\pi M^{1/2}r_0^{1/2}(r_0-3M)^{1/2}(2\ell-1)(2\ell+3)} - (2\ell+1)|\Delta r|\frac{2M^{1/2}}{r_0(r_0-3M)^{1/2}} \\ &+ \Delta r\left(\frac{32(r_0-2M)^{1/2}[(r_0-4M)\mathcal{E} - (r_0-5M)\mathcal{K}]}{\pi M^{1/2}r_0^{3/2}(r_0-3M)^{1/2}}\Lambda_1\right. \\ &\left.\left. - \frac{24[(r_0-2M)(2r_0-9M)\mathcal{E} - 2(11M^2 - 7Mr_0 + r_0^2)\mathcal{K}]}{\pi M^{1/2}r_0^{3/2}(r_0-3M)^{1/2}(r_0-2M)^{1/2}(2\ell-1)(2\ell+3)}\right)\right].\end{aligned} \tag{C4}$$



$$\begin{aligned}\bar{h}^{(5)\mathrm{P}}_{\ell m} = \ell(\ell+1)f(r)[D^{\ell}_{m,2} + D^{\ell}_{m,-2}]&\sqrt{\frac{4\pi}{2\ell+1}}\sqrt{\frac{1}{(\ell-1)\ell(\ell+1)(\ell+2)}} \\ &\times\left[\frac{256[(4r_0-11M)(r_0-2M)\mathcal{E} - (4r_0-9M)(r_0-3M)\mathcal{K})}{\pi M(r_0-2M)^{1/2}(r_0-3M)^{1/2}}\Lambda_2\right. \\ &\left. - \frac{(\ell-1)(\ell+2)}{(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)}\frac{640[(8r_0-23M)(r_0-2M)\mathcal{E} - (8r_0-19M)(r_0-3M)\mathcal{K})}{\pi M(r_0-2M)^{1/2}(r_0-3M)^{1/2}}\right] \\ &+ f(r)D^{\ell}_{m,0}\sqrt{\frac{4\pi}{2\ell+1}}\left[\frac{64[-(r_0-2M)\mathcal{E} + (r_0-3M)\mathcal{K})}{\pi(r_0-2M)^{1/2}(r_0-3M)^{1/2}}\Lambda_1\right],\end{aligned} \tag{C5}$$

$$\begin{aligned}\bar{h}^{(6)\mathrm{P}}_{\ell m} = \frac{1}{r}D^{\ell}_{m,0}\sqrt{\frac{4\pi}{2\ell+1}}&\left[\frac{8Mr_0\mathcal{K}}{\pi(r_0-2M)^{1/2}(r_0-3M)^{1/2}} - (2\ell+1)|\Delta r|\frac{2Mr_0^{1/2}}{(r_0-2M)(r_0-3M)^{1/2}}\right. \\ &\left. + \Delta r\frac{4M(\mathcal{E}+2\mathcal{K})}{\pi(r_0-2M)^{1/2}(r_0-3M)^{1/2}}\right],\end{aligned} \tag{C6}$$

$$\begin{aligned}\bar{h}^{(7)\mathrm{P}}_{\ell m} = \frac{(\ell-1)\ell(\ell+1)(\ell+2)}{r}&[D^{\ell}_{m,-2} + D^{\ell}_{m,2}]\sqrt{\frac{4\pi}{2\ell+1}}\sqrt{\frac{1}{(\ell-1)\ell(\ell+1)(\ell+2)}} \\ &\times\left[-\frac{128r_0[4(r_0-2M)(2r_0-5M)\mathcal{E} - (8r_0^2 - 40Mr_0 + 51M^2)\mathcal{K}]}{3\pi M(r_0-2M)^{1/2}(r_0-3M)^{1/2}}\Lambda_2\right. \\ &+ \frac{1}{(2\ell-1)(2\ell+3)}\frac{160r_0[8(r_0-2M)(2r_0-5M)\mathcal{E} - (4r_0-9M)(4r_0-11M)\mathcal{K}]}{3\pi M(r_0-2M)^{1/2}(r_0-3M)^{1/2}} \\ &- (2\ell+1)|\Delta r|\frac{Mr_0^{1/2}}{(r_0-2M)(r_0-3M)^{1/2}} + \Delta r\left(\frac{64[(8r_0^2 - 48Mr_0 + 67M^2)\mathcal{E} - 2(4r_0^2 - 26Mr_0 + 39M^2)\mathcal{K}]}{3\pi M(r_0-2M)^{1/2}(r_0-3M)^{1/2}}\Lambda_2\right. \\ &\left.\left. - \frac{1}{(2\ell-1)(2\ell+3)}\frac{80[(16r_0^2 - 96Mr_0 + 131M^2)\mathcal{E} - 2(8r_0^2 - 52Mr_0 + 81M^2)\mathcal{K}]}{3\pi M(r_0-2M)^{1/2}(r_0-3M)^{1/2}}\right)\right],\end{aligned} \tag{C7}$$

$$\begin{aligned}\bar{h}^{(8)\mathrm{P}}_{\ell m} = i\ell(\ell+1)[D^{\ell}_{m,1} + D^{\ell}_{m,-1}]\sqrt{\frac{4\pi}{2\ell+1}}\sqrt{\frac{1}{\ell(\ell+1)}}&\left[-\frac{64(r_0-2M)^{1/2}((r_0-2M)\mathcal{E} - (r_0-3M)\mathcal{K})}{\pi M^{1/2}r_0^{1/2}(r_0-3M)^{1/2}}\Lambda_1\right. \\ &+ \frac{1}{(2\ell-1)(2\ell+3)}\frac{48(r_0-2M)^{1/2}[2(r_0-2M)\mathcal{E} - (2r_0-5M)\mathcal{K}]}{\pi M^{1/2}r_0^{1/2}(r_0-3M)^{1/2}} + (2\ell+1)|\Delta r|\frac{2M^{1/2}}{r_0(r_0-3M)^{1/2}} \\ &+ \Delta r\left(\frac{32[(r_0-2M)(r_0-5M)\mathcal{E} - (r_0-3M)(r_0-4M)\mathcal{K}]}{\pi M^{1/2}r_0^{3/2}(r_0-2M)^{1/2}(r_0-3M)^{1/2}}\Lambda_1\right. \\ &\left.\left. - \frac{1}{(2\ell-1)(2\ell+3)}\frac{24[(r_0-2M)(2r_0-9M)\mathcal{E} - 2(11M^2 - 7Mr_0 + r_0^2)\mathcal{K}]}{\pi M^{1/2}r_0^{3/2}(r_0-3M)^{1/2}(r_0-2M)^{1/2}}\right)\right],\end{aligned} \tag{C8}$$

$$\begin{aligned}\bar{h}^{(9)\mathrm{P}}_{\ell m} = i\ell(\ell+1)f(r)[D^{\ell}_{m,2} - D^{\ell}_{m,-2}]&\sqrt{\frac{4\pi}{2\ell+1}}\sqrt{\frac{1}{(\ell-1)\ell(\ell+1)(\ell+2)}} \\ &\times\left[\frac{512(r_0-3M)^{1/2}[2(r_0-2M)\mathcal{E} - (2r_0-5M)\mathcal{K})}{\pi M(r_0-2M)^{1/2}}\Lambda_2\right. \\ &\left. - \frac{(\ell-1)(\ell+2)}{(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)}\frac{640[(8r_0-23M)(r_0-2M)\mathcal{E} - (8r_0-19M)(r_0-3M)\mathcal{K})}{\pi M(r_0-2M)^{1/2}(r_0-3M)^{1/2}}\right].\end{aligned} \tag{C9}$$

[4] B. Wardell and N. Warburton, Phys. Rev. D **92**, 084019 (2015).

# Puncture Field

The puncture components are given projected in the (1)-(10) basis[5], so we need to first translate these into the pure-spin basis of Martel and Poisson (almost trivial).

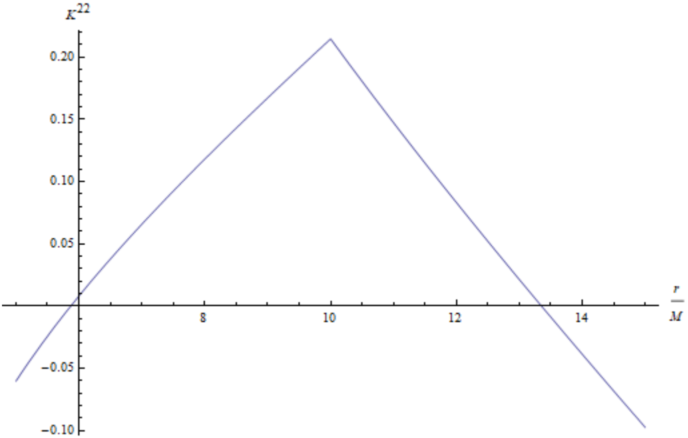
- Each projection of the puncture field is finite for a given $(\ell, m)$
- The rich structure is encapsulated in the $|\Delta r|$ pieces

For instance, we find that the Martel-Poisson $K^{\ell m}$ term scales as $\bar{h}^{(3)}$,

$$K^{\ell m}_{\text{not diff.}} \sim -\frac{r^2}{r-2M}(2\ell+1)|\Delta r|.$$

[5] L. Barack and C.O. Lousto, Phys. Rev. D **71**, 104003 (2005).

# Puncture Field



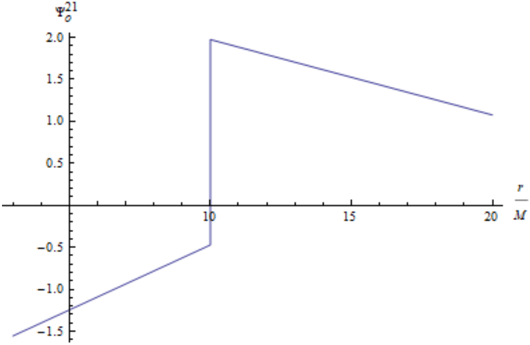
# Singular Master Functions

Now assume that locally, the master functions separate into singular and regular pieces,

$$\Psi^{\ell m}_{o/e} = \Psi^{\mathrm{R},\ell m}_{o/e} + \Psi^{\mathrm{P},\ell m}_{o/e},$$

and we construct,

$$\Psi^{\mathrm{P},\ell m}_{o} = \frac{r}{\lambda}\left[\partial_r h^{\mathrm{P}}_t - \partial_t h^{\mathrm{P}}_r - \frac{2}{r} h^{\mathrm{P}}_t\right].$$



# Effective Source Construction

Finally, operate on the singular master functions with the appropriate wave operators to generate the effective source for the residual master function,

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}-V_{e/o}^{\ell m}\right]\Psi_{e/o}^{\mathrm{R},\,\ell m}=\mathcal{S}_{e/o}^{\ell m}-\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}-V_{e/o}^{\ell m}\right]\Psi_{e/o}^{\mathrm{P},\,\ell m}$$
$$\equiv \mathcal{S}_{\mathrm{eff}}^{\ell m}$$

- The puncture field creates a source in the region of the particle off of the worldline.
- Radial derivatives hitting $|\Delta r|$ generate $\delta(r-R)$ and $\delta'(r-R)$ terms which contribute to the jump conditions across the orbit.

# Effective Source Construction

Write $|\Delta r|$ in a distributional sense,

$$|\Delta r| = \Delta r \; [2\,\Theta(\Delta r) - 1]\,,$$

- $\partial_r |\Delta r| = 2\,\Theta(\Delta r) - 1$,
- $\partial_r^2 |\Delta r| = 2\,\delta(\Delta r)$,
- $\partial_r^3 |\Delta r| = 2\,\delta'(\Delta r)$.

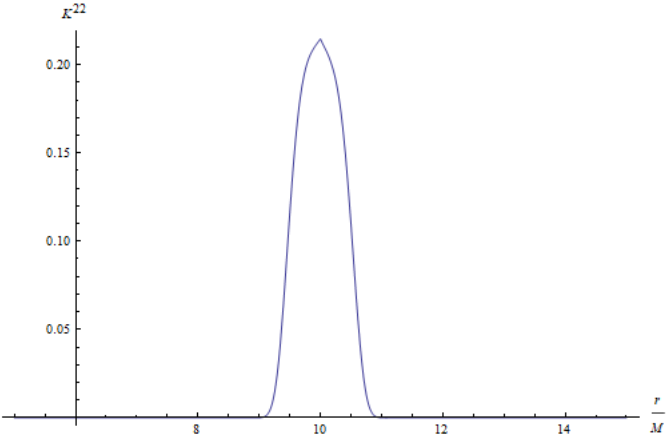
# Window Function

The puncture field is only valid locally (inside the radius of convergence for the series expansion), so we enforce this locality with a window function.

- First introduced by Vega and Detweiler[5] for the effective source problem, $W = \exp[-(r-R)^N/\sigma^N]$.
- Wardell and Warburton[6] show that both the window function and worldtube methods are equivalent (choose a step function for W).
- Use the window function introduced by Wardell and Warburton, $W = \exp[-8M^{-4}(r-R)^4]$.

[5] I. Vega and S. Detweiler, Phys. Rev. D **77**, 084008 (2008).
[6] N. Warburton and B. Wardell, Phys. Rev. D **89**, 044046 (2014).

# Window Function



# Results

# Results

Results will go here.

# Results

Results will go here.

(Eventually)

# Summary

- Take advantage of the gauge invariance of RWZ to calculate effective source from a puncture in any gauge.
- Higher-order punctures are necessary to regularize the $\delta'(r - R)$ terms.
- Gauge information returns during metric reconstruction.
- Get excited about Regge-Wheeler gauge (classes)!