Scalar self-force and QNM excitation for highly eccentric orbits in Kerr spacetime

Jonathan Thornburg

in collaboration with

Barry Wardell

Wardell, Vega, Thornburg, & Diener, PRD 85,104044 = arXiv:1112.6355

Thornburg & Wardell, PRD 95,084043 = arXiv:1610.09319

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- **Kerr**
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- [now] **scalar field**, **develop techniques for [future] gravitational field**

Overall Plan of the Computation

Effective-Source (puncture-function) regularization

- allows Kerr, arbitrary orbits
- less numerical cancellation than mode-sum/extended homogeneous solns

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- worldtube scheme
- worldtube moves in (r, θ) to follow the particle around the orbit
- (fixed) mesh refinement; finer grids follow the worldtube/particle
- hyperboloidal slices (reach horizon and $\mathcal{J}^+$) [Zenginoğlu, J. Comp. Phys. 230,2286 = arXiv:1008.3809]

Effective source (puncture function) regularization

Assume a δ-function particle with scalar charge q.

The particle's physical (retarded) scalar field φ satisfies $\Box\varphi = q\delta\big(x - x_{\text{particle}}(t)\big)$.
φ is singular at the particle.

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Instead we choose $\varphi_{\text{puncture}}$ so that it agrees with $\varphi_{\text{singular}}$ in the first n terms of a Laurent series in $|x - x_{\text{particle}}|$. Then $\varphi_{\text{residual}} := \varphi - \varphi_{\text{puncture}}$ is finite and "differentiable enough" at the particle, and

$$\Box\varphi_{\text{residual}} = \left.\begin{cases} 0 & \text{at the particle} \\ -\Box\varphi_{\text{puncture}} & \text{elsewhere} \end{cases}\right\} := S_{\text{effective}} \qquad \begin{bmatrix}\text{solve this} \\ \text{for } \varphi_{\text{residual}}\end{bmatrix}$$

The self-force is given by $F_a = (\nabla_a \varphi_{\text{residual}})\big|_{\text{particle}}$.
Note this is **exact** even though $\varphi_{\text{puncture}} \neq \varphi_{\text{singular}}$.

Puncture field and effective source

The choice of the puncture order n is a tradeoff:
Higher $n \Rightarrow \varphi_{\text{residual}}$ is smoother at the particle (good),
but $\varphi_{\text{puncture}}$ and $S_{\text{effective}}$ are more complicated (expensive) to compute.

We choose $n = 4 \Rightarrow \varphi_{\text{residual}}$ is C^2 at the particle.

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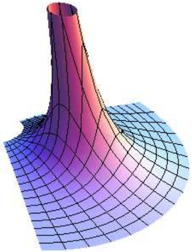
The actual computation of $\varphi_{\text{puncture}}$ and $S_{\text{effective}}$ uses a (lengthly) series expansion of the Synge world function in Mathematica, then machine-generated C code. [Wardell, Vega, Thornburg, & Diener, PRD 85,104044 = arXiv:1112.6355]
Computing $S_{\text{effective}}$ at a single event requires $\sim \frac{1}{2} \times 10^6$ arithmetic operations.

sample $\varphi_{\text{puncture}}$



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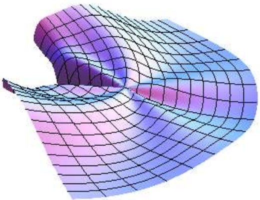

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Solution:
introduce finite worldtube containing the particle worldline

- define "numerical field" $\varphi_{\text{numerical}} = \begin{cases} \varphi_{\text{residual}} & \text{inside the worldtube} \\ \varphi & \text{outside the worldtube} \end{cases}$
- compute $\varphi_{\text{numerical}}$ by numerically solving

$$\Box \varphi_{\text{numerical}} = \begin{cases} S_{\text{effective}} & \text{inside the worldtube} \\ 0 & \text{outside the worldtube} \end{cases}$$

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- $\varphi_{\text{numerical}}$ has a $\pm\varphi_{\text{puncture}}$ jump discontinuity across worldtube boundary
  $\Rightarrow$ finite difference operators that cross the worldtube boundary
  must compensate for the jump discontinuity

m-mode decomposition

Instead of numerically solving $\Box \varphi_{\text{numerical}} = \begin{cases} S_{\text{effective}} & \text{inside the worldtube} \\ 0 & \text{outside the worldtube} \end{cases}$

in 3+1 dimensions, we Fourier-decompose into $e^{im\phi}$ modes and solve for each Fourier mode in 2+1 dimensions via

$$\Box_m \varphi_{\text{numerical},m} = \begin{cases} S_{\text{effective},m} & \text{inside the worldtube} \\ 0 & \text{outside the worldtube} \end{cases}$$

[numerically solve this for each m in 2+1D]

The self-force is given (exactly!) by $F_a = \sum_{m=0}^{\infty} (\nabla_a \varphi_{\text{numerical},m})\big|_{\text{particle}}$

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Comparison (vs. direct solution in 3+1 dimensions):

- ✓ can use different numerical parameters for different m
- ✓ each individual m's evolution is smaller ⇒ test/debug code on laptop
- ✓ "free" parallelization (run different m's evolutions in parallel)

- in practice compute $m \leq 20$ numerically, estimate $\sum_{m=21}^{\infty}$ via large-m tail series fitted to $m \in [12, 20]$ (separate fit & series at each point around the orbit)

Moving the worldtube

We actually do m-mode decomposition *before* introducing worldtube
$\Rightarrow$ worldtube "lives" in (t, r, θ) space, not full spacetime

The worldtube must contain the particle in (r, θ).
But for a non-circular orbit, the particle moves in (r, θ) during the orbit.

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Large eccentricity (say $e \gtrsim 0.3$):

- must move the worldtube in (r, θ) to follow the particle around the orbit
- recall that our numerically-evolved field is

$$\varphi_{\text{numerical}} := \begin{cases} \varphi - \varphi_{\text{puncture}} & \text{inside the worldtube} \\ \varphi & \text{outside the worldtube} \end{cases}$$

  this means then if we move the worldtube, we must adjust the evolved $\varphi_{\text{numerical}}$: add $\pm\varphi_{\text{puncture}}$ at spatial points which change from being inside the worldtube to being outside, or vice versa

Code Validation

Comparison with frequency-domain mode-sum results kindly provided by Niels Warburton

[Warburton & Barack, PRD 83,124038 = arXiv:1103.0287]

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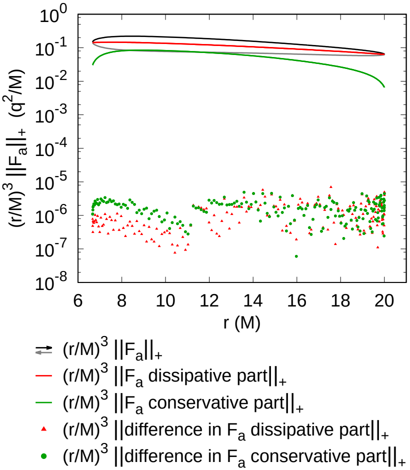
Typical example:
$(a, p, e) = (0.9, 10M, 0.5)$
$\Rightarrow$ results agree to $\sim 10^{-5}$ relative error

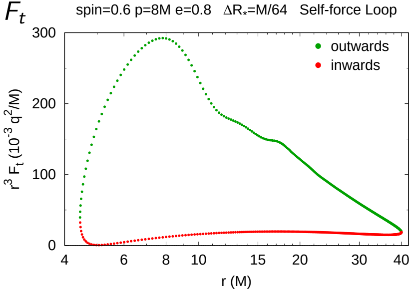
We have also compared a variety of other configurations, with fairly similar results



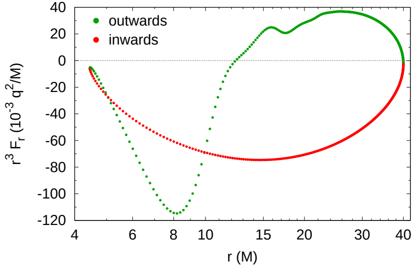
$e = 0.8$ orbit

$(a, p, e) = (0.6, 8M, 0.8)$

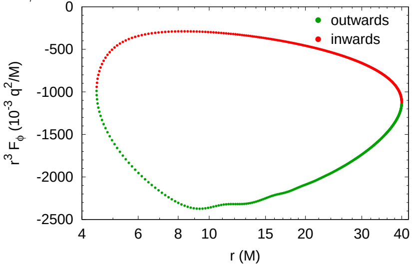
F_t



F_r

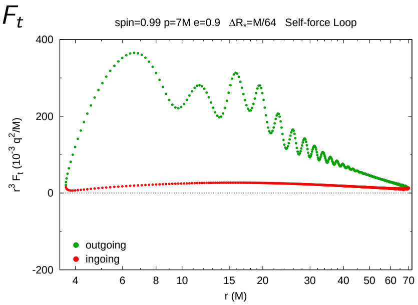


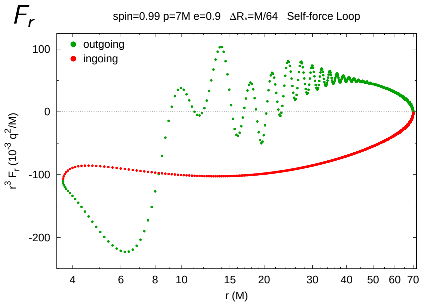
F_ϕ

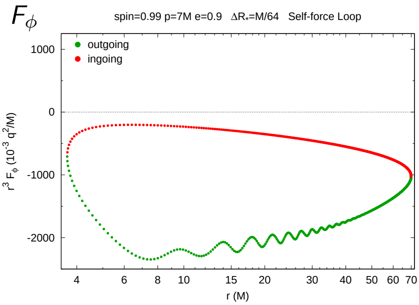


Wiggles!

Higher-eccentricity orbit:
$(a, p, e) = (0.99, 7M, 0.9)$







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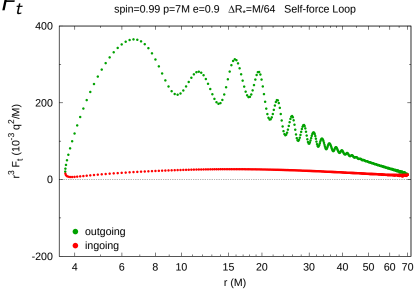
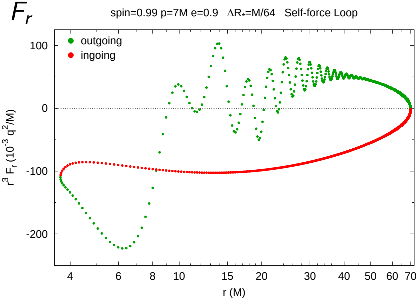
Key property:

- wiggles on **outgoing** leg of orbit
- wiggles **not** seen on **ingoing** leg

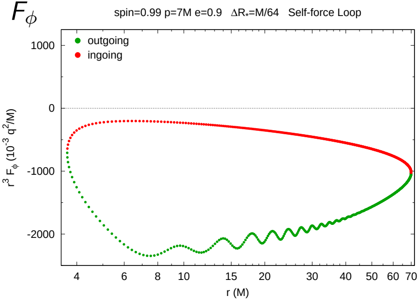
F_t



F_r



F_ϕ



Wiggles as Kerr Quasinormal Modes: Mode Fit

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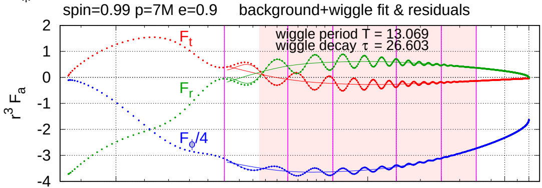

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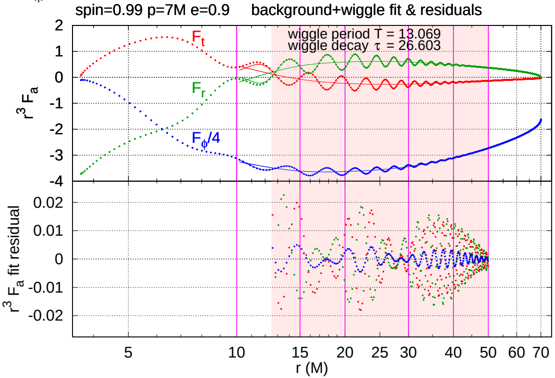

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Now compare wiggle-fit complex frequency $\omega := 2\pi/T - i/\tau$
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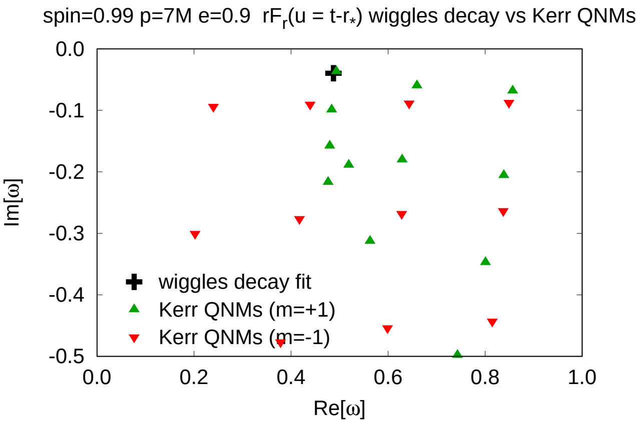
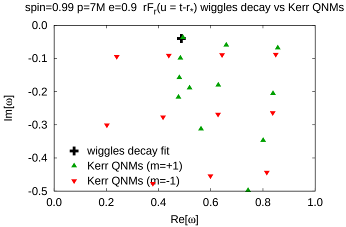


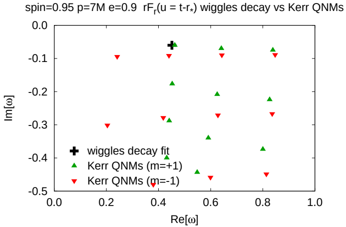
Wiggles as Kerr Quasinormal Modes: Varying BH Spin

Repeat wiggle-fit procedure for other Kerr spins (0.99, 0.95, 0.9, and 0.8)
$\Rightarrow$ Nice agreement with least-damped corotating QNM for all BH spins!

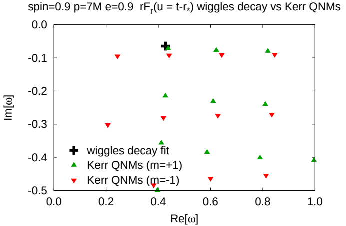
$a = 0.99$



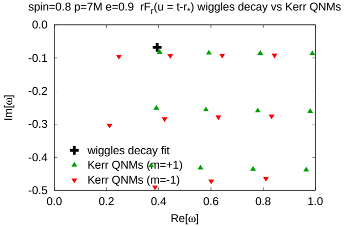
$a = 0.95$



$a = 0.9$



$a = 0.8$



For what orbits do wiggles occur?

Wiggles are quite generic: they occur whenever the configuration combines

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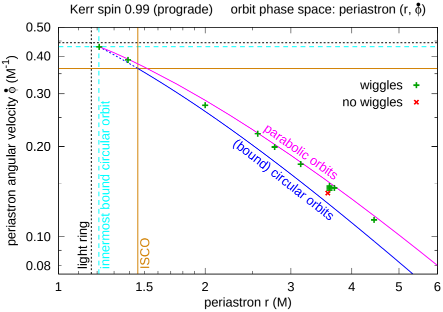

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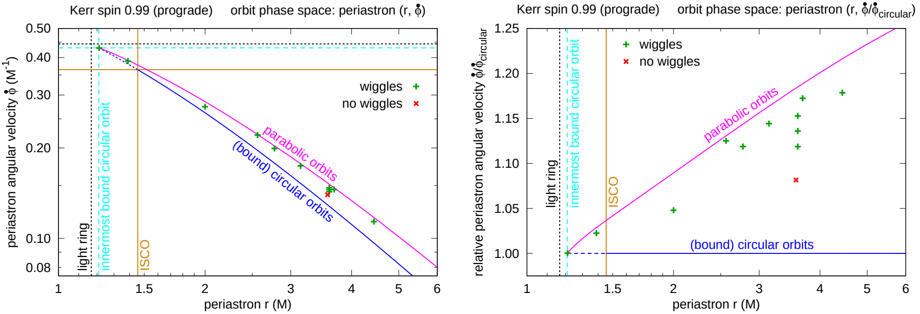

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- gravitation: Maarten van de Meent finds wiggles in Ψ_4 at $\mathcal{J}^+$
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  details (QNMs, phase space, gravitation): paper coming soon