First order gravitational self-force on generic bound orbits in Kerr spacetime

Maarten van de Meent

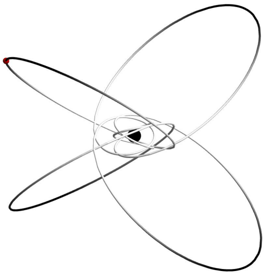
University of Southampton

Capra 20, Chapel Hill, 20 June 2016

Outline

Introduction

The trouble with Kerr



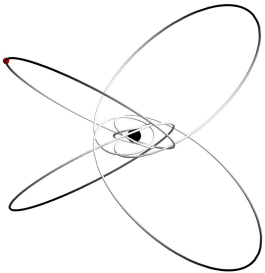
In short:

Kerr space time is not spherically symmetric!

Consequences:

- All equations are more complicated (by order of magnitude).
- Generic orbits are not planar (biperiodic, resonances).
- Linearized Einstein equation does not separate over spherical harmonics.

The trouble with Kerr



In short:

Kerr space time is not spherically symmetric!

Consequences:

- All equations are more complicated (by order of magnitude).
- Generic orbits are not planar (biperiodic, resonances).
- Linearized Einstein equation does not separate over spherical harmonics.

Time domain approaches

Strategy

- Use axisymmetry to decouple "m-mode".
- Solve 2+1D PDEs numerically.
- Use "effective source" regularization scheme

Scalar self-force (Thornburg)

Eccentric equatorial orbits.

Gravitational self-force (Dolan & Barack)

Circular equatorial orbits.
(problems with gauge instabilities)

Frequency domain (scalar field)

Klein-Gordon equation

The Klein-Gordon equation for a massless scalar field in Kerr spacetime can be separated into ODEs in the frequency domain.

Scalar self-force (Warburton&Barack)

Calculation of the scalar self-force in Kerr spacetime for a particle on eccentric equatorial and inclined circular orbits has been implemented by [Warburton& Barack, 2010-2014].

Generic orbits: See talk by Z. Nasipak this afternoon.

Frequency domain (scalar field)

Klein-Gordon equation

The Klein-Gordon equation for a massless scalar field in Kerr spacetime can be separated into ODEs in the frequency domain.

Scalar self-force (Warburton&Barack)

Calculation of the scalar self-force in Kerr spacetime for a particle on eccentric equatorial and inclined circular orbits has been implemented by [Warburton& Barack, 2010-2014].

Generic orbits: See talk by Z. Nasipak this afternoon.

Teukolsky

Equations of motion for the (linear) Weyl curvature scalars ψ_0 and ψ_4 on Type D backgrounds decouple from the other curvature scalars, and can be solve using separation of variables in the frequency domain.

Gravitational Flux

ψ_0 and ψ_4 contain sufficient information to determine the flux of GWs to infinity and into the BH.

Numerical calculation for completely generic bound orbits implemented by [Drasco& Hughes, 2006].

Gravitational Self-force?

In fact ψ_0 and ψ_4 contain most information about a metric perturbation.

Question:
Can the GSF be calculated from ψ_0 or ψ_4?

Frequency domain (gravity)

Teukolsky

Equations of motion for the (linear) Weyl curvature scalars ψ_0 and ψ_4 on Type D backgrounds decouple from the other curvature scalars, and can be solve using separation of variables in the frequency domain.

Gravitational Flux

ψ_0 and ψ_4 contain sufficient information to determine the flux of GWs to infinity and into the BH.

Numerical calculation for completely generic bound orbits implemented by [Drasco& Hughes, 2006].

Gravitational Self-force?

In fact ψ_0 and ψ_4 contain most information about a metric perturbation.

Question:
Can the GSF be calculated from ψ_0 or ψ_4?

Frequency domain (gravity)

Teukolsky

Equations of motion for the (linear) Weyl curvature scalars ψ_0 and ψ_4 on Type D backgrounds decouple from the other curvature scalars, and can be solve using separation of variables in the frequency domain.

Gravitational Flux

ψ_0 and ψ_4 contain sufficient information to determine the flux of GWs to infinity and into the BH.

Numerical calculation for completely generic bound orbits implemented by [Drasco& Hughes, 2006].

Gravitational Self-force?

In fact ψ_0 and ψ_4 contain most information about a metric perturbation.

Question:
Can the GSF be calculated from ψ_0 or ψ_4?

(Incomplete) History

Work	Details
[Barack & Ori, 2001]	Note difficulty due to irregularity of radiation gauge.
[Keidl, Friedman & Wiseman, 2007]	Metric for static particle flat space/Schwarzschild
[Keidl, Shah, Friedman, Kim & Price, 2010& 2011]	GSF and redshift for circular orbits in Schwarzschild
[Shah, Friedman & Keidl, 2012]	Redshift circular equatorial orbits Kerr
[Pound, Merlin, & Barack, 2013]	Rigorous formulation of GSF in radiation gauge
[MvdM & Shah, 2015]	Redshift for eccentric equatorial orbits in Kerr
[MvdM, 2016]	GSF for eccentric equatorial orbits in Kerr.

Teukolsky Equation

Teukolsky equation

Teukolsky equation

$$\hat{\mathcal{T}}_s^{(2)} \circ \Phi_s = \hat{\mathcal{S}}_s^{(2)}[T_{\mu\nu}]$$

Teukolsky variables

ψ_0: Teukolsky variable of spin-weight +2
$\rho^{-4}\psi_4$: Teukolsky variable of spin-weight -2

Separation of variables

$$\Phi_s = \sum_{lm\omega} {}_sR_{lm\omega}(r)\, {}_sS_{lm\omega}(\theta) e^{i(m\phi - \omega t)}$$

${}_sS_{lm\omega}(\theta)$: spin-weighted spheroidal harmonic
${}_sR_{lm\omega}(r)$: solution of radial Teukolsky equation

$$\left(\Delta^{-s}\frac{\mathrm{d}}{\mathrm{d}r}(\Delta^{s+1}\frac{\mathrm{d}}{\mathrm{d}r}) + {}_sV_{lmw}(r)\right) {}_sR_{lm\omega} = {}_sS_{lm\omega}[T^{\mu\nu}]$$

Teukolsky equation

$$\hat{\mathcal{T}}_s^{(2)} \circ \Phi_s = \hat{\mathcal{S}}_s^{(2)}[T_{\mu\nu}]$$

Teukolsky variables

ψ_0: Teukolsky variable of spin-weight +2
$\rho^{-4}\psi_4$: Teukolsky variable of spin-weight -2

Separation of variables

$$\Phi_s = \sum_{lm\omega} {}_sR_{lm\omega}(r)\,{}_sS_{lm\omega}(\theta)\,e^{i(m\phi-\omega t)}$$

${}_sS_{lm\omega}(\theta)$: spin-weighted spheroidal harmonic
${}_sR_{lm\omega}(r)$: solution of radial Teukolsky equation

$$\left(\Delta^{-s}\frac{\mathrm{d}}{\mathrm{d}r}(\Delta^{s+1}\frac{\mathrm{d}}{\mathrm{d}r}) + {}_sV_{lmw}(r)\right){}_sR_{lm\omega} = {}_sS_{lm\omega}[T^{\mu\nu}]$$

MST method

Series solution

$$ {}_sR_{lm\omega}(r) = \mathcal{C} \sum_{n=-\infty}^{\infty} a_n^\nu \; F_n^\nu(r) $$

- $F_n^\nu(r)$: Hypergeometric function
- a_n^ν satisfies $\alpha_n^\nu a_{n-1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n+1}^\nu = 0$
- Two independent solutions for ν give rise to independent homogeneous solutions

Advantages

- Analytic implementation of boundary conditions
- Arbitrary precision implementation possible
- Numerical implementation for generic orbits.[Fujita et al., 2009]

MST method

Series solution

$$ {}_sR_{lm\omega}(r) = \mathcal{C} \sum_{n=-\infty}^{\infty} a_n^\nu \, F_n^\nu(r) $$

- $F_n^\nu(r)$: Hypergeometric function
- a_n^ν satisfies $\alpha_n^\nu a_{n-1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n+1}^\nu = 0$
- Two independent solutions for ν give rise to independent homogeneous solutions

Advantages

- Analytic implementation of boundary conditions
- Arbitrary precision implementation possible
- Numerical implementation for generic orbits.[Fujita et al., 2009]

Metric reconstruction

Metric reconstruction [Chrzanowski, Cohen,& Kegeles, Wald, 1974-1979]

$$
\begin{tikzcd}
\text{sym. 2-tensors } h_{\mu\nu} \arrow[r, "\hat{\mathcal{G}}"] \arrow[d, "\hat{\Phi}_{\pm2}"] & \text{sym. 2-tensors } T_{\mu\nu} \arrow[d, "\hat{\mathcal{S}}_{\pm2}"] \\
s = \pm2 \text{ fields} \arrow[r, "\hat{\mathcal{T}}_{\pm2}"] & s = \mp2 \text{ fields}
\end{tikzcd}
$$

$$\hat{\mathcal{T}}_{\pm s} \circ \hat{\Phi}_{\pm s} = \hat{\mathcal{S}}_{\pm s} \circ \hat{\mathcal{G}}$$

$$
\begin{tikzcd}
\text{sym. 2-tensors} & \text{sym. 2-tensors} \arrow[l, "\hat{\mathcal{G}}^\dagger = \hat{\mathcal{G}}"'] \\
s = \pm2 \text{ fields} \arrow[u, "\hat{\Phi}^\dagger_{\pm s}"] & s = \mp2 \text{ fields} \arrow[l, "\hat{\mathcal{T}}^\dagger_{\pm2} = \hat{\mathcal{T}}_{\mp2}"'] \arrow[u, "\hat{\mathcal{S}}^\dagger_{\pm2}"']
\end{tikzcd}
$$

$$\hat{\Phi}^\dagger_{\pm s} \circ \hat{\mathcal{T}}_{\mp s} = \hat{\mathcal{G}} \circ \hat{\mathcal{S}}^\dagger_{\pm s}$$

Conclusion

$\hat{\mathcal{S}}^\dagger_{\pm2}$ maps vacuum solutions of the spin ∓2 Teukolsky equation into vacuum solutions of the lin. Einstein equation.

$h_{\mu\nu} = \hat{\mathcal{S}}^\dagger_{\pm2}[\Phi_{\mp2}]$ satisfies the IRG/ORG gauge condition.

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{sym.}\\ \text{2-tensors}\\ h_{\mu\nu}\end{array}} \arrow[r, "\hat{\mathcal{G}}"] \arrow[d, "\hat{\Phi}_{\pm 2}"] & \boxed{\begin{array}{c}\text{sym.}\\ \text{2-tensors}\\ T_{\mu\nu}\end{array}} \arrow[d, "\hat{\mathcal{S}}_{\pm 2}"] \\
\boxed{\begin{array}{c}s = \pm 2\\ \text{fields}\end{array}} \arrow[r, "\hat{\mathcal{T}}_{\pm 2}"] & \boxed{\begin{array}{c}s = \mp 2\\ \text{fields}\end{array}}
\end{tikzcd}$$

$$\hat{\mathcal{T}}_{\pm s} \circ \hat{\Phi}_{\pm s} = \hat{\mathcal{S}}_{\pm s} \circ \hat{\mathcal{G}}$$

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{sym.}\\ \text{2-tensors}\end{array}} & \boxed{\begin{array}{c}\text{sym.}\\ \text{2-tensors}\end{array}} \arrow[l, "\hat{\mathcal{G}}^\dagger = \hat{\mathcal{G}}"'] \\
\boxed{\begin{array}{c}s = \pm 2\\ \text{fields}\end{array}} \arrow[u, "\hat{\Phi}^\dagger_{\pm s}"] & \boxed{\begin{array}{c}s = \mp 2\\ \text{fields}\end{array}} \arrow[l, "\hat{\mathcal{T}}^\dagger_{\pm 2} = \hat{\mathcal{T}}_{\mp 2}"] \arrow[u, "\hat{\mathcal{S}}^\dagger_{\pm 2}"']
\end{tikzcd}$$

$$\hat{\Phi}^\dagger_{\pm s} \circ \hat{\mathcal{T}}_{\mp s} = \hat{\mathcal{G}} \circ \hat{\mathcal{S}}^\dagger_{\pm s}$$

Conclusion

$\hat{\mathcal{S}}^\dagger_{\pm 2}$ maps vacuum solutions of the spin ∓ 2 Teukolsky equation into vacuum solutions of the lin. Einstein equation.

$h_{\mu\nu} = \hat{\mathcal{S}}^\dagger_{\pm 2}[\Phi_{\mp 2}]$ satisfies the IRG/ORG gauge condition.

Metric reconstruction [Chrzanowski, Cohen,& Kegeles, Wald, 1974-1979]

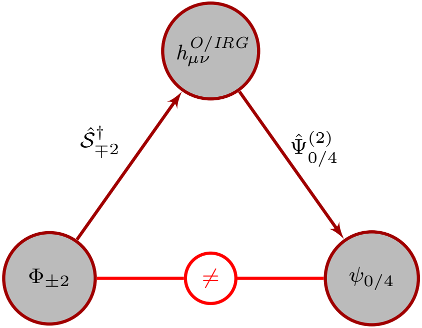
$$\begin{tikzcd}[row sep=huge, column sep=huge]
\begin{array}{c}\text{sym.}\\ \text{2-tensors}\\ h_{\mu\nu}\end{array} \arrow[r, "\hat{\mathcal{G}}"] \arrow[d, "\hat{\Phi}_{\pm 2}"] & \begin{array}{c}\text{sym.}\\ \text{2-tensors}\\ T_{\mu\nu}\end{array} \arrow[d, "\hat{\mathcal{S}}_{\pm 2}"] \\
\begin{array}{c}s = \pm 2\\ \text{fields}\end{array} \arrow[r, "\hat{\mathcal{T}}_{\pm 2}"] & \begin{array}{c}s = \mp 2\\ \text{fields}\end{array}
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=huge, column sep=huge]
\begin{array}{c}\text{sym.}\\ \text{2-tensors}\end{array} & \begin{array}{c}\text{sym.}\\ \text{2-tensors}\end{array} \arrow[l, "\hat{\mathcal{G}}^\dagger = \hat{\mathcal{G}}"'] \\
\begin{array}{c}s = \pm 2\\ \text{fields}\end{array} \arrow[u, "\hat{\Phi}^\dagger_{\pm s}"] & \begin{array}{c}s = \mp 2\\ \text{fields}\end{array} \arrow[l, "\hat{\mathcal{T}}^\dagger_{\pm 2} = \hat{\mathcal{T}}_{\mp 2}"'] \arrow[u, "\hat{\mathcal{S}}^\dagger_{\pm 2}"']
\end{tikzcd}$$

$$\hat{\mathcal{T}}_{\pm s} \circ \hat{\Phi}_{\pm s} = \hat{\mathcal{S}}_{\pm s} \circ \hat{\mathcal{G}} \qquad \hat{\Phi}^\dagger_{\pm s} \circ \hat{\mathcal{T}}_{\mp s} = \hat{\mathcal{G}} \circ \hat{\mathcal{S}}^\dagger_{\pm s}$$

Conclusion

$\hat{\mathcal{S}}^\dagger_{\pm 2}$ maps vacuum solutions of the spin ∓ 2 Teukolsky equation into vacuum solutions of the lin. Einstein equation.

$h_{\mu\nu} = \hat{\mathcal{S}}^\dagger_{\pm 2}[\Phi_{\mp 2}]$ satisfies the IRG/ORG gauge condition.



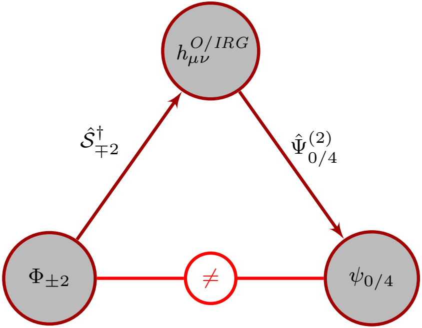
Hertz Potential

The fields $\Phi_{\pm}$ are referred to as Hertz potentials and DO NOT coincide with ψ_0 and ψ_4.

Inversion [Ori, 2001][Keidl et al, 2010]

However, the operators $\hat{\Psi}_s \circ \hat{\mathcal{S}}^{\dagger}_{s'}$ can be inverted algebraically on a mode-by-mode basis.

Hertz potential



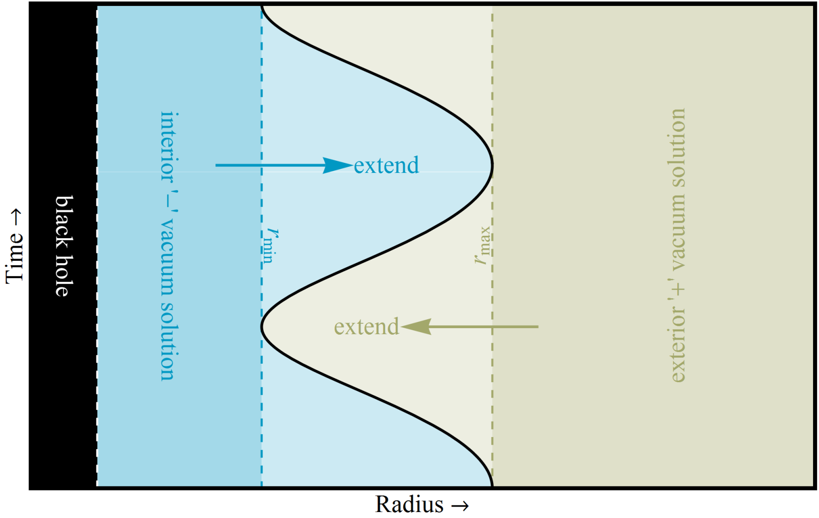
Hertz Potential

The fields $\Phi_{\pm}$ are referred to as Hertz potentials and DO NOT coincide with ψ_0 and ψ_4.

Inversion [Ori, 2001][Keidl et al, 2010]

However, the operators $\hat{\Psi}_s \circ \hat{\mathcal{S}}^{\dagger}_{s'}$ can be inverted algebraically on a mode-by-mode basis.

Adding a particle source: method of extended homogeneous solutions



Completion problem ~~($l = 0, 1$ problem)~~

$$h_{\mu\nu} \xrightarrow{\hat{\Psi}^{(2)}_{0/4}} \psi_{0/4}$$

Theorem [Wald, 1974]

For any vacuum Type-D (i.e $\psi_0 = \psi_4 = 0$) background spacetime

$$\ker \hat{\Psi}^{(2)}_{0/4} = \begin{cases} \text{Gauge modes} \\ \text{Perturbations of the background within the class} \\ \text{of vacuum type-D spacetimes} \\ \text{(i.e. } \delta M,\ \delta J,\ \delta\alpha\text{ , or } \delta Q_{NUT}\text{)} \end{cases}$$

Completion problem ~~($l = 0, 1$ problem)~~

$$h_{\mu\nu} \xrightarrow{\hat{\Psi}^{(2)}_{0/4}} \psi_{0/4}$$

Theorem [Wald, 1974]

For any vacuum Type-D (i.e $\psi_0 = \psi_4 = 0$) background spacetime

$$\ker \hat{\Psi}^{(2)}_{0/4} = \begin{cases} \text{Gauge modes} \\ \text{Perturbations of the background within the class} \\ \text{of vacuum type-D spacetimes} \\ \text{(i.e. } \delta M,\ \delta J,\ \delta\alpha \text{, or } \delta Q_{NUT}\text{)} \end{cases}$$

Completion problem ~~($l = 0, 1$ problem)~~

$$h_{\mu\nu} \xrightarrow{\hat{\Psi}^{(2)}_{0/4}} \psi_{0/4}$$

Theorem [Wald, 1974]

For any vacuum Type-D (i.e $\psi_0 = \psi_4 = 0$) background spacetime

$$\ker \hat{\Psi}^{(2)}_{0/4} = \begin{cases} \text{Gauge modes} \\ \text{Perturbations of the background within the class} \\ \text{of vacuum type-D spacetimes} \\ \text{(i.e. } \delta M,\ \delta J,\ \cancel{\delta\alpha}\text{, or } \cancel{\delta Q_{NUT}}\text{)} \end{cases}$$

Gauge invariant content (mass and angular momentum)

Mass and Angular momentum perturbations

$$h_{\mu\nu}^{M,J} = \delta M^{\pm} \frac{\partial g_{\mu\nu}^{\mathrm{Kerr}}}{\partial M} + \delta J^{\pm} \frac{\partial g_{\mu\nu}^{\mathrm{Kerr}}}{\partial J}$$

Abbott-Deser invariants [Abbott& Deser,1981]

$$F_{\alpha\beta}[k^\mu] := \frac{1}{8\pi}\left(k^\lambda \nabla_{[\alpha}\bar{h}_{\beta]\lambda} + \bar{h}_{\lambda[\alpha}\nabla_{\beta]}k^\lambda - k_{[\alpha}\nabla^\lambda \bar{h}_{\beta]\lambda}\right)$$

Main feature, $-\nabla_\lambda F^{\lambda\alpha} = k_\lambda T^{\lambda\alpha}$, implies:

1. $\mathcal{Q}[h_{\mu\nu}, k^\mu, \mathcal{S}] := \int_{\mathcal{S}} F^{\alpha\beta}[k^\mu]\,\mathrm{d}\mathcal{S}_{\alpha\beta}$ is a topological invariant of vacuum metric perturbations.
2. $\mathcal{Q}[h_{\mu\nu}, k^\mu, \mathcal{S}_1] - \mathcal{Q}[h_{\mu\nu}, k^\mu, \mathcal{S}_2]$ equals the total Killing charge of any sources between $\mathcal{S}_1$ and $\mathcal{S}_2$

Gauge invariant content (mass and angular momentum)

Mass and Angular momentum perturbations

$$h^{M,J}_{\mu\nu} = \delta M^{\pm} \frac{\partial g^{\text{Kerr}}_{\mu\nu}}{\partial M} + \delta J^{\pm} \frac{\partial g^{\text{Kerr}}_{\mu\nu}}{\partial J}$$

Abbott-Deser invariants [Abbott& Deser,1981]

$$F_{\alpha\beta}[k^{\mu}] := \frac{1}{8\pi}\left(k^{\lambda}\nabla_{[\alpha}\bar{h}_{\beta]\lambda} + \bar{h}_{\lambda[\alpha}\nabla_{\beta]}k^{\lambda} - k_{[\alpha}\nabla^{\lambda}\bar{h}_{\beta]\lambda}\right)$$

Main feature, $-\nabla_{\lambda}F^{\lambda\alpha} = k_{\lambda}T^{\lambda\alpha}$, implies:

1. $\mathcal{Q}[h_{\mu\nu}, k^{\mu}, \mathcal{S}] := \int_{\mathcal{S}} F^{\alpha\beta}[k^{\mu}]\,\mathrm{d}\mathcal{S}_{\alpha\beta}$ is a topological invariant of vacuum metric perturbations.
2. $\mathcal{Q}[h_{\mu\nu}, k^{\mu}, \mathcal{S}_1] - \mathcal{Q}[h_{\mu\nu}, k^{\mu}, \mathcal{S}_2]$ equals the total Killing charge of any sources between $\mathcal{S}_1$ and $\mathcal{S}_2$

Gauge invariant content (mass and angular momentum)

Mass and Angular momentum perturbations

$$h_{\mu\nu}^{M,J} = \delta M^{\pm} \frac{\partial g_{\mu\nu}^{\mathrm{Kerr}}}{\partial M} + \delta J^{\pm} \frac{\partial g_{\mu\nu}^{\mathrm{Kerr}}}{\partial J}$$

Abbott-Deser invariants [Abbott& Deser,1981]

$$F_{\alpha\beta}[k^\mu] := \frac{1}{8\pi}\left(k^\lambda \nabla_{[\alpha}\bar{h}_{\beta]\lambda} + \bar{h}_{\lambda[\alpha}\nabla_{\beta]}k^\lambda - k_{[\alpha}\nabla^\lambda \bar{h}_{\beta]\lambda}\right)$$

Main feature, $-\nabla_\lambda F^{\lambda\alpha} = k_\lambda T^{\lambda\alpha}$, implies:

1. $\mathcal{Q}[h_{\mu\nu}, k^\mu, \mathcal{S}] := \int_{\mathcal{S}} F^{\alpha\beta}[k^\mu]\, \mathrm{d}\mathcal{S}_{\alpha\beta}$ is a topological invariant of vacuum metric perturbations.
2. $\mathcal{Q}[h_{\mu\nu}, k^\mu, \mathcal{S}_1] - \mathcal{Q}[h_{\mu\nu}, k^\mu, \mathcal{S}_2]$ equals the total Killing charge of any sources between $\mathcal{S}_1$ and $\mathcal{S}_2$

A general lemma

Lemma [MvdM,2017]

Let ${}_s\Phi_{m\omega}(r,z)e^{i(m\phi-\omega t)}$ be a smooth solution of the homogeneous $(s=\pm 2)$-Teukolsky equation. Then all Abbott-Deser charges of the corresponding reconstructed vacuum metric, $h_{\mu\nu}$, vanish.

Corollary

$$\delta M^{+} = E, \quad \delta M^{-} = 0,$$
$$\delta J^{+} = L, \quad \delta J^{-} = 0,$$

A general lemma

Lemma [MvdM,2017]

Let ${}_s\Phi_{m\omega}(r,z)e^{i(m\phi-\omega t)}$ be a smooth solution of the homogeneous $(s=\pm2)$-Teukolsky equation. Then all Abbott-Deser charges of the corresponding reconstructed vacuum metric, $h_{\mu\nu}$, vanish.

Corollary

$$
\begin{aligned}
\delta M^{+} &= E, & \delta M^{-} &= 0,\\
\delta J^{+} &= L, & \delta J^{-} &= 0,
\end{aligned}
$$

Quasi-gauge invariant content

Quasi-gauge invariants

Commonly used “invariants” such as redshift and spin precession are sensitive to gauge transformations of the form:

$$\xi^\mu = \alpha t k_t^\mu + \beta t k_\phi^\mu$$

SAS gauge transformations

$$\xi^\mu = (\alpha t + \xi_t(r,z), \xi_r(r,z), \xi_z(r,z), \beta t + \xi_\phi(r,z))$$

$$\begin{pmatrix} [\alpha,\beta,\xi_r,\xi_z] & [\xi_t,\xi_\phi] & [\xi_t,\xi_\phi] & [\alpha,\beta,\xi_r,\xi_z] \\ [\xi_t,\xi_\phi] & [\xi_r,\xi_z] & [\xi_r,\xi_z] & [\xi_t,\xi_\phi] \\ [\xi_t,\xi_\phi] & [\xi_r,\xi_z] & [\xi_r,\xi_z] & [\xi_t,\xi_\phi] \\ [\alpha,\beta,\xi_r] & [\xi_t,\xi_\phi] & [\xi_t,\xi_\phi] & [\alpha,\beta,\xi_r,\xi_z] \end{pmatrix}$$

Reference gauge

Chandrasekhar gauge $\delta\Psi_2 = 0$ uniquely fixes ξ_r and ξ_z.

- Gauge invariant fields: $\tilde{h}_{rr}^{SAS}$, $\tilde{h}_{rz}^{SAS}$, and $\tilde{h}_{zz}^{SAS}$. (Can be used to find mass and angular momentum perturbation [Merlin et al., 2016])
- Quasi-invariant fields: $\tilde{h}_{tt}^{SAS}$, $\tilde{h}_{t\phi}^{SAS}$, and $\tilde{h}_{\phi\phi}^{SAS}$

Quasi-gauge invariant content

Quasi-gauge invariants

Commonly used "invariants" such as redshift and spin precession are sensitive to gauge transformations of the form:

$$\xi^\mu = \alpha t k_t^\mu + \beta t k_\phi^\mu$$

SAS gauge transformations

$$\xi^\mu = (\alpha t + \xi_t(r,z), \xi_r(r,z), \xi_z(r,z), \beta t + \xi_\phi(r,z))$$

$$\begin{pmatrix} [\alpha,\beta,\xi_r,\xi_z] & [\xi_t,\xi_\phi] & [\xi_t,\xi_\phi] & [\alpha,\beta,\xi_r,\xi_z] \\ [\xi_t,\xi_\phi] & [\xi_r,\xi_z] & [\xi_r,\xi_z] & [\xi_t,\xi_\phi] \\ [\xi_t,\xi_\phi] & [\xi_r,\xi_z] & [\xi_r,\xi_z] & [\xi_t,\xi_\phi] \\ [\alpha,\beta,\xi_r] & [\xi_t,\xi_\phi] & [\xi_t,\xi_\phi] & [\alpha,\beta,\xi_r,\xi_z] \end{pmatrix}$$

Reference gauge

Chandrasekhar gauge $\delta\Psi_2 = 0$ uniquely fixes ξ_r and ξ_z.

- Gauge invariant fields: $\tilde{h}^{SAS}_{rr}$, $\tilde{h}^{SAS}_{rz}$, and $\tilde{h}^{SAS}_{zz}$. (Can be used to find mass and angular momentum perturbation [Merlin et al., 2016])
- Quasi-invariant fields: $\tilde{h}^{SAS}_{tt}$, $\tilde{h}^{SAS}_{t\phi}$, and $\tilde{h}^{SAS}_{\phi\phi}$

Quasi-gauge invariant content

Quasi-gauge invariants

Commonly used "invariants" such as redshift and spin precession are sensitive to gauge transformations of the form:

$$\xi^\mu = \alpha t k_t^\mu + \beta t k_\phi^\mu$$

SAS gauge transformations

$$\xi^\mu = (\alpha t + \xi_t(r,z), \xi_r(r,z), \xi_z(r,z), \beta t + \xi_\phi(r,z))$$

$$\begin{pmatrix} [\alpha,\beta,\xi_r,\xi_z] & [\xi_t,\xi_\phi] & [\xi_t,\xi_\phi] & [\alpha,\beta,\xi_r,\xi_z] \\ [\xi_t,\xi_\phi] & [\xi_r,\xi_z] & [\xi_r,\xi_z] & [\xi_t,\xi_\phi] \\ [\xi_t,\xi_\phi] & [\xi_r,\xi_z] & [\xi_r,\xi_z] & [\xi_t,\xi_\phi] \\ [\alpha,\beta,\xi_r] & [\xi_t,\xi_\phi] & [\xi_t,\xi_\phi] & [\alpha,\beta,\xi_r,\xi_z] \end{pmatrix}$$

Reference gauge

Chandrasekhar gauge $\delta\Psi_2 = 0$ uniquely fixes ξ_r and ξ_z.

- Gauge invariant fields: $\tilde{h}_{rr}^{SAS}$, $\tilde{h}_{rz}^{SAS}$, and $\tilde{h}_{zz}^{SAS}$. (Can be used to find mass and angular momentum perturbation [Merlin et al., 2016])
- Quasi-invariant fields: $\tilde{h}_{tt}^{SAS}$, $\tilde{h}_{t\phi}^{SAS}$, and $\tilde{h}_{\phi\phi}^{SAS}$

Fixing α and β [MvdM, Pound& Shah, ?2017?]

Strategy

Require analyticity of quasi-invariant fields $\tilde{h}^{SAS}_{tt}$, $\tilde{h}^{SAS}_{t\phi}$, and $\tilde{h}^{SAS}_{\phi\phi}$, away from source.

Method of partial spheres

$$T^{SAS}_{\mu\nu}(r,z) = \int_{r_{min}}^{r_{max}} T^{SAS}_{\mu\nu}[r_0](r,z)\,\mathrm{d}r_0$$

- The jump in the reconstructed field $\Delta\tilde{h}^{SAS}_{\mu\nu}$ can be calculated analytically for each $T^{SAS}_{\mu\nu}[r_0](r,z)$.
- Requiring continuity for the tt, $t\phi$ and $\phi\phi$ gives $\Delta\alpha[r_0]$ and $\Delta\beta[r_0]$.
- Method of extended homogeneous solutions then gives total jumps $\Delta\alpha$ and $\Delta\beta$

Conclusion

We can reconstruct the metric perturbation generated by a point particle on any bound geodesic around a Kerr BH, including all gauge effects that effect quasi-invariants.

Fixing α and β [MvdM, Pound& Shah, ?2017?]

Strategy

Require analyticity of quasi-invariant fields $\tilde{h}_{tt}^{SAS}$, $\tilde{h}_{t\phi}^{SAS}$, and $\tilde{h}_{\phi\phi}^{SAS}$, away from source.

Method of partial spheres

$$T_{\mu\nu}^{SAS}(r,z) = \int_{r_{min}}^{r_{max}} T_{\mu\nu}^{SAS}[r_0](r,z)\,\mathrm{d}r_0$$

- The jump in the reconstructed field $\Delta\tilde{h}_{\mu\nu}^{SAS}$ can be calculated analytically for each $T_{\mu\nu}^{SAS}[r_0](r,z)$.
- Requiring continuity for the tt, $t\phi$ and $\phi\phi$ gives $\Delta\alpha[r_0]$ and $\Delta\beta[r_0]$.
- Method of extended homogeneous solutions then gives total jumps $\Delta\alpha$ and $\Delta\beta$

Conclusion

We can reconstruct the metric perturbation generated by a point particle on any bound geodesic around a Kerr BH, including all gauge effects that effect quasi-invariants.

Fixing α and β [MvdM, Pound& Shah, ?2017?]

Strategy

Require analyticity of quasi-invariant fields $\tilde{h}^{SAS}_{tt}$, $\tilde{h}^{SAS}_{t\phi}$, and $\tilde{h}^{SAS}_{\phi\phi}$, away from source.

Method of partial spheres

$$T^{SAS}_{\mu\nu}(r,z) = \int_{r_{min}}^{r_{max}} T^{SAS}_{\mu\nu}[r_0](r,z)\,\mathrm{d}r_0$$

- The jump in the reconstructed field $\Delta\tilde{h}^{SAS}_{\mu\nu}$ can be calculated analytically for each $T^{SAS}_{\mu\nu}[r_0](r,z)$.
- Requiring continuity for the tt, $t\phi$ and $\phi\phi$ gives $\Delta\alpha[r_0]$ and $\Delta\beta[r_0]$.
- Method of extended homogeneous solutions then gives total jumps $\Delta\alpha$ and $\Delta\beta$

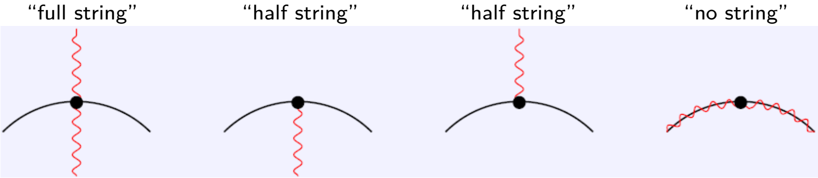
Conclusion

We can reconstruct the metric perturbation generated by a point particle on any bound geodesic around a Kerr BH, including all gauge effects that effect quasi-invariants.

Self-force in radiation gauge

Self-force in radiation gauge [Pound,Merlin&Barack, 2013]

Radiation gauge metric perturbations have (gauge) string singularities.
[Barack& Ori, 2001]



Credit: C. Merlin

[Pound,Merlin&Barack, 2013]

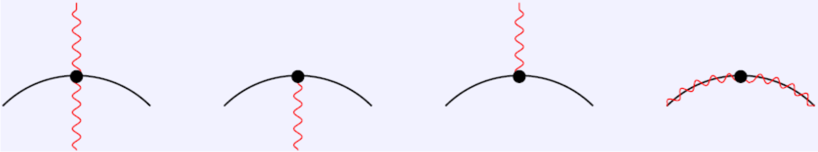
- GSF can be calculated in “half-string” gauges, but non-zero corrections to regularization parameters occur.
- Corrections to Lorenz gauge RPs cancel in “no string” gauge:

$$F^\mu = \left(\sum_{l=0}^{\infty} \frac{F^{\mu,+}_{l,\mathrm{Rad}} + F^{\mu,-}_{l,\mathrm{Rad}}}{2} - B^\mu_{\mathrm{Lor}} - \frac{C^\mu_{\mathrm{Lor}}}{L} \right) - D^\mu_{\mathrm{Lor}}.$$

Self-force in radiation gauge [Pound,Merlin&Barack, 2013]

Radiation gauge metric perturbations have (gauge) string singularities.
[Barack& Ori, 2001]

“full string” “half string” “half string” “no string”



Credit: C. Merlin

[Pound,Merlin&Barack, 2013]

- GSF can be calculated in “half-string” gauges, but non-zero corrections to regularization parameters occur.
- Corrections to Lorenz gauge RPs cancel in “no string” gauge:

$$F^{\mu} = \left(\sum_{l=0}^{\infty} \frac{F^{\mu,+}_{l,\mathrm{Rad}} + F^{\mu,-}_{l,\mathrm{Rad}}}{2} - B^{\mu}_{\mathrm{Lor}} - \frac{C^{\mu}_{\mathrm{Lor}}}{L} \right) - D^{\mu}_{\mathrm{Lor}}.$$

Gravitational self-force (extension)

Compute self-force from Hertz potential:

$$\begin{aligned}\mathcal{F}^{\mu,\pm}_{\text{Rad}} &= \hat{\mathcal{F}}^{\mu,(1)} \circ \hat{\mathcal{S}}^{\dagger}_{-2} \circ \Phi^{\pm}_{ORG} \\ &= \hat{\mathcal{F}}^{\mu,(1)} \circ \hat{\mathcal{S}}^{\dagger}_{-2} \circ \sum_{m\omega\mathfrak{l}} \Psi^{\pm}_{\mathfrak{l}m\omega}\, {}_2R^{\pm}_{\mathfrak{l}m\omega}(r)\, {}_2S_{lm\omega}(z)\, e^{im\phi - i\omega t} + c.c.,\end{aligned}$$

Almost l-mode decomposition, but...

1. Wrong modes for mode-sum
2. Coefficients depend on z

Gravitational self-force (extension)

Compute self-force from Hertz potential:

$$\mathcal{F}^{\mu,\pm}_{\mathrm{Rad}} = \hat{\mathcal{F}}^{\mu,(1)} \circ \hat{\mathcal{S}}^{\dagger}_{-2} \circ \Phi^{\pm}_{ORG}$$
$$= \hat{\mathcal{F}}^{\mu,(1)} \circ \hat{\mathcal{S}}^{\dagger}_{-2} \circ \sum_{\substack{m\omega \\ l\mathfrak{l}}} \Psi^{\pm}_{\mathfrak{l}m\omega}\, {}_2R^{\pm}_{\mathfrak{l}m\omega}(r) ({}_2b_{m\omega})^{\mathfrak{l}}_{l}\, {}_2Y_{lm}(z) e^{im\phi - i\omega t} + c.c.,$$

Almost l-mode decomposition, but...

1. Wrong modes for mode-sum
2. Coefficients depend on z

Gravitational self-force (extension)

Compute self-force from Hertz potential:

$$\mathcal{F}^{\mu,\pm}_{\mathrm{Rad}} = \hat{\mathcal{F}}^{\mu,(1)} \circ \hat{\mathcal{S}}^{\dagger}_{-2} \circ \Phi^{\pm}_{ORG}$$
$$= \sum_{\substack{m\omega si \\ l\mathfrak{l}}} \left(\mathcal{C}^{\mu}_{lm\omega si}(r,z) \Psi^{\pm}_{\mathfrak{l}m\omega}\, {}_2R^{\pm,(\mathrm{i})}_{\mathfrak{l}m\omega}(r) ({}_2b_{m\omega})^{\mathfrak{l}}_{l} e^{im\phi - i\omega t} + c.c. \right) {}_sY_{lm}(z),$$

Almost l-mode decomposition, but...

1. Wrong modes for mode-sum
2. Coefficients depend on z

Gravitational self-force (extension)

Compute self-force from Hertz potential:

$$
\begin{aligned}
\mathcal{F}^{\mu,\pm}_{\mathrm{Rad}} &= \hat{\mathcal{F}}^{\mu,(1)} \circ \hat{\mathcal{S}}^{\dagger}_{-2} \circ \Phi^{\pm}_{ORG} \\
&= \sum_{\substack{m\omega si \\ l_1 l_2 \mathfrak{l}}} \left(\mathcal{C}^{\mu}_{m\omega si}(r,z) \Psi^{\pm}_{\mathfrak{l}m\omega}\, {}_2R^{\pm,(\mathrm{i})}_{\mathfrak{l}m\omega}(r) ({}_2b_{m\omega})^{\mathfrak{l}}_{l_1}\, {}^{m}_{s}\mathcal{A}^{l_1}_{l_2}\, e^{im\phi - i\omega t} + c.c. \right) Y_{l_2 m}(z),
\end{aligned}
$$

Almost l-mode decomposition, but...

1. ~~Wrong modes for mode-sum~~ use $(1-z^2)^{|s|/2}\,{}_sY_{l_1 m}(\theta) = \sum_{l_2} {}^{m}_{s}\mathcal{A}^{l_1}_{l_2}\, Y_{l_2 m}(z)$
2. Coefficients depend on z

Gravitational self-force (extension)

Compute self-force from Hertz potential:

$$\begin{aligned}\mathcal{F}^{\mu,\pm}_{\mathrm{Rad}} &= \hat{\mathcal{F}}^{\mu,(1)} \circ \hat{\mathcal{S}}^{\dagger}_{-2} \circ \Phi^{\pm}_{ORG} \\ &= \sum_{\substack{m\omega sij \\ l_1 l_2 \mathfrak{l}}} \left(\mathcal{C}^{\mu,j}_{m\omega si} \Psi^{\pm}_{\mathfrak{l}m\omega}\, {}_2R^{\pm,(\mathrm{i})}_{\mathfrak{l}m\omega} ({}_2b_{m\omega})^{\mathfrak{l}}_{l_1}\, {}^{m}_{s}\mathcal{A}^{l_1}_{l_2}\, e^{im\phi - i\omega t} + c.c. \right) (z - z_0)^j\ Y_{l_2 m}(z),\end{aligned}$$

Almost l-mode decomposition, but...

1. ~~Wrong modes for mode-sum~~ use $(1-z^2)^{|s|/2}\, {}_sY_{l_1 m}(\theta) = \sum_{l_2} {}^{m}_{s}\mathcal{A}^{l_1}_{l_2}\ Y_{l_2 m}(z)$
2. ~~Coefficients depend on z~~

Compute self-force from Hertz potential:

$$
\begin{aligned}
\mathcal{F}^{\mu,\pm}_{\text{Rad}} &= \hat{\mathcal{F}}^{\mu,(1)} \circ \hat{\mathcal{S}}^{\dagger}_{-2} \circ \Phi^{\pm}_{ORG} \\
&= \sum_{\substack{m\omega sij \\ l_1 l_2 l_3 \mathfrak{l}}} \left(\mathcal{C}^{\mu,j}_{m\omega si} \Psi^{\pm}_{\mathfrak{l}m\omega}\, {}_2R^{\pm,(\mathrm{i})}_{\mathfrak{l}m\omega} ({}_2b_{m\omega})^{\mathfrak{l}}_{l_1}\, {}^{m}_{s}\mathcal{A}^{l_1}_{l_2}\, {}^{jm}\mathcal{B}^{l_2}_{l_3}\, e^{im\phi - i\omega t} + c.c. \right) Y_{l_3 m}(z),
\end{aligned}
$$

Almost l-mode decomposition, but...

1. ~~Wrong modes for mode-sum~~ use $(1-z^2)^{|s|/2}\, {}_sY_{l_1 m}(\theta) = \sum_{l_2} {}^{m}_{s}\mathcal{A}^{l_1}_{l_2}\, Y_{l_2 m}(z)$
2. ~~Coefficients depend on z~~ Taylor expand around z_0 and reexpand using

   $z^j\, Y_{l_1 m}(z) = \sum_{l_2} {}^{jm}\mathcal{B}^{l_1}_{l_2}\, Y_{l_2 m}(z)$

Summary of method

ψ_4	MST method
$\downarrow$	
Φ_{ORG}	Radial inversion
$\downarrow$	
F^{μ}_{l}, $h_{uu,l}$	Re-expansion to spherical harmonics
$\downarrow$	
Mode sum	Lorenz gauge parameters, accelerate convergence by “tail fit” methods
$\downarrow$	
Completion	δM,δJ,α, and β

Summary of method

ψ_4	MST method
Φ_{ORG}	Radial inversion
F^{μ}_{l}, $h_{uu,l}$	Re-expansion to spherical harmonics
Mode sum	Lorenz gauge parameters, accelerate convergence by “tail fit” methods
Completion	δM,δJ,α, and β

Summary of method

ψ_4 — MST method

$\downarrow$

Φ_{ORG} — Radial inversion

$\downarrow$

F^{μ}_{l}, $h_{uu,l}$ — Re-expansion to spherical harmonics

$\downarrow$

Mode sum — Lorenz gauge parameters,
accelerate convergence by "tail fit" methods

$\downarrow$

Completion — δM,δJ,α, and β

Summary of method

ψ_4	MST method
Φ_{ORG}	Radial inversion
F^μ_l, $h_{uu,l}$	Re-expansion to spherical harmonics
Mode sum	Lorenz gauge parameters, accelerate convergence by "tail fit" methods
Completion	δM,δJ,α, and β

ψ_4	MST method
Φ_{ORG}	Radial inversion
F^{μ}_{l}, $h_{uu,l}$	Re-expansion to spherical harmonics
Mode sum	Lorenz gauge parameters, accelerate convergence by “tail fit” methods
Completion	δM,δJ,α, and β

Summary of method

ψ_4 — MST method

$\downarrow$

Φ_{ORG} — Radial inversion

$\downarrow$

F^{μ}_{l}, $h_{uu,l}$ — Re-expansion to spherical harmonics

$\downarrow$

Mode sum — Lorenz gauge parameters,
accelerate convergence by "tail fit" methods

$\downarrow$

Completion — δM,δJ,α, and β

Results

Test orbit

$a = 0.5M$, $p = 10M$, $e = 0.1$, and $z_{max} = 0.1$
40000 CPU hours, 1 TB of raw data.

Summary and Outlook

Summary

Calculation of the GSF corrections to generic orbits is almost a reality. Some testing and debugging required.

Outlook

- Calculate new invariants such as the ISSO shift.
- Explore resonances

Thank you for listening!

Acknowledgments

This work has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 705229.