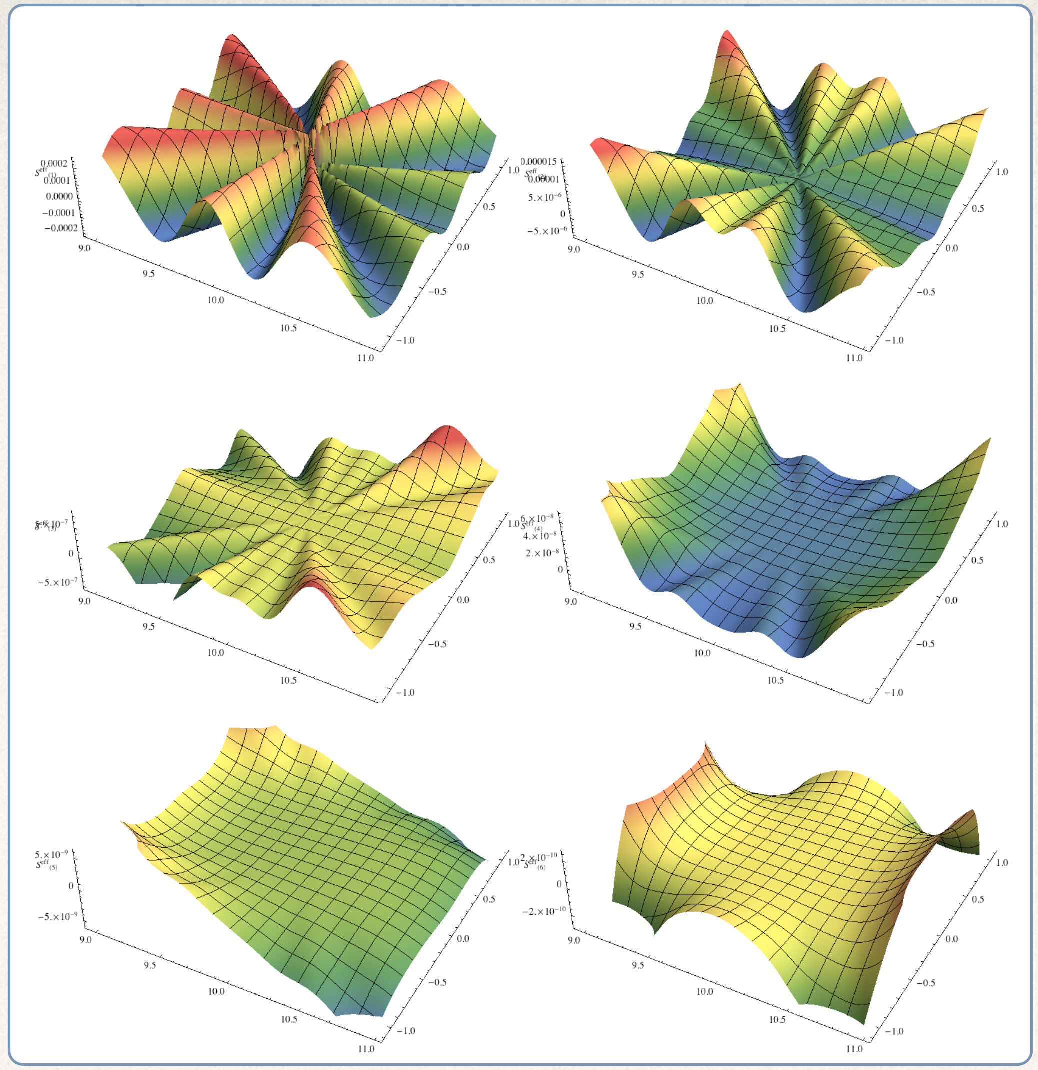


Effective Source Calculations Through Second Perturbative Order

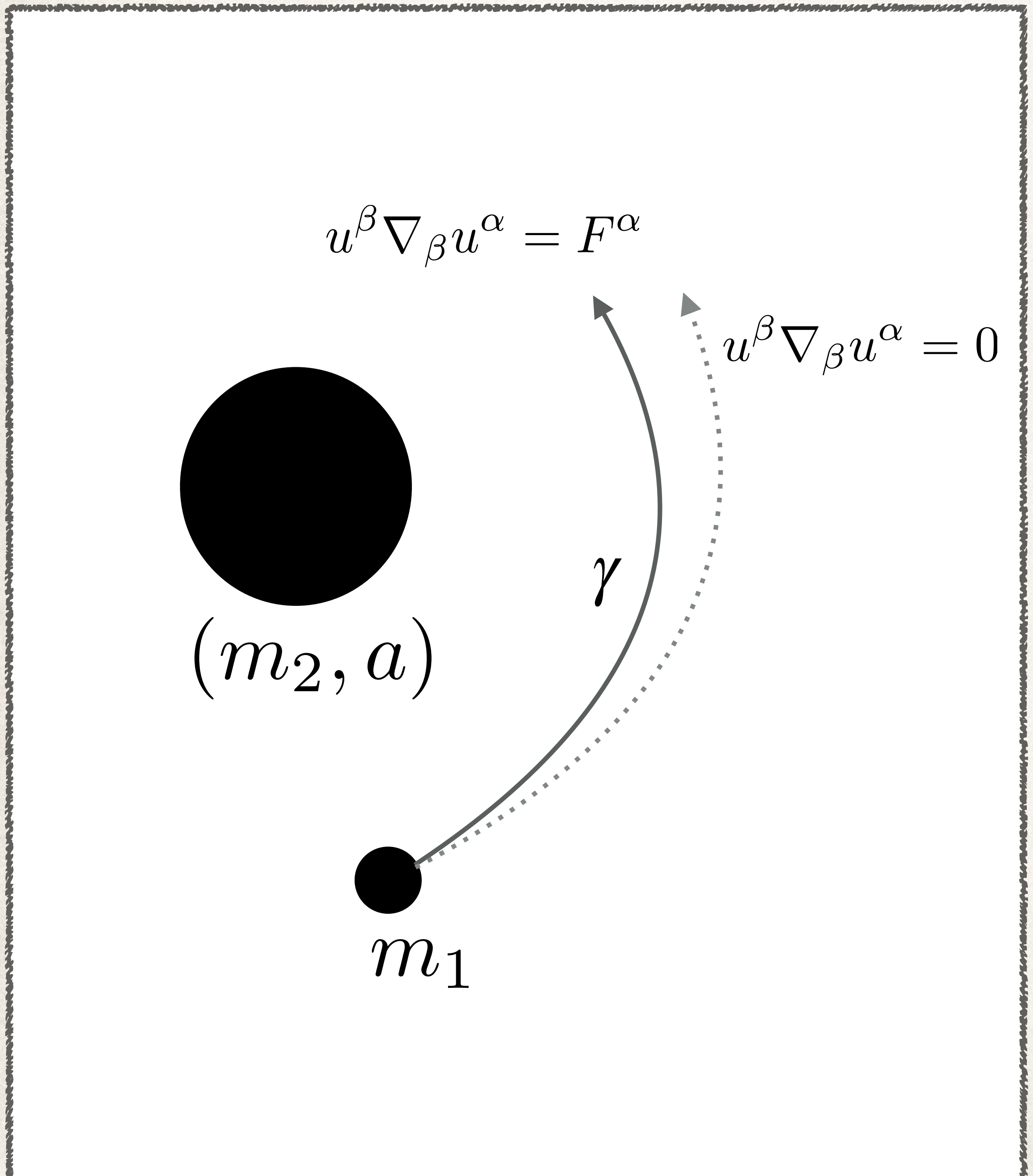
Barry Wardell

University College Dublin

20^{th} Capra Meeting, Chapel Hill, 21^{st} June 2017

Phys. Rev. D 95, 104056; Phys. Rev. D 94, 104018; Phys. Rev. D 92, 104047; Phys. Rev. D 92, 084019; Phys. Rev. D 90, 084039; Phys. Rev. D 89, 104020; Phys. Rev. D 89, 044046; Phys. Rev. D 86, 084019

Gravitational Self-force



Expand the metric into a background plus a perturbation

$$\boxed{\mathbf{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^1_{\mu\nu} + \epsilon^2 h^2_{\mu\nu} + \cdots}$$

Substitute expansion into Einstein equation

$$\boxed{G_{\mu\nu}[\mathbf{g}] = 8\pi T_{\mu\nu}}$$

Obtain equations at each order in ϵ, which we can solve for $h^1_{\mu\nu}$, $h^2_{\mu\nu}$, ..., along with equations of motion for a worldline γ.

Self-force at First Order

- Compute first order metric perturbation sourced by a point particle with retarded (outgoing radiation) boundary conditions at horizon and infinity.
- Subtract Detweiler-Whiting singular (S) field from retarded (ret) field to obtain finite regular (R) field.

$$\Box \bar{h}^{\text{ret}}_{ab} + 2C_a{}^c{}_b{}^d \bar{h}^{\text{ret}}_{cd} + g_{ab} Z_d{}^{;d} - 2Z_{(a;b)} = -16\pi\mu \int g_{a'(a} u^{a'} g_{b)b'} u^{b'} \sqrt{-g}\, \delta_4(x, z(\tau))\, d\tau$$

$$\bar{h}^1_{\text{R}} = \bar{h}^1_{\text{ret}} - \bar{h}^1_{\text{S}}$$

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$$\boxed{E_{\mu\nu}[h^1_{\rm ret}] = -16\pi \bar{T}_{\mu\nu}[\gamma]}$$

$$\boxed{\bar{h}^1_{\rm R} = \bar{h}^1_{\rm ret} - \bar{h}^1_{\rm S}}$$

Self-force at Second Order

- Compute second order metric perturbation sourced by first order metric perturbation with *appropriate* boundary conditions at horizon and infinity.
- Subtract second order singular (S) field from retarded (ret) field to obtain finite regular (R) field.

$$\boxed{\begin{aligned} E_{\alpha\beta}[h^2_{\mathrm{ret}}] = 2\Big[&-\frac{1}{2}h^{\mu\nu}(2h_{\mu(\alpha;\beta)\nu} - h_{\alpha\beta;\mu\nu} - h_{\mu\nu;\alpha\beta}) \\ &+\frac{1}{4}h^{\mu\nu}{}_{;\alpha}h_{\mu\nu;\beta}+\frac{1}{2}h^{\mu}{}_{\beta}{}^{;\nu}(h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}) \\ &-\frac{1}{2}\bar{h}^{\mu\nu}{}_{;\nu}(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu})\Big] \end{aligned}}$$

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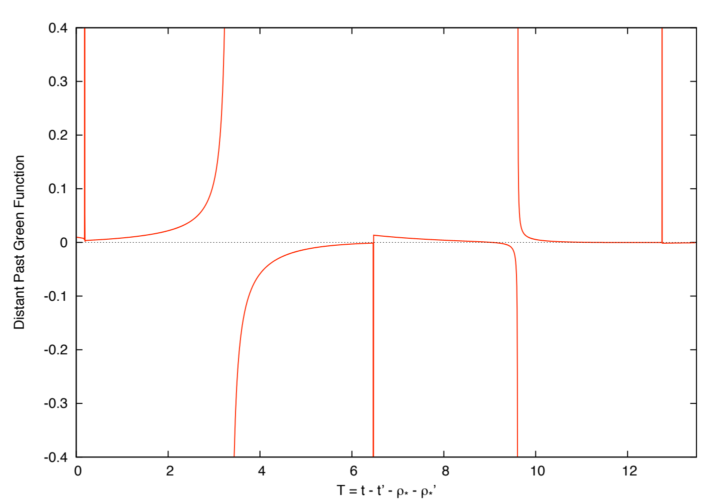
$$E_{\mu\nu}[h^2_{\text{ret}}] = 2\delta^2 R_{\mu\nu}[h^1_{\text{ret}}, h^1_{\text{ret}}]$$

$$\bar{h}^2_{\text{R}} = \bar{h}^2_{\text{ret}} - \bar{h}^2_{\text{S}}$$

Self-force computation strategies

- Several methods have emerged for computing h^{1R}_{ab}, dealing with the numerical issues of point sources, singular fields.
- These broadly fall into three different categories (+ dissipative approx)

Worldline convolution

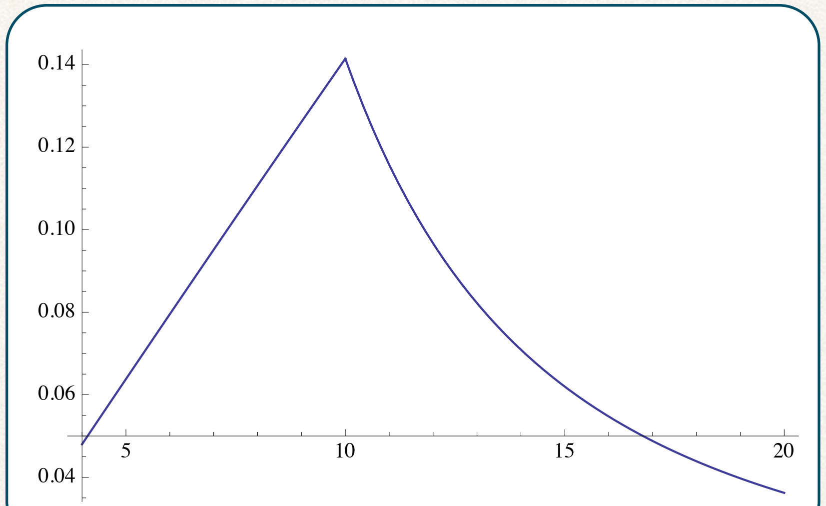


Earliest proposed, latest implemented

Arbitrary motion

Geometric interpretation

Mode-sum

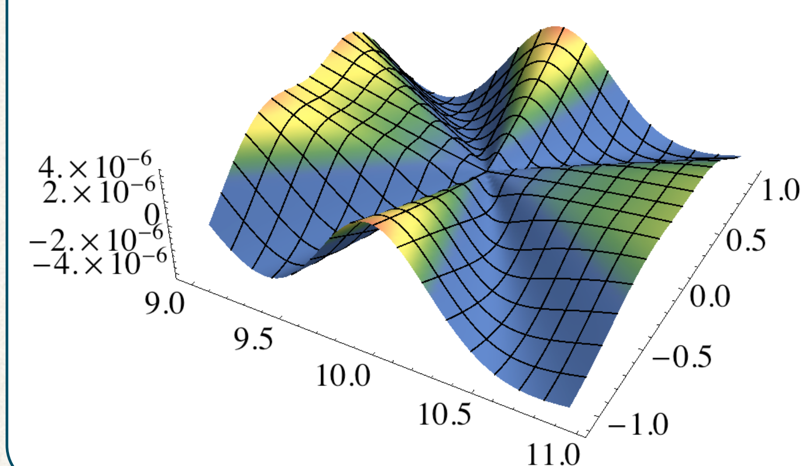


First implemented

Most accurate

Easiest to implement

Effective source



Latest proposed

Well suited to evolving orbits

Well-defined at second order

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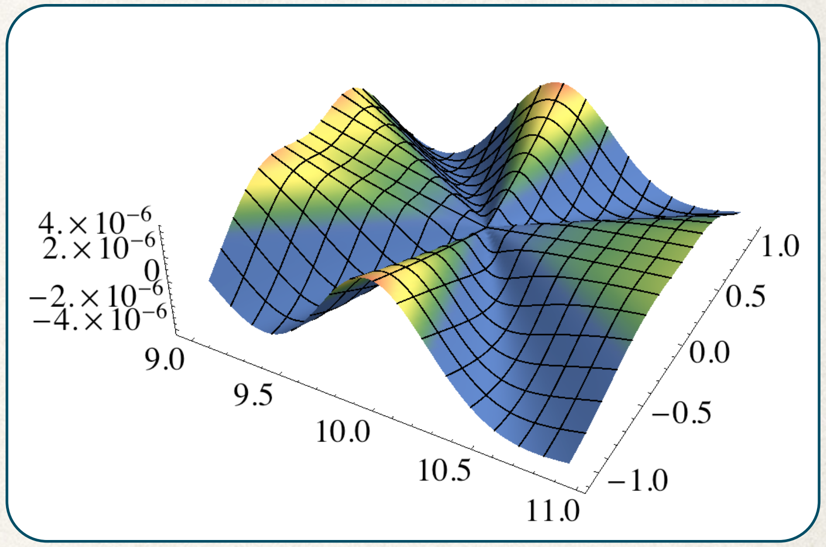

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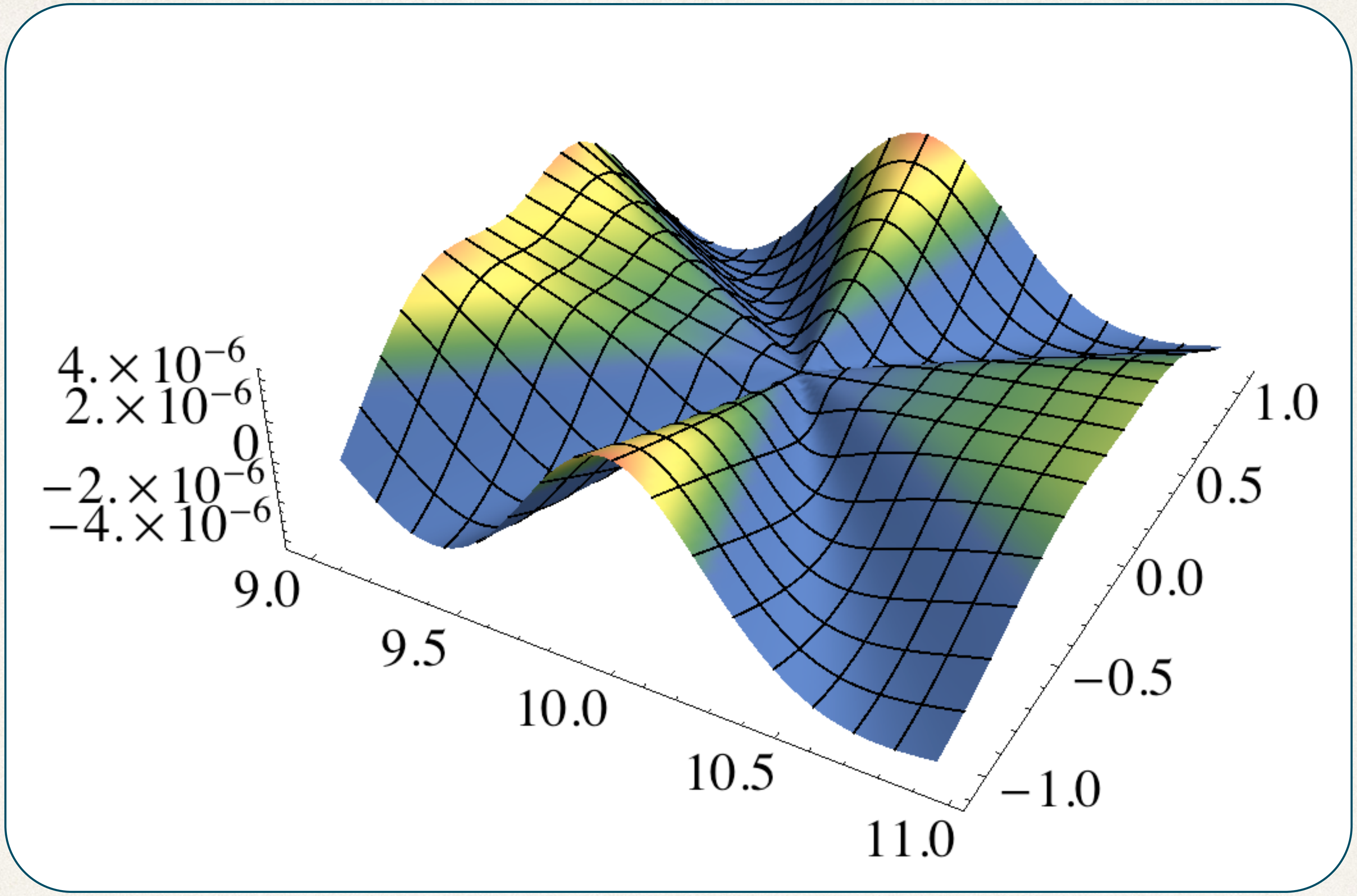

Effective Source Regularisation

Effective Source at First Order

- Derive an evolution equation for $\bar{h}^1_{\rm R}$ by moving $\bar{h}^1_{\rm S}$ to right hand side and treating it as an *effective* source.
  Barack and Golbourn, Phys. Rev. D 76, 044020
  Detweiler and Vega, Phys. Rev. D 77, 084008

$$\boxed{E_{\mu\nu}[h^1_{\rm R}] = -16\pi \bar{T}_{\mu\nu}[\gamma] - E_{\mu\nu}[h^1_{\rm S}]}$$

- Always work with $\bar{h}^1_{\rm R}$ instead of $\bar{h}^1_{\rm ret}$
- No distributional sources and no singular fields.

$$\boxed{\frac{Du^\mu}{d\tau} = (g^{\mu\nu} + u^\mu u^\nu)(2h^{1\ \rm R}_{\nu\lambda;\rho} - h^{1\ \rm R}_{\lambda\rho;\nu})u^\lambda u^\rho}$$

- If $\bar{h}^1_{\rm S}$ is chosen appropriately, then we can directly use $\bar{h}^1_{\rm R}$ in the worldline equations of motion.

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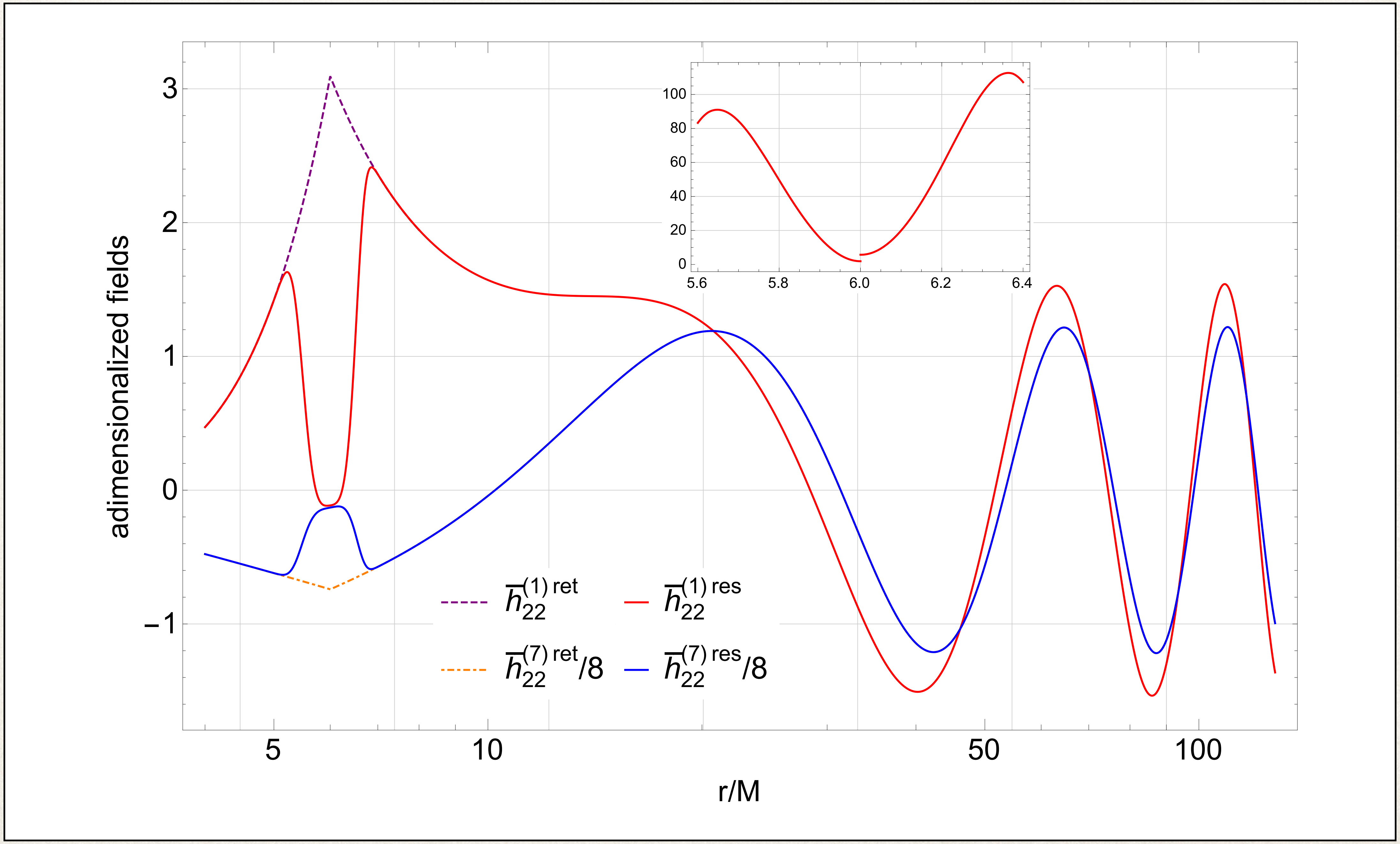
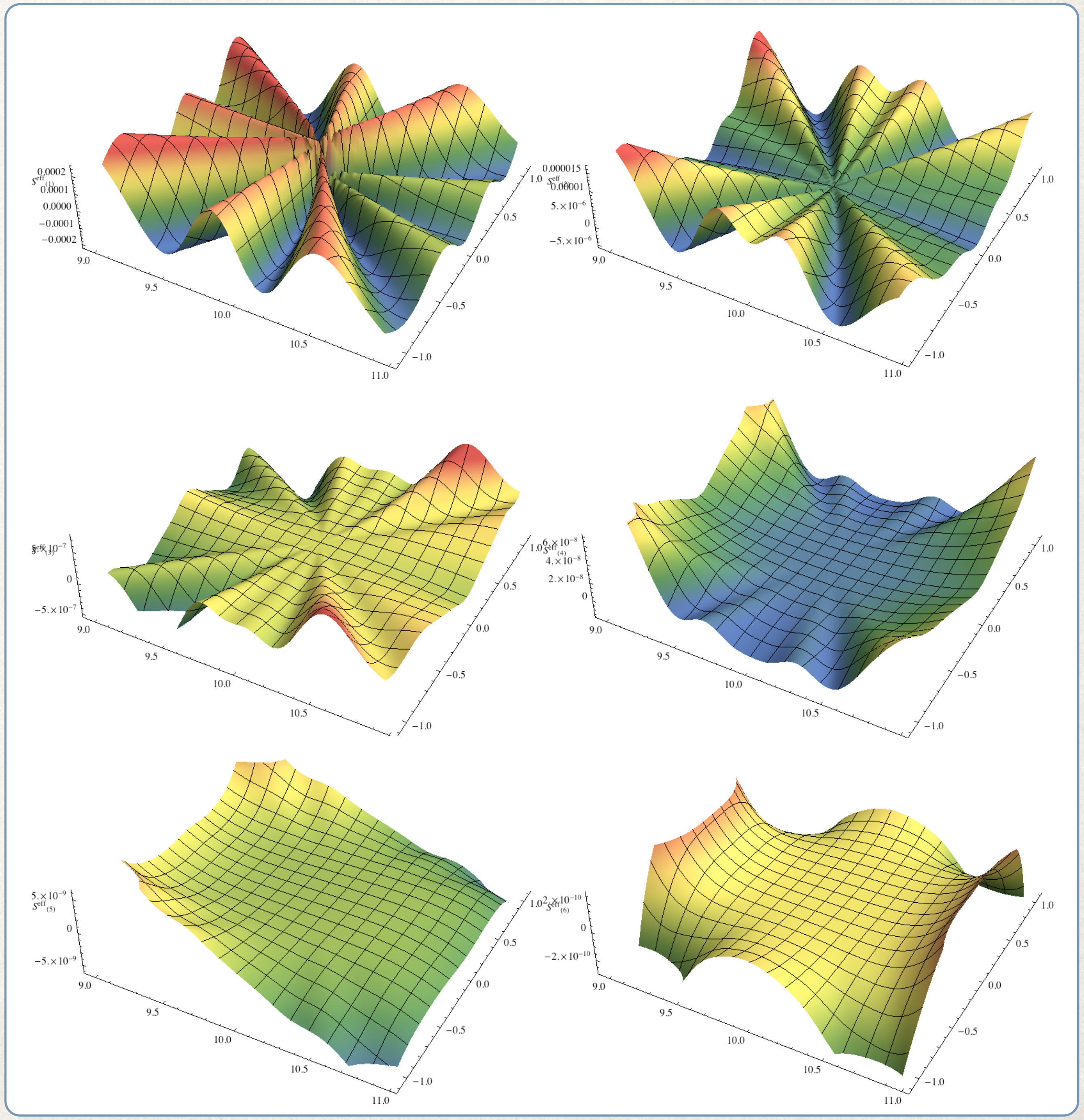
First Order Residual Field



Wardell & Warburton, Phys. Rev. D 92, 084019

Smoothness of Effective Source

- If $\bar{h}^1_{\mathrm{S}}$ is exactly the Detweiler-Whiting singular field, Φ^{R} is a solution of the homogeneous wave equation.
- If $\bar{h}^1_{\mathrm{S}}$ is only approximately the Detweiler-Whiting singular field, then the equation for $\bar{h}^1_{\mathrm{R}}$ has an effective source, $S^{\mathrm{eff}}_{\mu\nu}[h^1_{\mathrm{S}}]$
- $S^{\mathrm{eff}}_{\mu\nu}[h^1_{\mathrm{S}}]$ finite, but of limited differentiability on worldline.



Wardell, Vega, Thornburg, Diener, Phys. Rev. D 85, 104044

Window function and Worldtube

- Detweiler-Whiting singular field defined through a Hadamard form which is not defined globally.
- Need to introduce a method for restricting the singular field to a region near the particle.
- Two equivalent [Phys. Rev. D 89, 044046] approaches: window function and worldtube.

- Window function: multiply the singular field by a function which is 1 at the particle and goes to 0 far away:

$$\Box\Phi^{\mathrm{R}} = -\Box(W\Phi^{\mathrm{S}})$$

- Worldtube: Solve for regular field inside, outside solve for retarded field. Boundary condition at edge of tube:

$$\bar{h}^2_{\mathrm{R}} = \bar{h}^2_{\mathrm{ret}} - \bar{h}^2_{\mathrm{S}}$$

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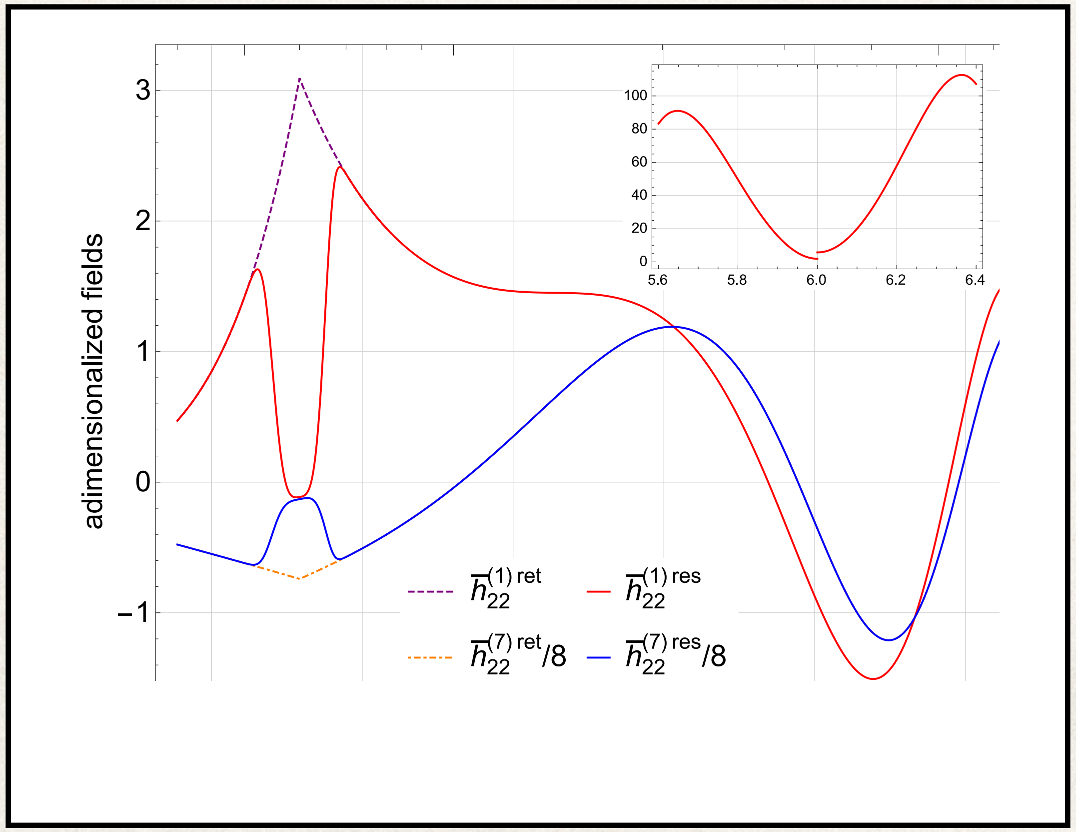

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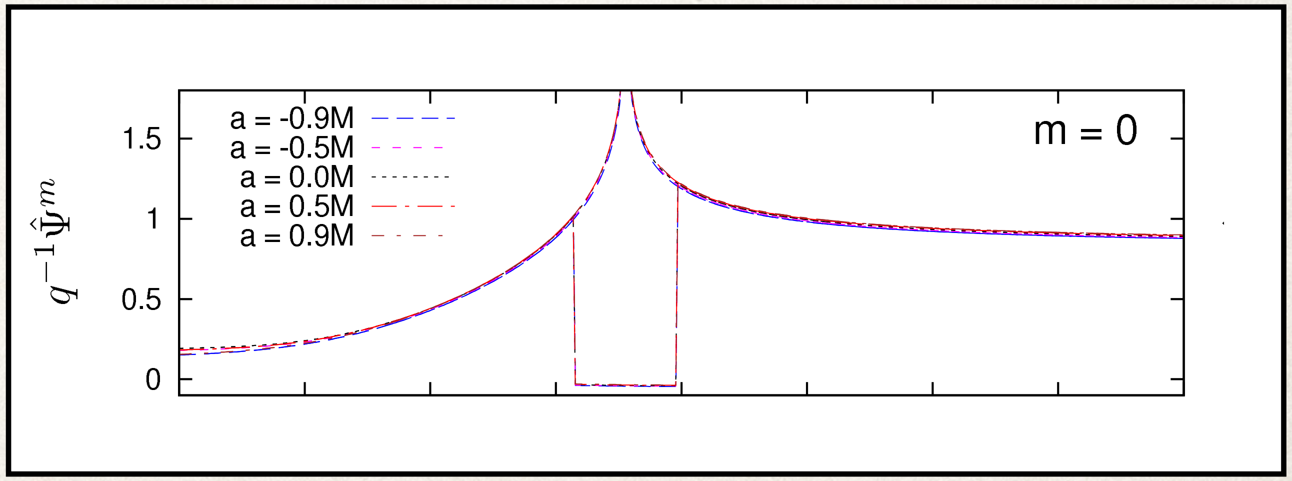

Effective Source at First Order: Results - Numerical Techniques

Authors	Reference	Method	Case
Barack & Golbourn	Phys. Rev. D 76, 044020	2+1 Worldtube	Scalar
Vega & Detweiler	Phys. Rev. D 77, 084008	1+1 Window	Scalar
Lousto & Nakano	Class. Quantum Grav. 25, 145018	2+1 Window (not based on Detweiler-Whiting)	Scalar
Barack, Golbourn, Sago	Phys. Rev. D 76, 124036	2+1 Worldtube	Gravity
Vega, Diener, Tichy, Detweiler	Phys.Rev. D 80, 084021	3+1 Window	Scalar
Warburton & Wardell	Phys. Rev. D 89, 044046	Frequency Domain Worldtube & Window	Schwarzschild scalar
Wardell & Warburton	Phys. Rev. D 92, 084019	Frequency Domain Worldtube & Window	Schwarzschild gravity

Effective Source at First Order: Results - Analytical Techniques

Authors	Reference	Case
Vega, Wardell, Diener	Class. Quantum Grav. 28 134010	Scalar, geodesic
Wardell, Vega, Thornburg, Diener	Phys. Rev. D 85, 104044	Kerr, gravity, geodesic
Heffernan, Ottewill, Warburton, Wardell, Diener	Unpublished, see Anna Heffernan's talk	Non-geodesic/accelerated scalar

Effective Source at First Order: Results - Applications

Authors	Reference	Details
Dolan & Barack	Phys. Rev. D 83, 024019	2+1 Worldtube, Schwarzschild, Scalar, Circular
Dolan, Barack, Wardell	Phys. Rev. D 84, 084001	2+1 Worldtube, Kerr, Scalar, Circular
Dolan & Barack	Phys. Rev. D 87, 084066	2+1 Worldtube, Schwarzschild, Gravity, Circular Instability in non-radiative modes
Dolan, Barack & Wardell	Unpublished Dolan Capra 16	2+1 Worldtube, Kerr, Gravity, Circular Instability in non-radiative modes
Diener, Vega, Wardell, Detweiler	Phys. Rev. Lett. 108, 191102	3+1 Window Function, Schwarzschild, Scalar, Self-consistent Evolution
Vega, Wardell, Diener, Cupp, Haas	Phys. Rev. D 88, 084021	3+1 Window Function, Schwarzschild, Scalar, Eccentric Geodesic
Thornburg & Wardell	Phys. Rev. D 95, 084043	2+1 Worldtube, Kerr, Scalar Highly-eccentric Geodesics, Self-force Wiggles

Effective Source: Interesting Future Directions at First Order

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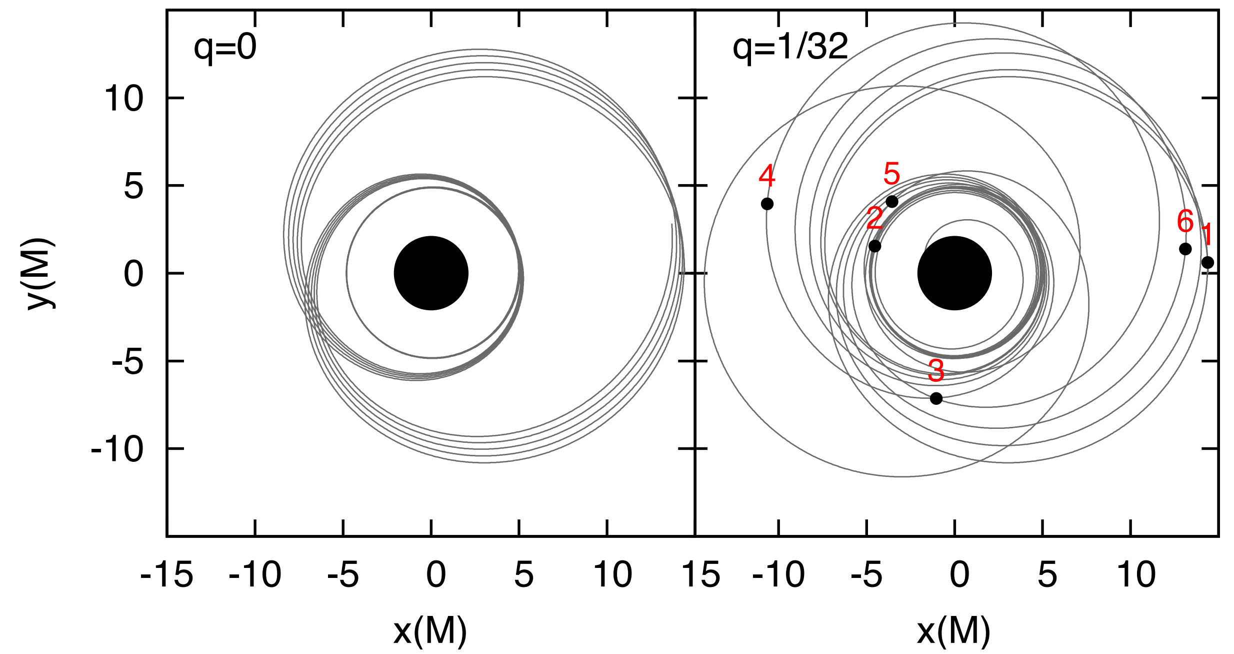
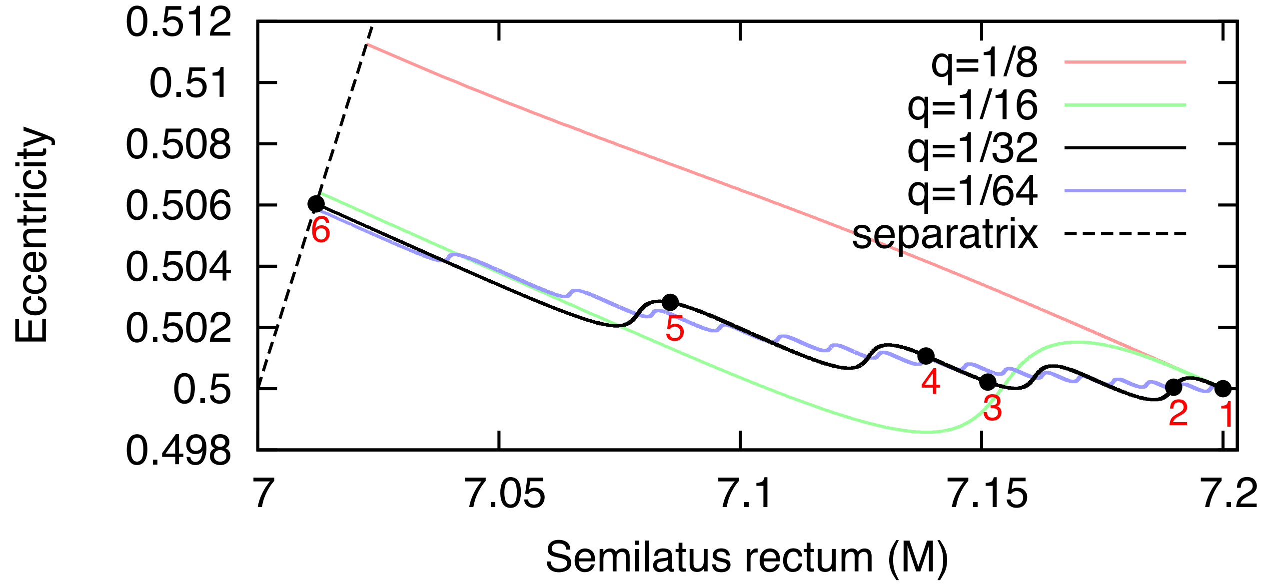
- Self-consistent evolutions (see talks by Peter Diener, Niels Warburton).





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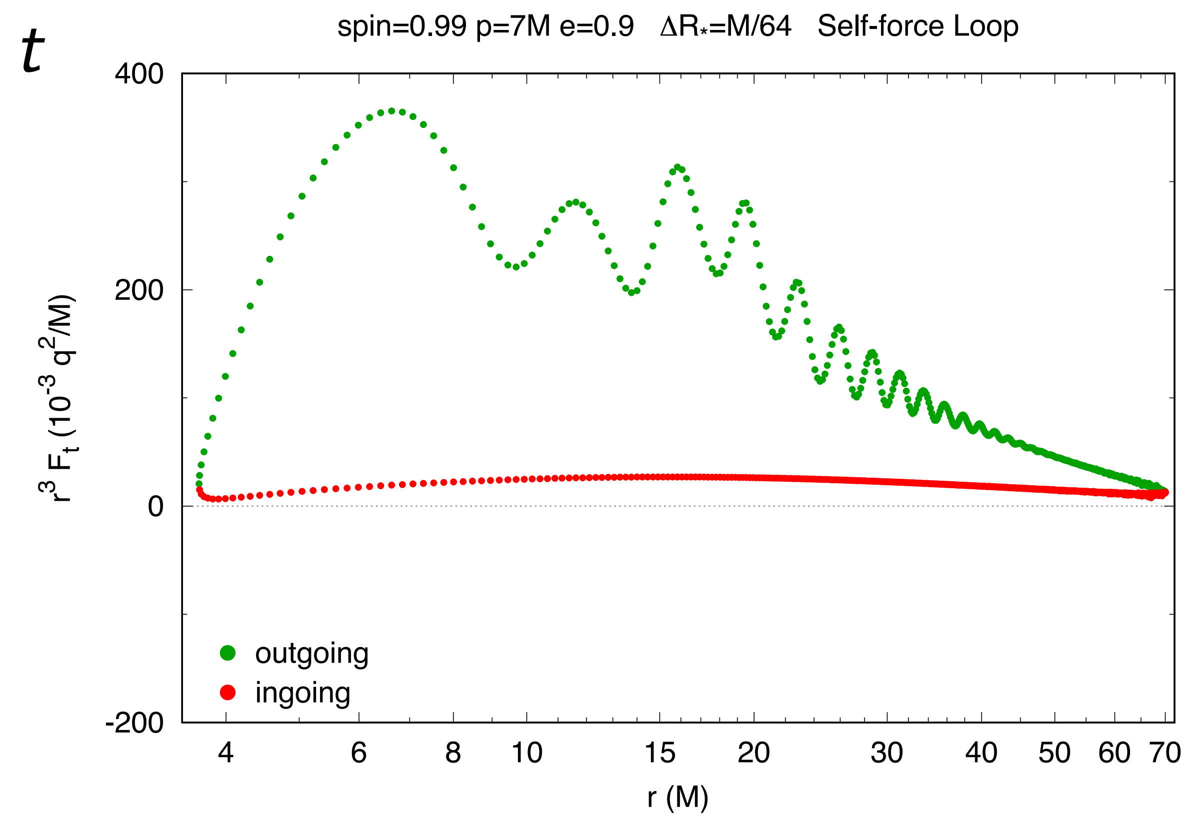

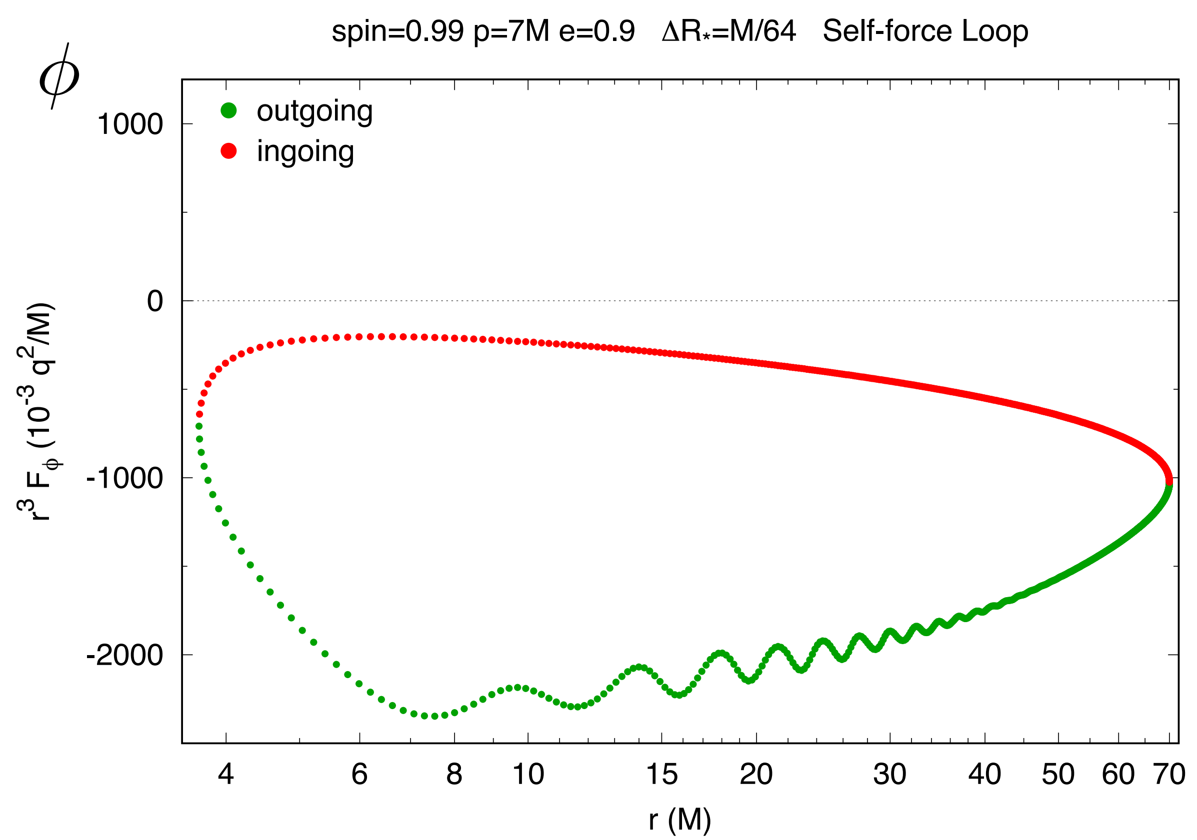

Effective Source: Interesting Future Directions at First Order

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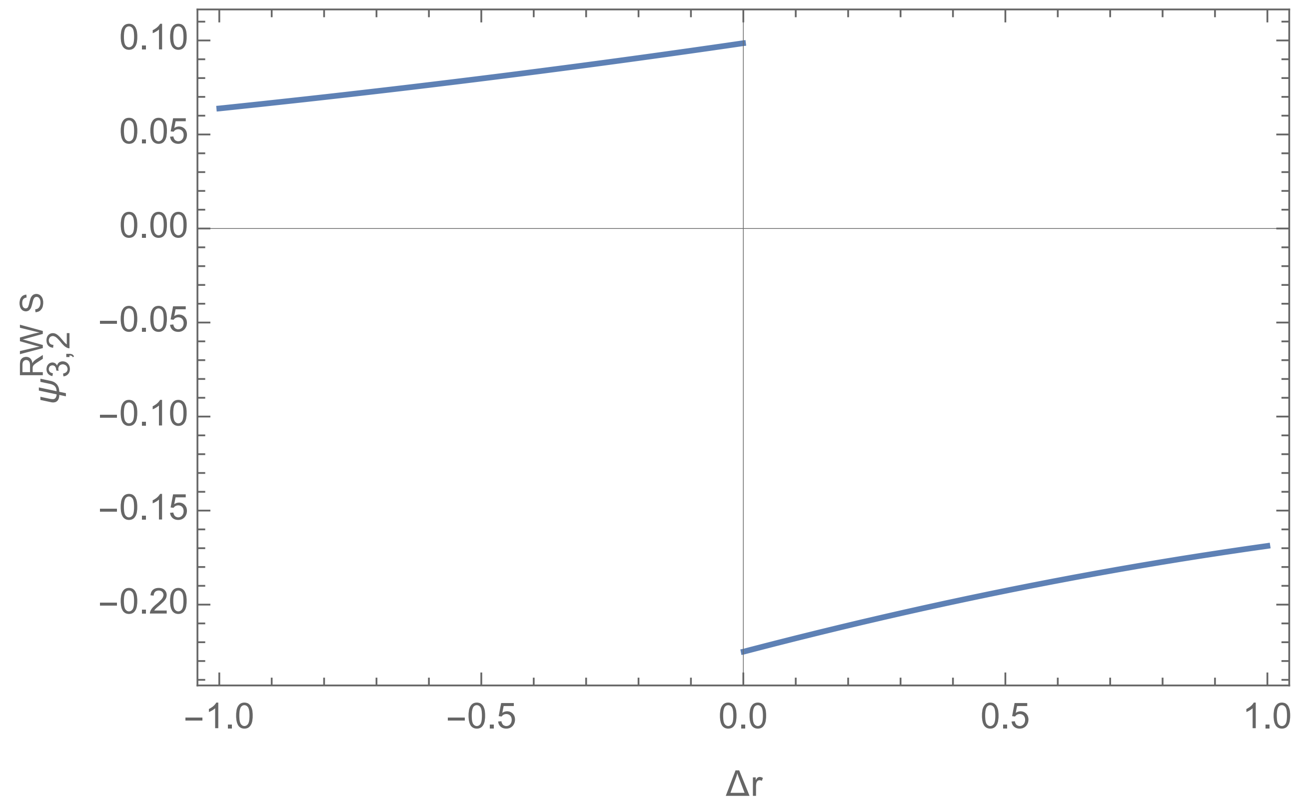

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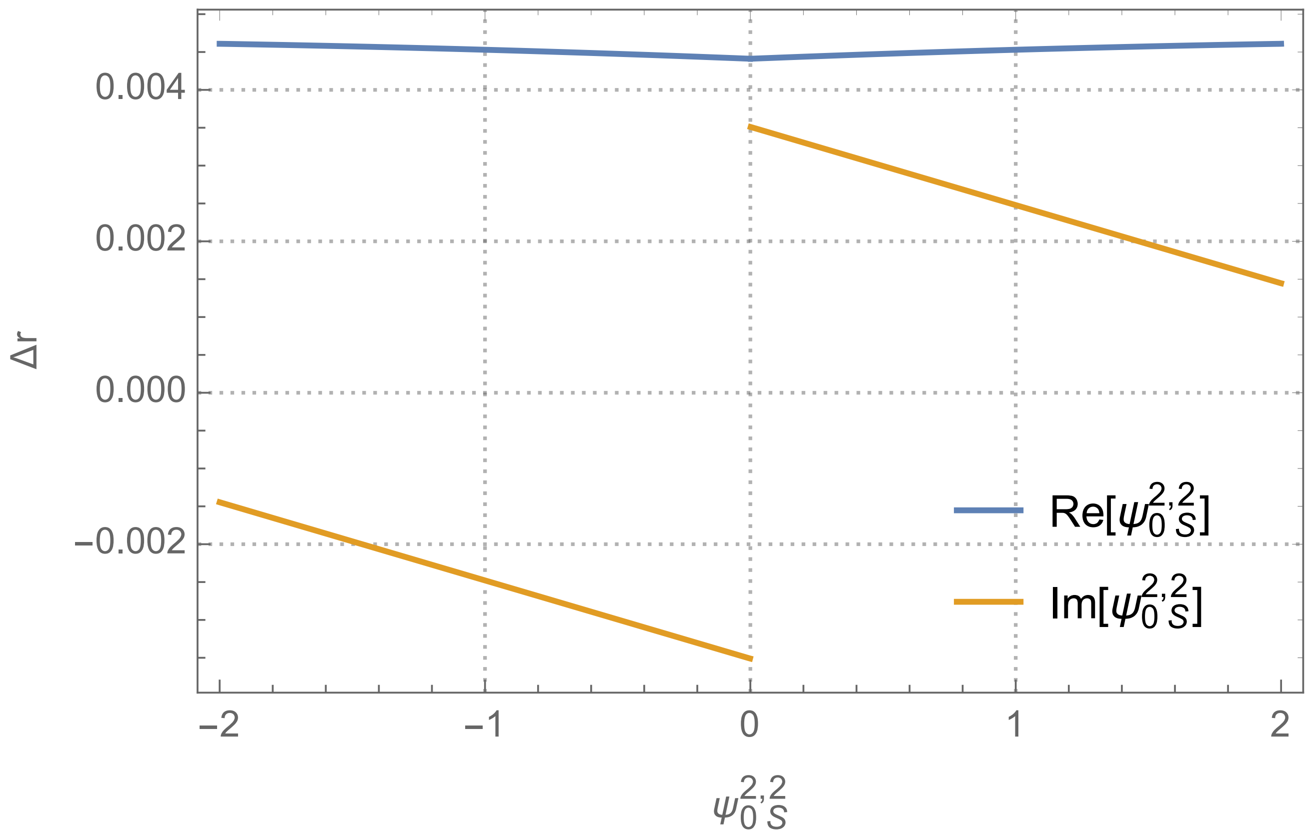

Effective Source: Interesting Future Directions at First Order

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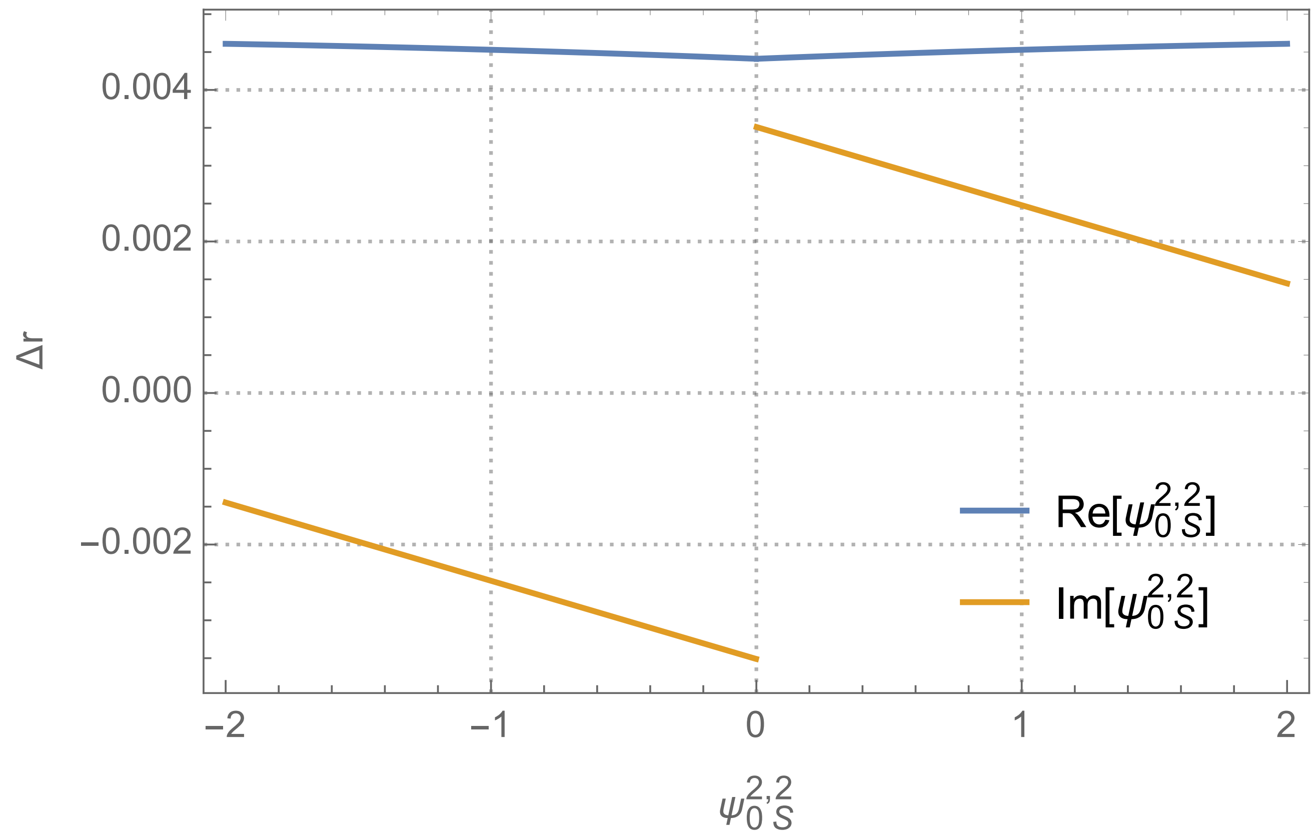
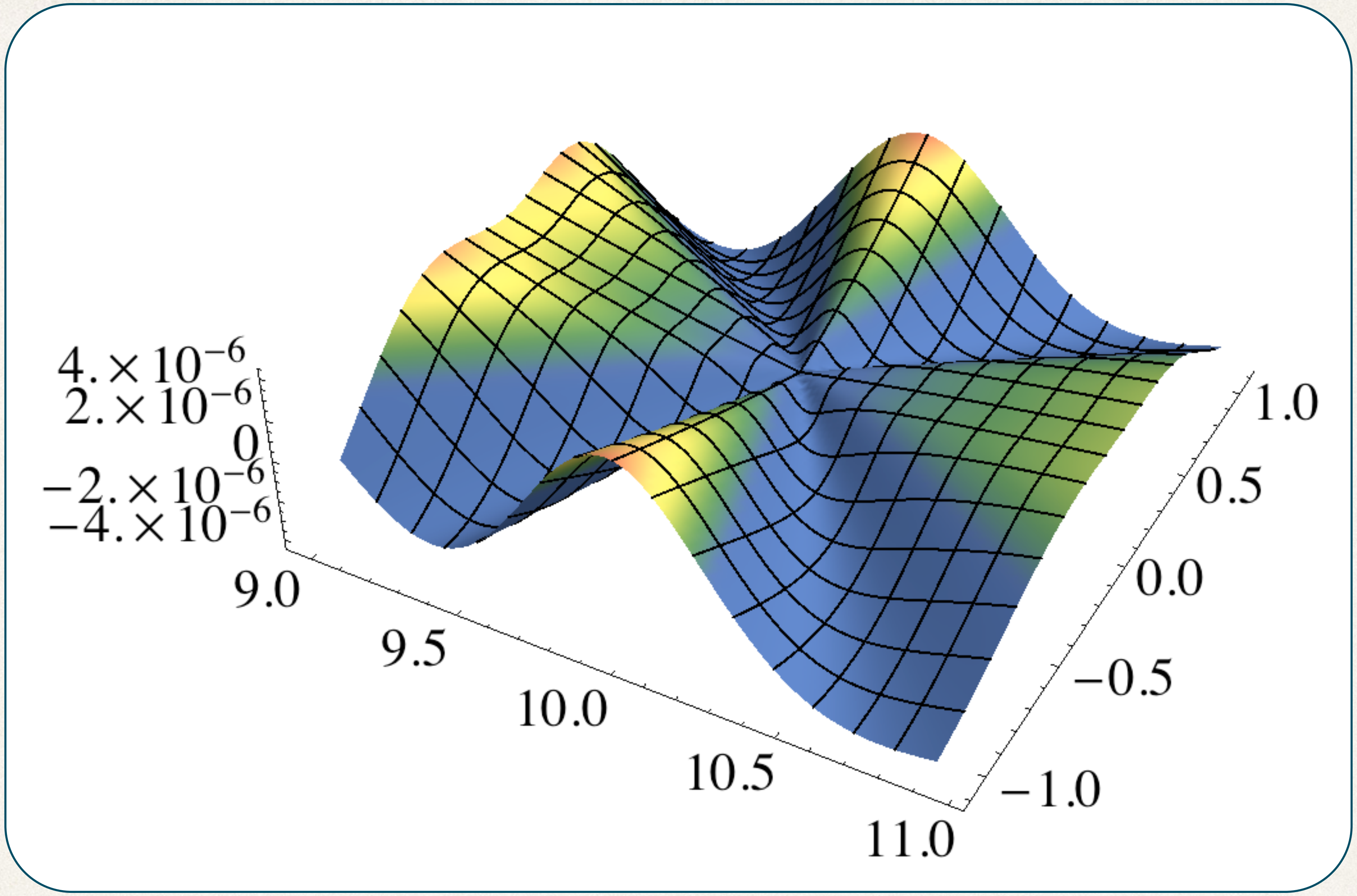
- Effective source for Hertz potential (Ψ, see talk by Leor Barack)?





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$$\boxed{\begin{aligned} E_{\mu\nu}[h^1_{\rm ret}] &= -16\pi \bar{T}_{\mu\nu}[\gamma] \\ E_{\mu\nu}[h^2_{\rm R}] &= 2\delta^2 R_{\mu\nu}[h^1_{\rm ret}, h^1_{\rm ret}] - E_{\mu\nu}[h^2_{\rm S}] \end{aligned}}$$

$$\boxed{\begin{aligned} \delta^2 R_{\mu\nu}[h,h] \equiv & -\tfrac{1}{2} h^{\mu\nu}\left(2h_{\mu(\alpha;\beta)\nu} - h_{\alpha\beta;\mu\nu} - h_{\mu\nu;\alpha\beta}\right) \\ & + \tfrac{1}{4} h^{\mu\nu}{}_{;\alpha} h_{\mu\nu;\beta} + \tfrac{1}{2} h^{\mu}{}_{\beta}{}^{;\nu}\left(h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}\right) \\ & - \tfrac{1}{2} \bar{h}^{\mu\nu}{}_{;\nu}\left(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}\right) \end{aligned}}$$

Second Order Self-force: Timeline

Full Booking Details

BOOKING REFERENCE: D1J8VK

Date	Flight No	Route	Depart	Arrive
Wed 07 Nov 2012 Operated by Flybe Essentials	BE389	**Dublin to Southampton**	20:20	21:45
Fri 09 Nov 2012 Operated by Flybe Essentials	BE388	**Southampton to Dublin**	18:30	19:55

Dr. Barry Wardell

Flight	From	To	Seat	Baggage	Insurance	Advance Passenger Information (API)	Checked In	Change itinerary
BE389	Dublin	Southampton	--	0Kg		Not Required		
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Frequency domain effective source implementation   Attachments x

Niels Warburton <nielsw@gmail.com>   29/11/2012

to Leor, Adam, me

Hi all,

Barry and myself have been working on implementing the effective source approach in the FD and now have some results to share. As is customary we have initially considered a scalar charge moving in a circular orbit in Schwarzschild spacetime. Barry computed an effective source from his puncture and then, once we obtain the homogeneous radial fields in the usual way, we make use of the method of variation of parameters to compute the

Challenges at Second order

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- Second order gravitational self-force will require high accuracy $\Rightarrow$ Frequency domain.

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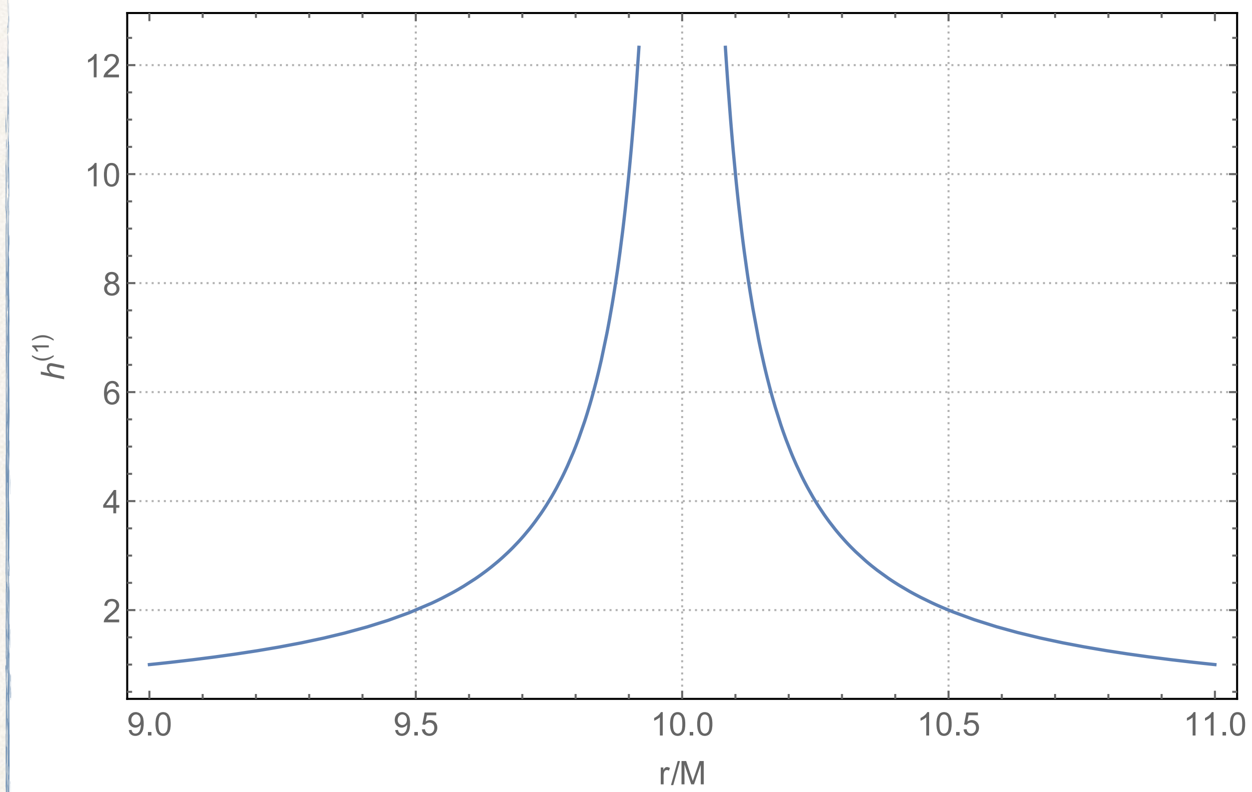
- Second order gravitational self-force will require high accuracy $\Rightarrow$ Frequency domain.
- Spherical harmonic modes at first order finite on world line $\Rightarrow$ mode-sum regularisation.



First order metric perturbation ~ $1/(r-r_0)$

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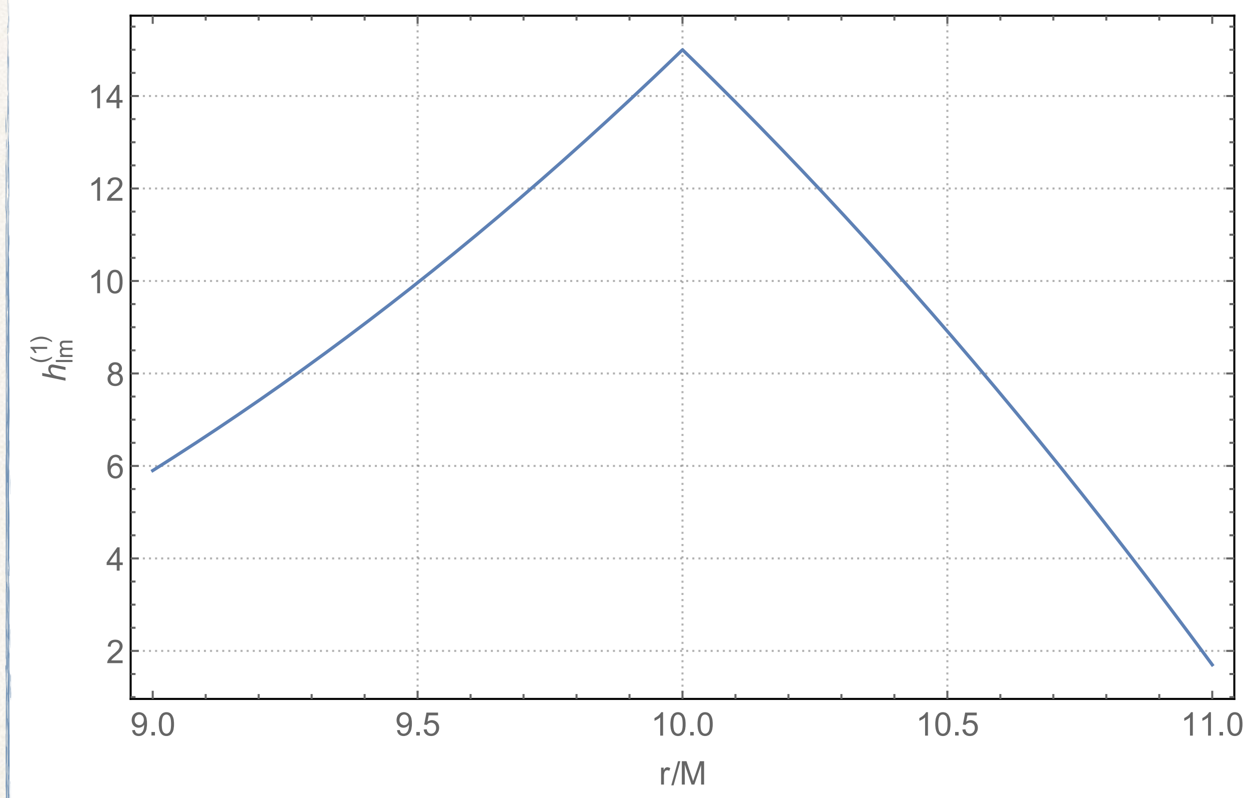
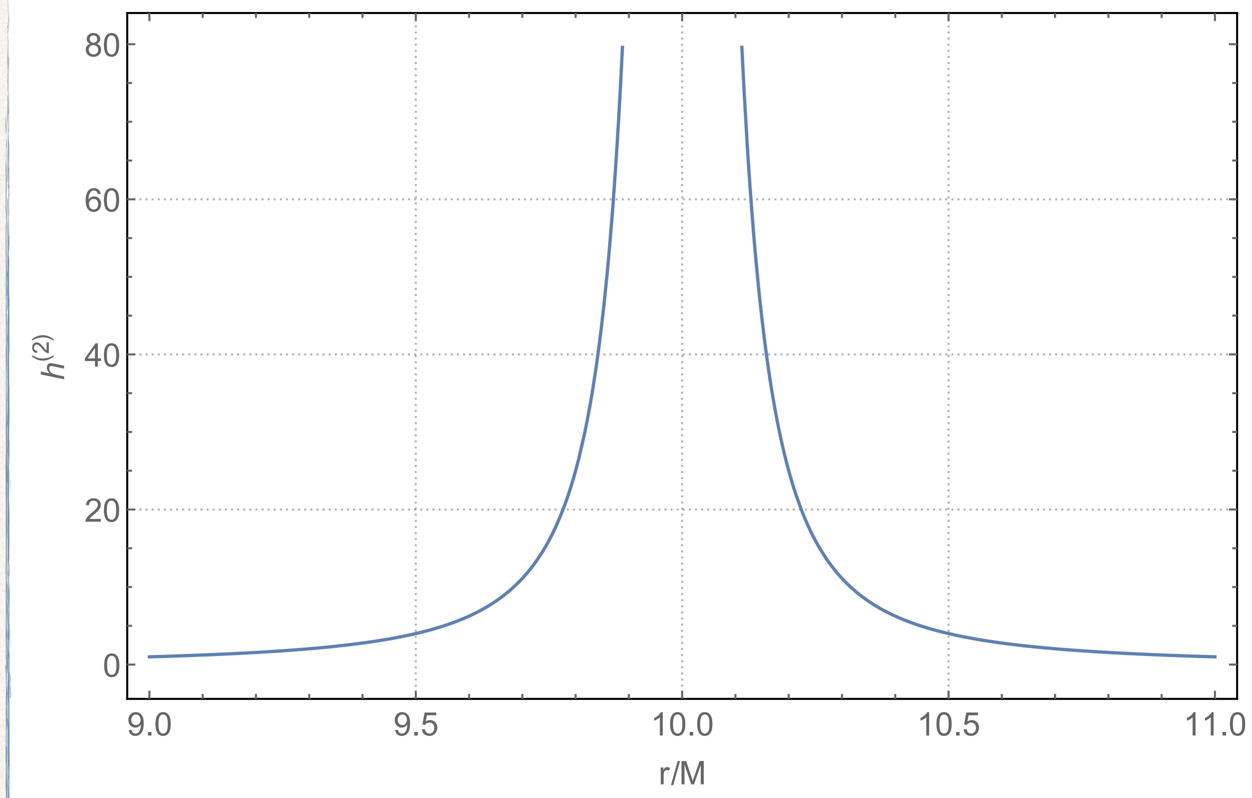


First order modes ~ $|r-r_0|/(r-r_0)$

Challenges at Second order

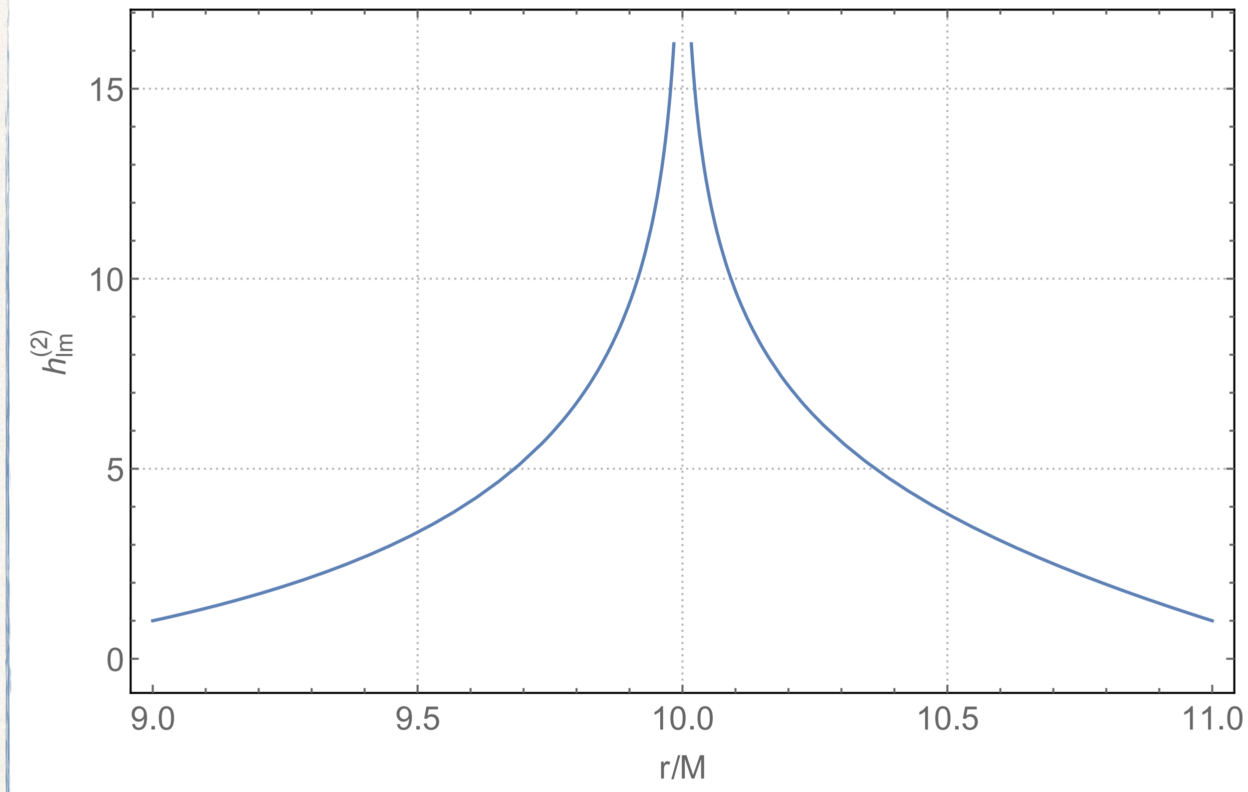
- Second order gravitational self-force will require high accuracy $\Rightarrow$ Frequency domain.
- Spherical harmonic modes at first order finite on world line $\Rightarrow$ mode-sum regularisation.
- Second order metric more singular.



Second order perturbation ~ $1/(r-r_0)^2$

Challenges at Second order

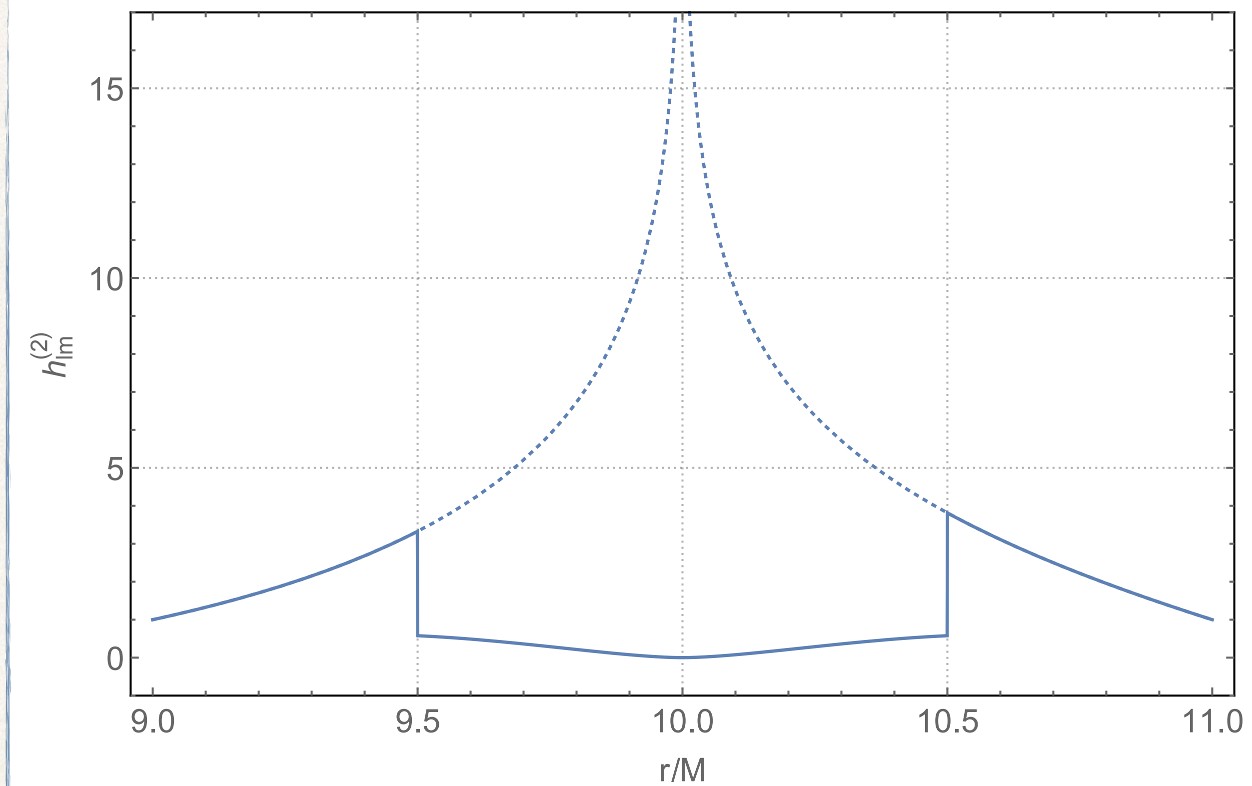
- Second order gravitational self-force will require high accuracy $\Rightarrow$ Frequency domain.
- Spherical harmonic modes at first order finite on world line $\Rightarrow$ mode-sum regularisation.
- Second order metric more singular.
- Second order modes diverge logarithmically.



Second order modes ~ log | r-r$_0$ |

Challenges at Second order

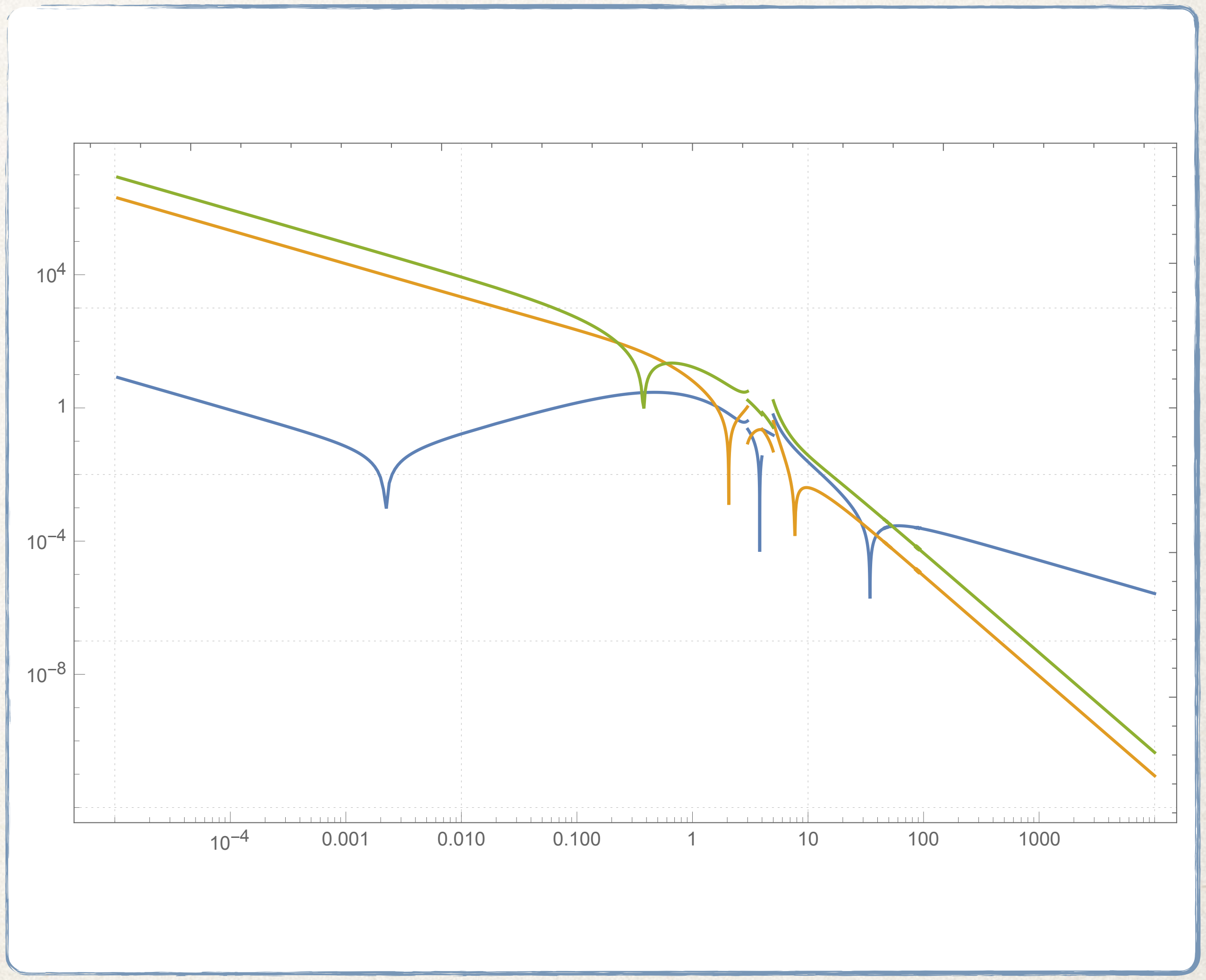
- Second order gravitational self-force will require high accuracy $\Rightarrow$ Frequency domain.
- Spherical harmonic modes at first order finite on world line $\Rightarrow$ mode-sum regularisation.
- Second order metric more singular.
- Second order modes diverge logarithmically.
- Avoid computing retarded field on world line $\Rightarrow$ effective source.



Second order modes ~ log | r-r_0 |

Challenges at Second order

- Also have problems on the horizon and at infinity, where the source does not fall off fast enough.
- Need punctures near horizon and infinity in order to obtain a finite result.
- Derive punctures at infinity from a post-Minskowski expansion.
- Punctures near horizon currently derived ad-hoc by requiring that they match the singular behaviour.



Nonlinear gravitational self-force. I. Field outside a small body

Adam Pound[1]
[1]*School of Mathematics, University of Southampton, Southampton, United Kingdom, SO17 1BJ*
(Dated: September 5, 2012)

A small extended body moving through an external spacetime $g_{\alpha\beta}$ creates a metric perturbation $h_{\alpha\beta}$, which forces the body away from geodesic motion in $g_{\alpha\beta}$. The foundations of this effect, called the gravitational self-force, are now well established, but concrete results have mostly been limited to linear order. Accurately modeling the dynamics of compact binaries requires proceeding to nonlinear orders. To that end, I show how to obtain the metric perturbation outside the body

Nonlinear gravitational self-force: second-order equation of motion

Adam Pound
Mathematical Sciences and STAG Research Centre,
University of Southampton, Southampton, United Kingdom, SO17 1BJ
(Dated: May 25, 2017)

When a small, uncharged, compact object is immersed in an external background spacetime, at zeroth order in its mass it moves as a test particle in the background. At linear order, its own gravitational field alters the geometry around it, and it moves instead as a test particle in a certain effective metric satisfying the linearized vacuum Einstein equation. In the letter [Phys. Rev. Lett. 109, 051101 (2012)], using a method of matched asymptotic expansions, I showed that the same statement holds true at second order: if the object's leading-order spin and quadrupole moment vanish, then through second order in its mass it moves on a geodesic of a certain smooth, locally causal vacuum metric defined in its local neighbourhood. Here I present the complete details of the derivation of that result. In addition, I extend the result, which had previously been derived in

A practical, covariant puncture for second-order self-force calculations

Adam Pound and Jeremy Miller
Mathematical Sciences, University of Southampton, Southampton, United Kingdom, SO17 1BJ
(Dated: May 5, 2014)

Accurately modeling an extreme-mass-ratio inspiral requires knowledge of the second-order gravitational self-force on the inspiraling small object. Recently, numerical puncture schemes have been formulated to calculate this force, and their essential analytical ingredients have been derived from first principles. However, the *puncture*, a local representation of the small object's self-field, in each of these schemes has been presented only in a local coordinate system centered on the small object, while a numerical implementation will require the puncture in coordinates covering the entire numerical domain. In this paper we provide an explicit covariant self-field as a local expansion in

Conservative effect of the second-order gravitational self-force on quasicircular orbits in Schwarzschild spacetime

Adam Pound
Mathematical Sciences, University of Southampton, Southampton, United Kingdom, SO17 1BJ
(Dated: October 14, 2014)

A compact object moving on a quasicircular orbit about a Schwarzschild black hole gradually spirals inward due to the dissipative action of its gravitational self-force. But in addition to driving the inspiral, the self-force has a conservative piece. Within a second-order self-force formalism, I derive a second-order generalization of Detweiler's redshift variable, which provides a gauge-invariant measure of conservative effects on quasicircular orbits. I sketch a frequency-domain numerical

Applying the effective-source approach to frequency-domain self-force calculations

Niels Warburton[1] and Barry Wardell[2,1]
[1]*School of Mathematical Sciences and Complex & Adaptive Systems Laboratory,*
University College Dublin, Belfield, Dublin 4, Ireland
[2]*Department of Astronomy, Cornell University, Ithaca, NY 14853, USA*
(Dated: 28th October 2014)

The equations of motion of a point particle interacting with its own field are defined in terms of a certain regularized self-field. Two of the leading methods for computing this regularized field are the mode-sum and effective-source approaches. In this work we unite these two distinct regularization schemes by generalizing traditional frequency-domain mode-sum calculations to incorporate effective-source techniques. For a toy scalar-field model we analytically compute an appropriate puncture field from which the regularized residual field can be calculated. To demonstrate the

Applying the effective-source approach to frequency-domain self-force calculations: Lorenz-gauge gravitational perturbations

Barry Wardell[1,2] and Niels Warburton[3,2]
[1]*Department of Astronomy, Cornell University, Ithaca, NY 14853, USA*
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University College Dublin, Belfield, Dublin 4, Ireland
[3]*MIT Kavli Institute for Astrophysics and Space Research,*
Massachusetts Institute of Technology, Cambridge, MA 02139, USA

With a view to developing a formalism that will be applicable at second perturbative order, we devise a new practical scheme for computing the gravitational self-force experienced by a point mass moving in a curved background spacetime. Our method works in the frequency domain and employs the effective-source approach, in which a distributional source for the retarded metric perturbation is replaced with an effective source for a certain regularized self-field. A key ingredient of the calculation is the analytic determination of an appropriate puncture field from which the effective source and regularized residual field can be calculated. In addition to its application in our effective-

Second-order perturbation theory: problems on large scales

Adam Pound
Mathematical Sciences, University of Southampton, Southampton, United Kingdom, SO17 1BJ
(Dated: October 20, 2015)

In general-relativistic perturbation theory, a point mass accelerates away from geodesic motion due to its gravitational self-force. Because the self-force is small, one can often approximate the motion as geodesic. However, it is well known that self-force effects accumulate over time, making the geodesic approximation fail on long timescales. It is less well known that this failure at large times translates to a failure at large distances as well. At second perturbative order, two large-distance pathologies arise: spurious secular growth and infrared-divergent retarded integrals. Both stand in the way of practical computations of second-order self-force effects.
Utilizing a simple flat-space scalar toy model, I develop methods to overcome these obstacles. The

Second-order perturbation theory: the problem of infinite mode coupling

Jeremy Miller,[1] Barry Wardell,[2,3] and Adam Pound[1]
[1]*Mathematical Sciences and STAG Research Centre,*
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(Dated: August 25, 2016)

Second-order self-force computations, which will be essential in modeling extreme-mass-ratio inspirals, involve two major new difficulties that were not present at first order. One is the problem of large scales, discussed in [Phys. Rev. D 92, 104047 (2015)]. Here we discuss the second difficulty, which occurs instead on small scales: if we expand the field equations in spherical harmonics, then because the first-order field contains a singularity, we require an arbitrarily large number of first-order modes to accurately compute even a single second-order mode. This is a generic feature

High Accuracy Numerical Methods

Applying the effective-source approach to frequency-domain self-force calculations

Niels Warburton[1] and Barry Wardell[2,1]

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High Accuracy Numerical Methods

Lorenz gauge

$$\boxed{\nabla_a h^{ab} = 0}$$

$$\Downarrow$$

$$\boxed{\Box\, \bar{h}_{ab} + 2C_a{}^c{}_b{}^d \bar{h}_{cd} = -16\pi T_{ab}}$$

Spherical-harmonic and Fourier mode decomposition

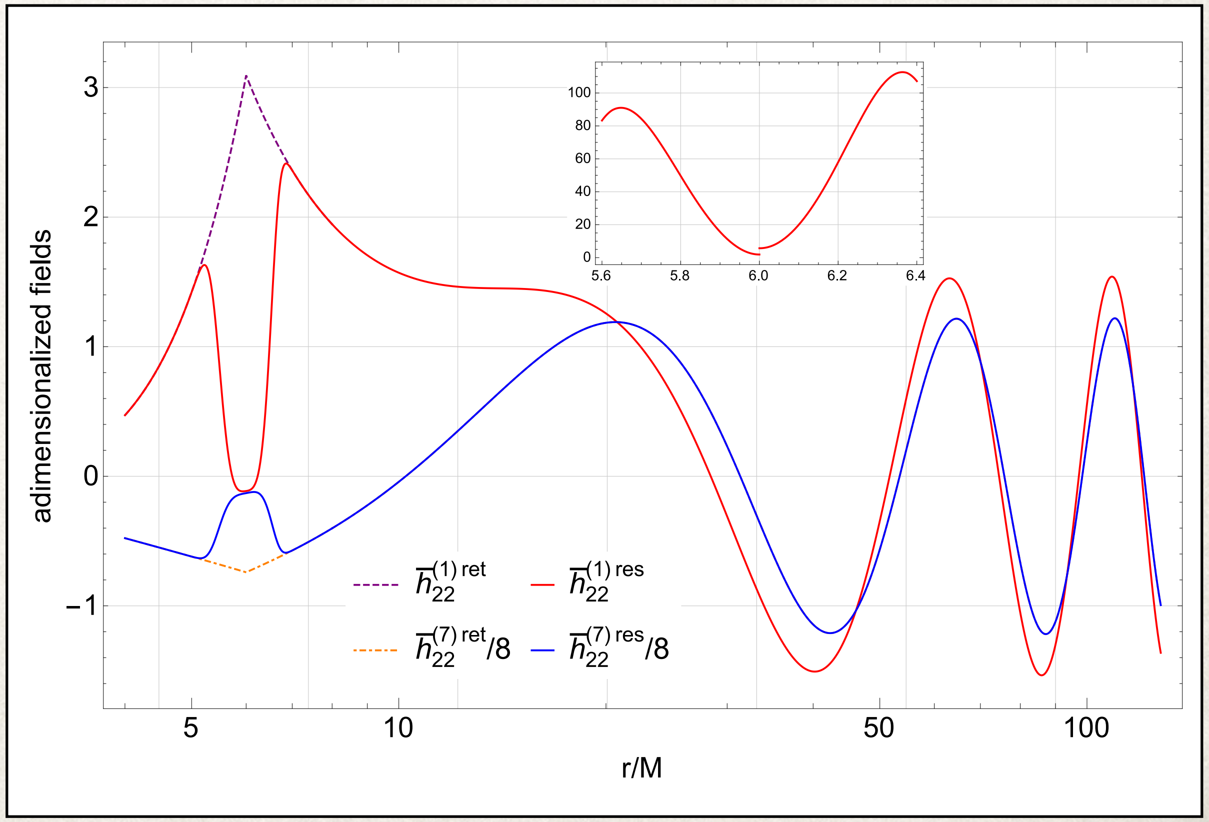
$$\boxed{\bar{h}^{(i)}_{\ell m}(t,r) = \frac{r}{\mu\, a^{(i)}_{\ell}} \int_0^{2\pi}\int_0^{\pi} \bar{h}_{\tau\kappa}\eta^{\tau\mu}\eta^{\kappa\nu} Y^{(i)\ell m *}_{\mu\nu}\, d\Omega}$$

$$\boxed{\bar{h}^{(i)}_{\ell m}(t,r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{h}^{(i)}_{\ell m}(\omega, r) e^{-i\omega t} dt}$$

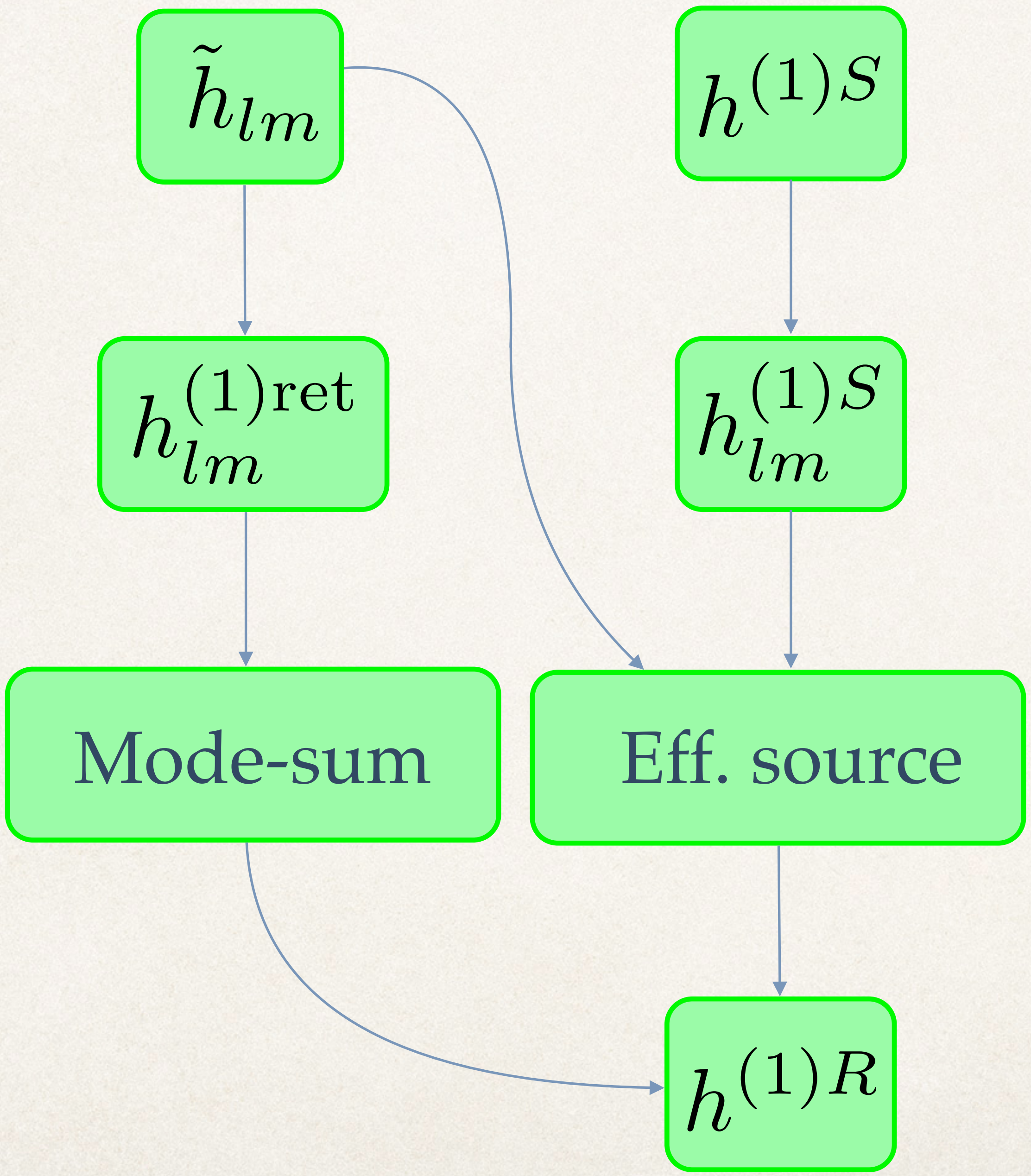
Frequency-domain Lorenz gauge equations with effective source

$$\boxed{\Box^{sc}_{\ell m}\, \bar{h}^{(i)}_{\ell m} - 4f^{-2}\mathscr{M}^{(i)}{}_{(j)}\bar{h}^{(j)}_{\ell m} = \mathcal{S}^{(i)}_{\ell m}}$$

Solve for $\bar{h}_{\mathrm{R}}$



Towards second order self-force



Second Order Punctures and Effective Source

Nonlinear gravitational self-force. I. Field outside a small body

Adam Pound[1]
[1] *School of Mathematics, University of Southampton, Southampton, United Kingdom, SO17 1BJ*
(Dated: September 5, 2012)

A small extended body moving through an external spacetime $g_{\alpha\beta}$ creates a metric perturbation $h_{\alpha\beta}$, which forces the body away from geodesic motion in $g_{\alpha\beta}$. The foundations of this effect, called the gravitational self-force, are now well established, but concrete results have mostly been limited to linear order. Accurately modeling the dynamics of compact binaries requires proceeding to nonlinear orders. To that end, I show how to obtain the metric perturbation outside the body

Nonlinear gravitational self-force: second-order equation of motion

Adam Pound
Mathematical Sciences and STAG Research Centre,
University of Southampton, Southampton, United Kingdom, SO17 1BJ
(Dated: May 25, 2017)

When a small, uncharged, compact object is immersed in an external background spacetime, at zeroth order in its mass it moves as a test particle in the background. At linear order, its own gravitational field alters the geometry around it, and it moves instead as a test particle in a certain effective metric satisfying the linearized vacuum Einstein equation. In the letter [Phys. Rev. Lett. 109, 051101 (2012)], using a method of matched asymptotic expansions, I showed that the same statement holds true at second order: if the object's leading-order spin and quadrupole moment vanish, then through second order in its mass it moves on a geodesic of a certain smooth, locally causal vacuum metric defined in its local neighbourhood. Here I present the complete details of the derivation of that result. In addition, I extend the result, which had previously been derived in

PHYSICAL REVIEW D **89,** 104020 (2014)

Practical, covariant puncture for second-order self-force calculations

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(Received 7 March 2014; published 13 May 2014)

Accurately modeling an extreme-mass-ratio inspiral requires knowledge of the second-order gravitational self-force on the inspiraling small object. Recently, numerical puncture schemes have been formulated to calculate this force, and their essential analytical ingredients have been derived from first principles. However, the "puncture," a local representation of the small object's self-field, in each of these schemes has been presented only in a local coordinate system centered on the small object, while a numerical implementation will require the puncture in coordinates covering the entire numerical domain. In this paper we provide an explicit covariant self-field as a local expansion in terms of Synge's world function. The self-field is written in the Lorenz gauge, in an arbitrary vacuum background, and in forms suitable for both self-consistent and Gralla-Wald-type representations of the object's trajectory. We illustrate the local expansion's utility by sketching the procedure of constructing from it a numerically practical puncture in any chosen coordinate system.

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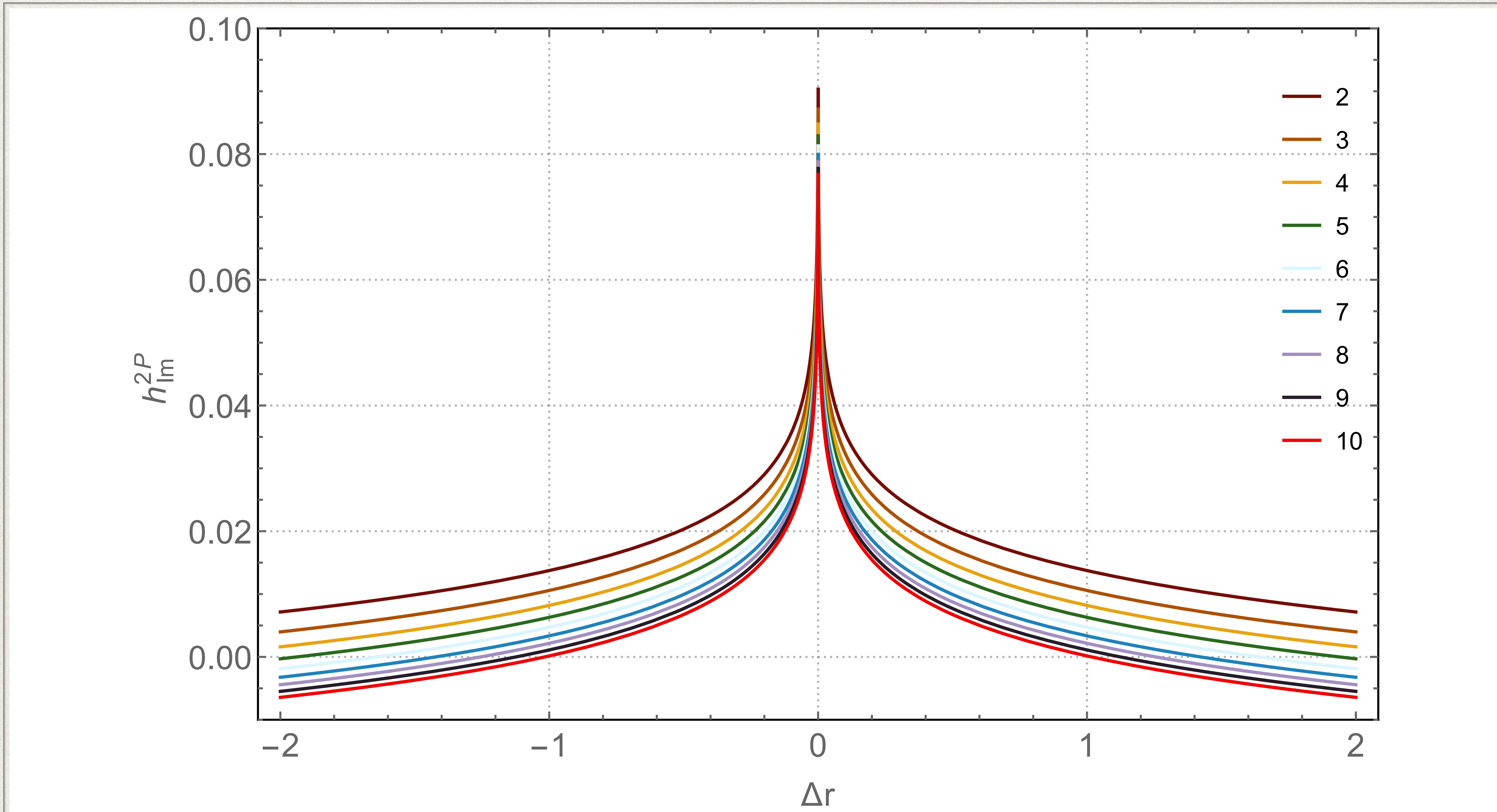
Second-order perturbation theory: problems on large scales

Adam Pound
Mathematical Sciences, University of Southampton, Southampton, United Kingdom, SO17 1BJ
(Dated: October 20, 2015)

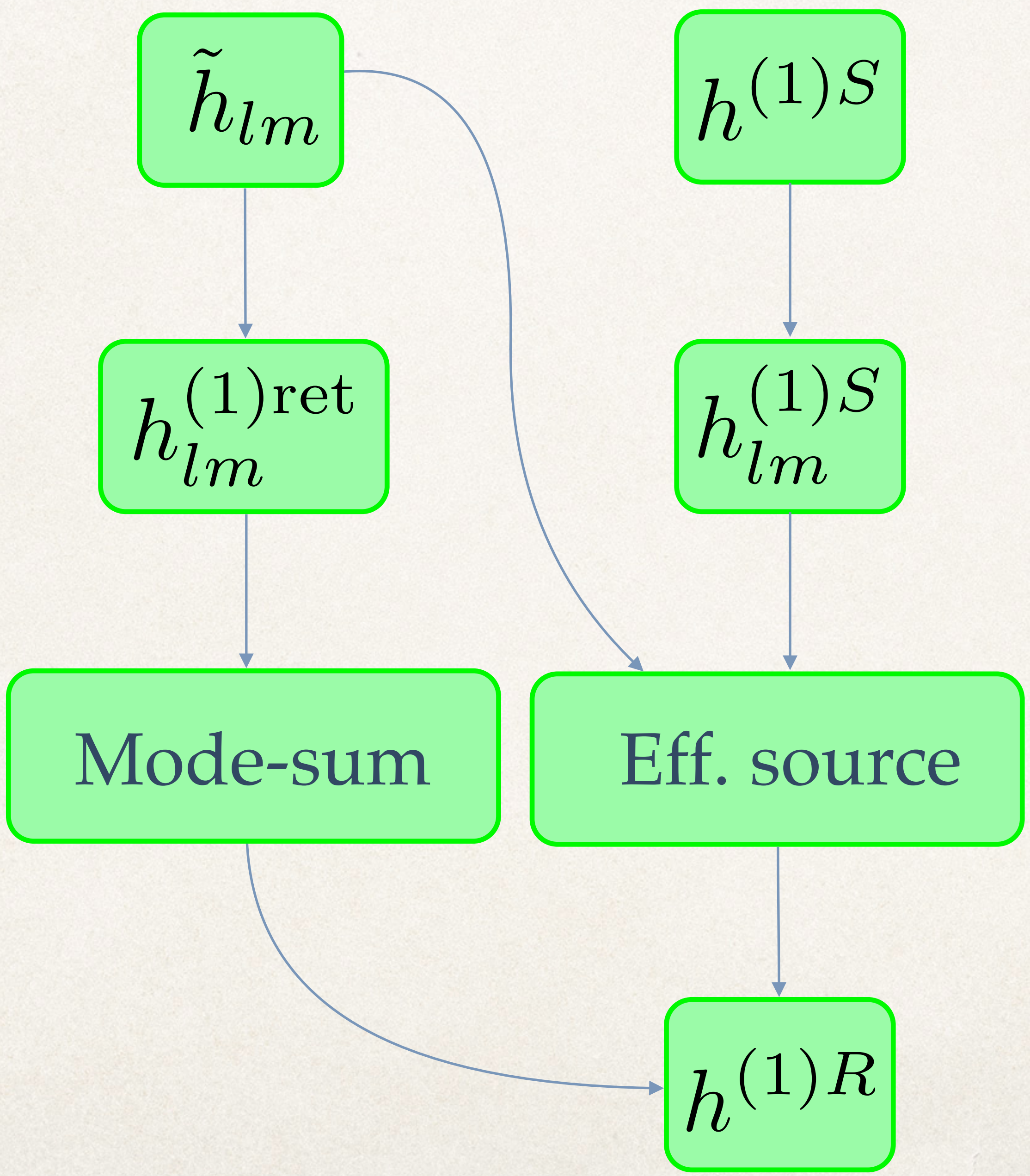
In general-relativistic perturbation theory, a point mass accelerates away from geodesic motion due to its gravitational self-force. Because the self-force is small, one can often approximate the motion as geodesic. However, it is well known that self-force effects accumulate over time, making the geodesic approximation fail on long timescales. It is less well known that this failure at large times translates to a failure at large distances as well. At second perturbative order, two large-distance pathologies arise: spurious secular growth and infrared-divergent retarded integrals. Both stand in the way of practical computations of second-order self-force effects.

Utilizing a simple flat-space scalar toy model, I develop methods to overcome these obstacles. The

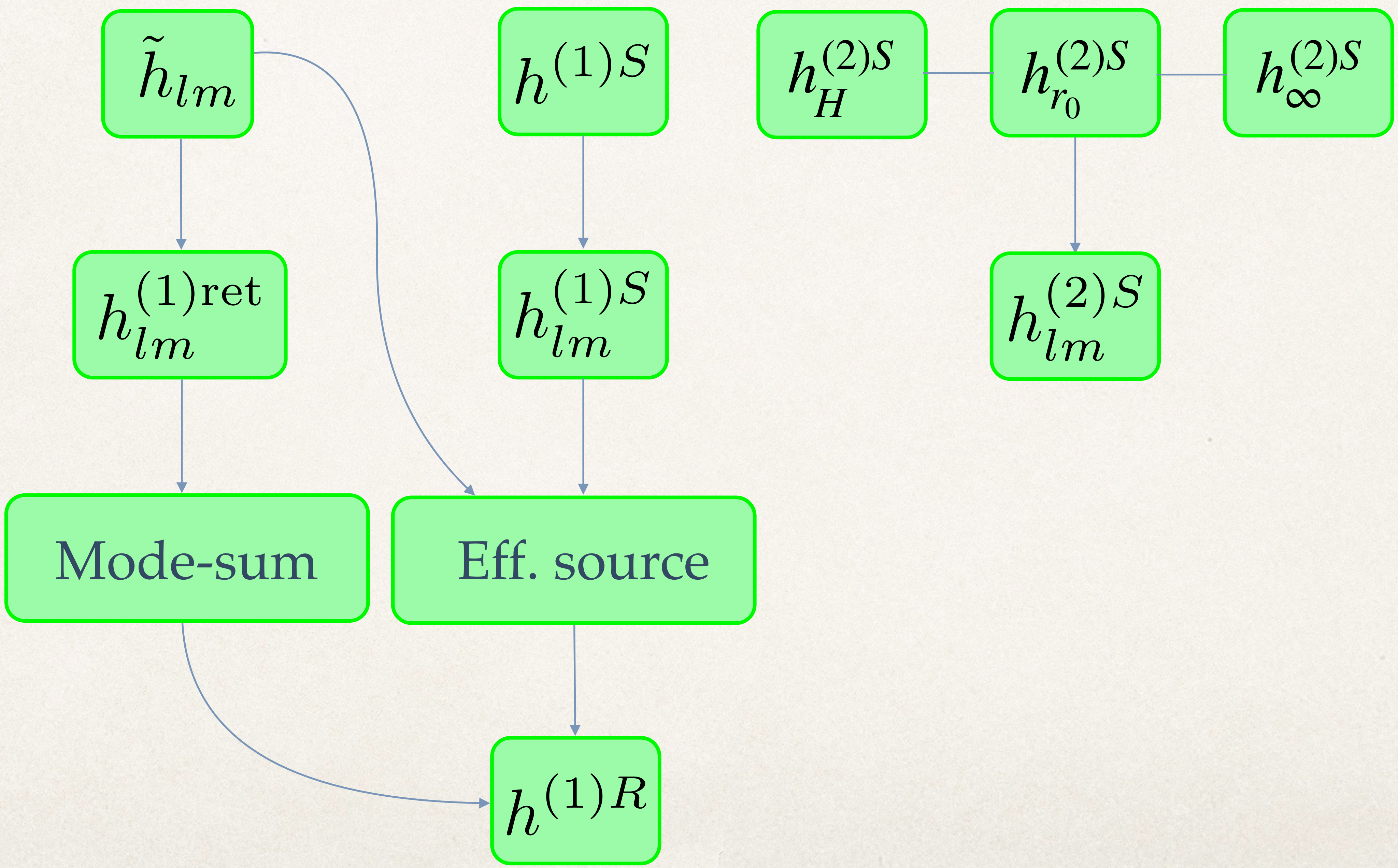
Second Order Puncture Modes



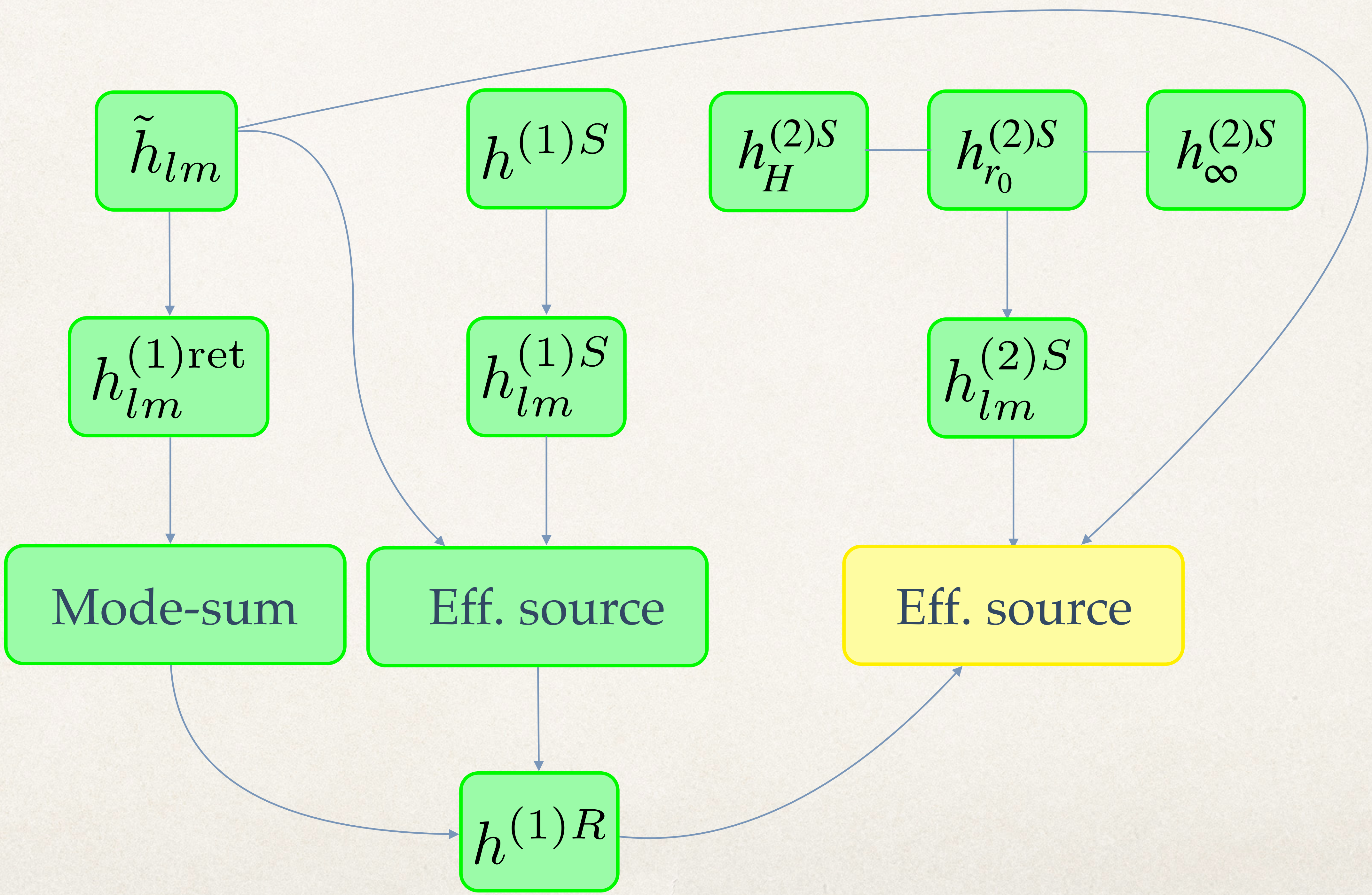
Towards second order self-force



Towards second order self-force



Towards second order self-force



Second order effective source

$$E_{\mu\nu}[h^1_{\text{ret}}] = -16\pi\bar{T}_{\mu\nu}[\gamma]$$

$$E_{\mu\nu}[h^2_{\text{R}}] = 2\delta^2 R_{\mu\nu}[h^1_{\text{ret}}, h^1_{\text{ret}}] - E_{\mu\nu}[h^2_{\text{S}}]$$

$$\begin{aligned}\delta^2 R_{\alpha\beta}[h,h] \equiv & -\tfrac{1}{2}h^{\mu\nu}\left(2h_{\mu(\alpha;\beta)\nu} - h_{\alpha\beta;\mu\nu} - h_{\mu\nu;\alpha\beta}\right) \\ & + \tfrac{1}{4}h^{\mu\nu}{}_{;\alpha}h_{\mu\nu;\beta} + \tfrac{1}{2}h^{\mu}{}_{\beta}{}^{;\nu}\left(h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}\right) \\ & - \tfrac{1}{2}\bar{h}^{\mu\nu}{}_{;\nu}\left(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}\right)\end{aligned}$$

Second order effective source

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& - \tfrac{1}{2} \bar{h}^{\mu\nu}{}_{;\nu}\left(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}\right)
\end{aligned}$$

Second order effective source

$$
\begin{aligned}
E_{\mu\nu}[h^1_{\rm ret}] &= -16\pi \bar{T}_{\mu\nu}[\gamma] \\
E_{\mu\nu}[h^2_{\rm R}] &= 2\delta^2 R_{\mu\nu}[h^1_{\rm ret}, h^1_{\rm ret}] - E_{\mu\nu}[h^2_{\rm S}]
\end{aligned}
$$

$$
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& - \tfrac{1}{2} \bar{h}^{\mu\nu}{}_{;\nu}\left(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}\right)
\end{aligned}
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Second order effective source

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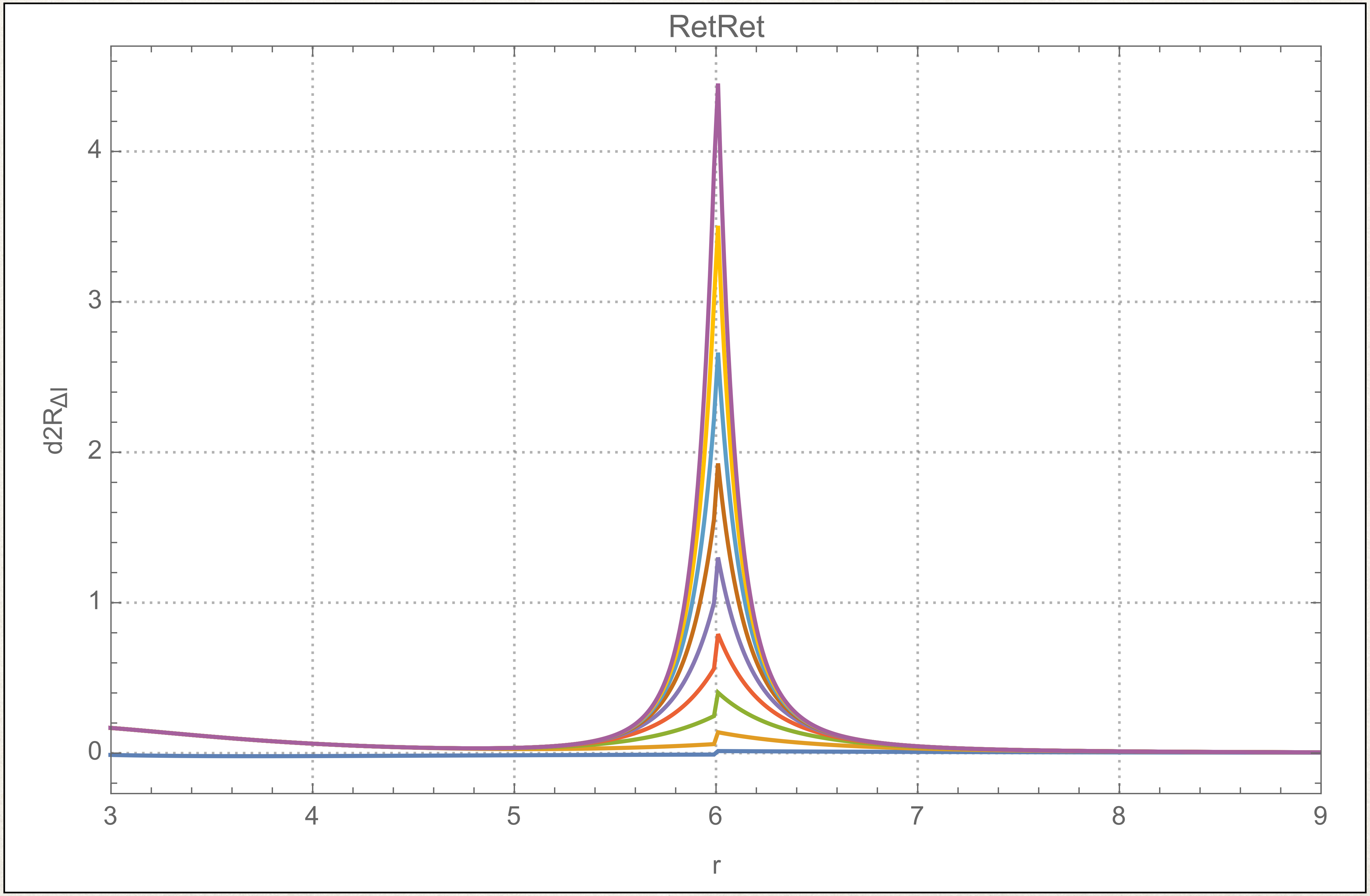
Second order Ricci tensor

$$\boxed{\delta^2 R_{\mu\nu}[h^1, h^1] \;=\; \sum_{i\ell m} \delta^2 R_{i\ell m}(r;\hat{r}) e^{-im\Omega t} Y^{i\ell m}_{\mu\nu}(r,\theta^A)}$$

$$\boxed{\delta^2 R_{i\ell m} = \sum_{\substack{i'\ell' m' \\ i''\ell'' m''}} \mathcal{D}^{i'\ell' m' i''\ell'' m''}_{i\ell m}\left[h_{1i'\ell' m'}, h_{1i''\ell'' m''}\right]}$$

Problem: infinite mode coupling



Mode coupling

Second-order perturbation theory: the problem of infinite mode coupling

Jeremy Miller,[1] Barry Wardell,[2,3] and Adam Pound[1]

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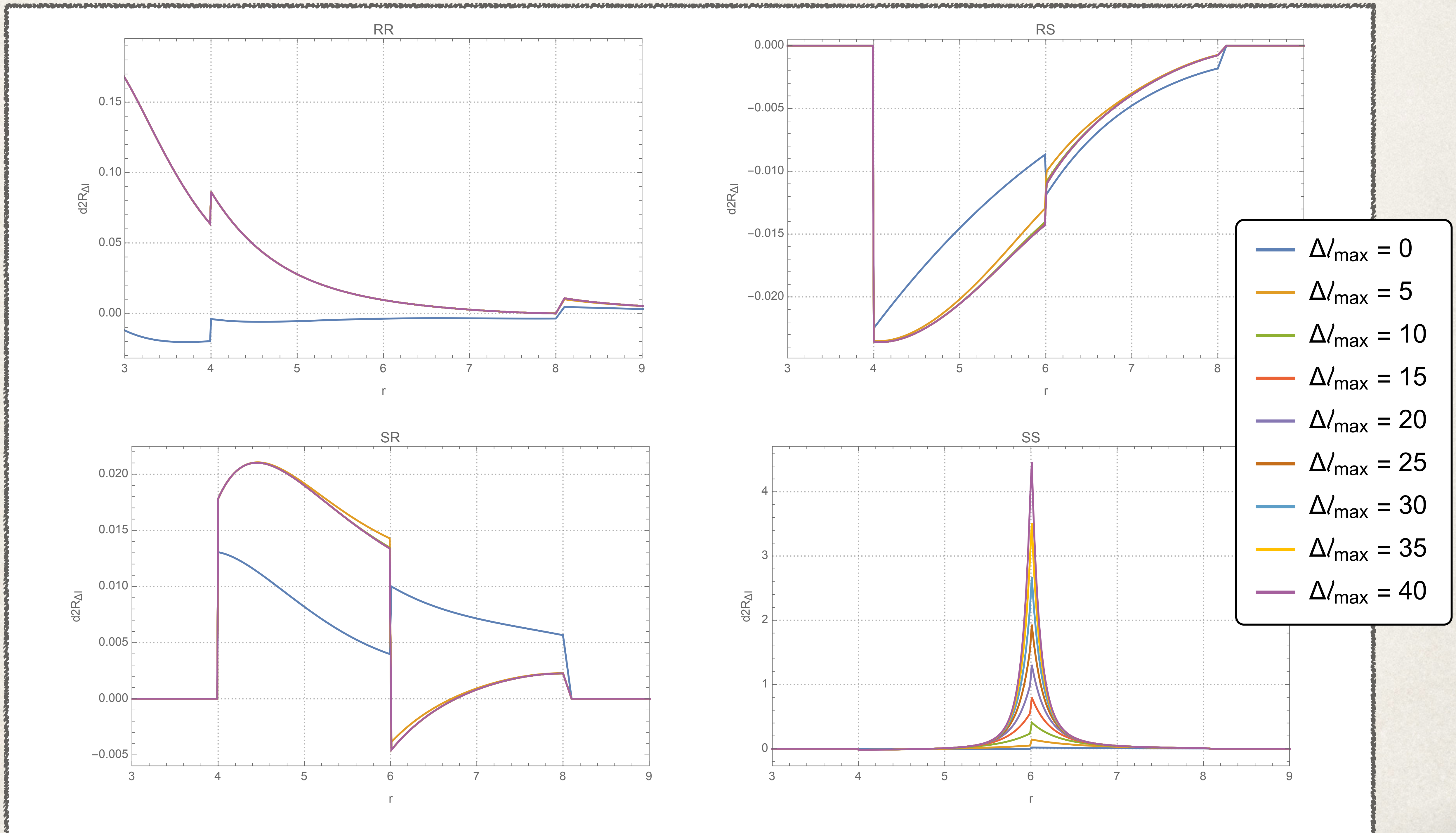
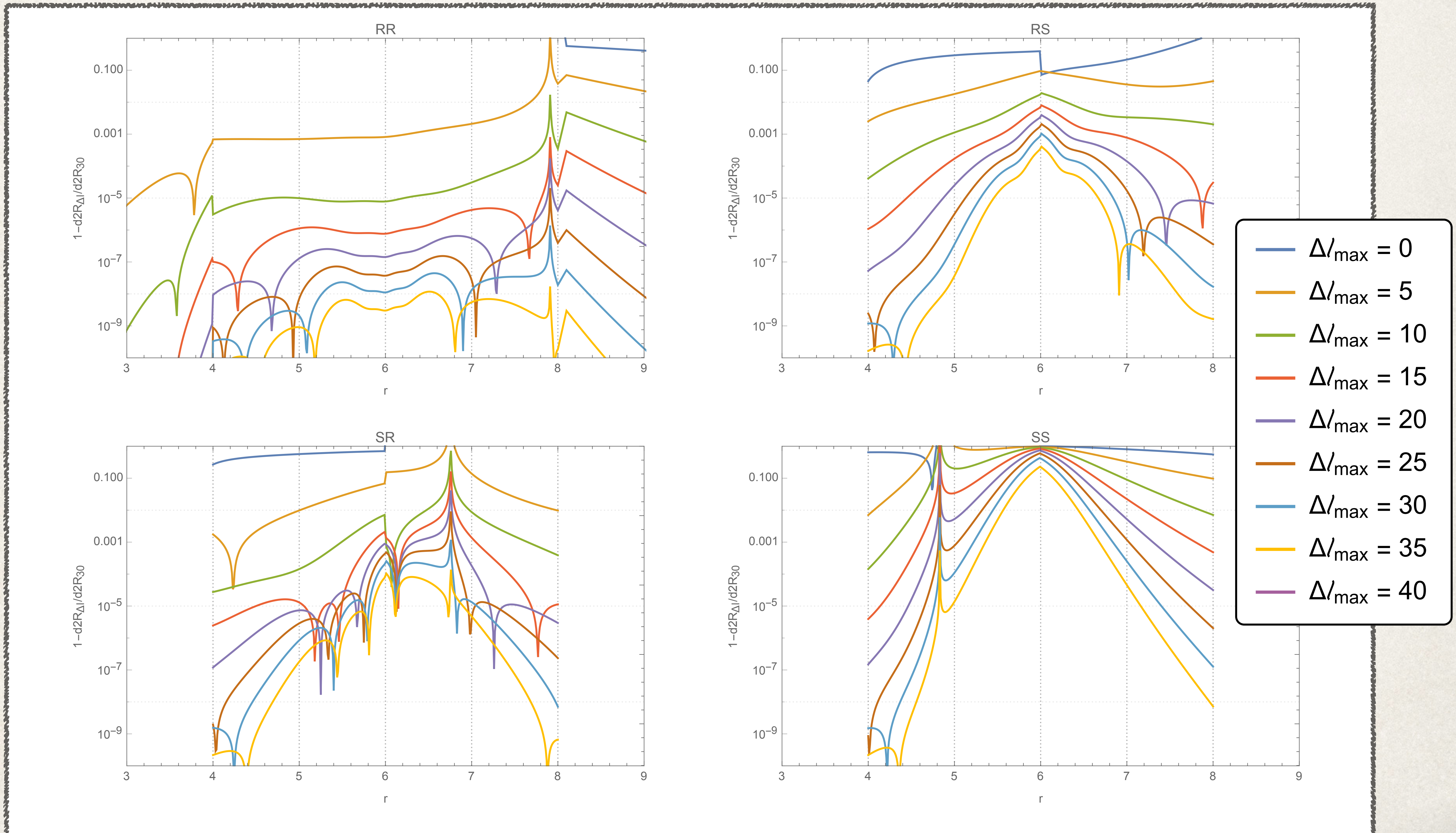
Mode coupling



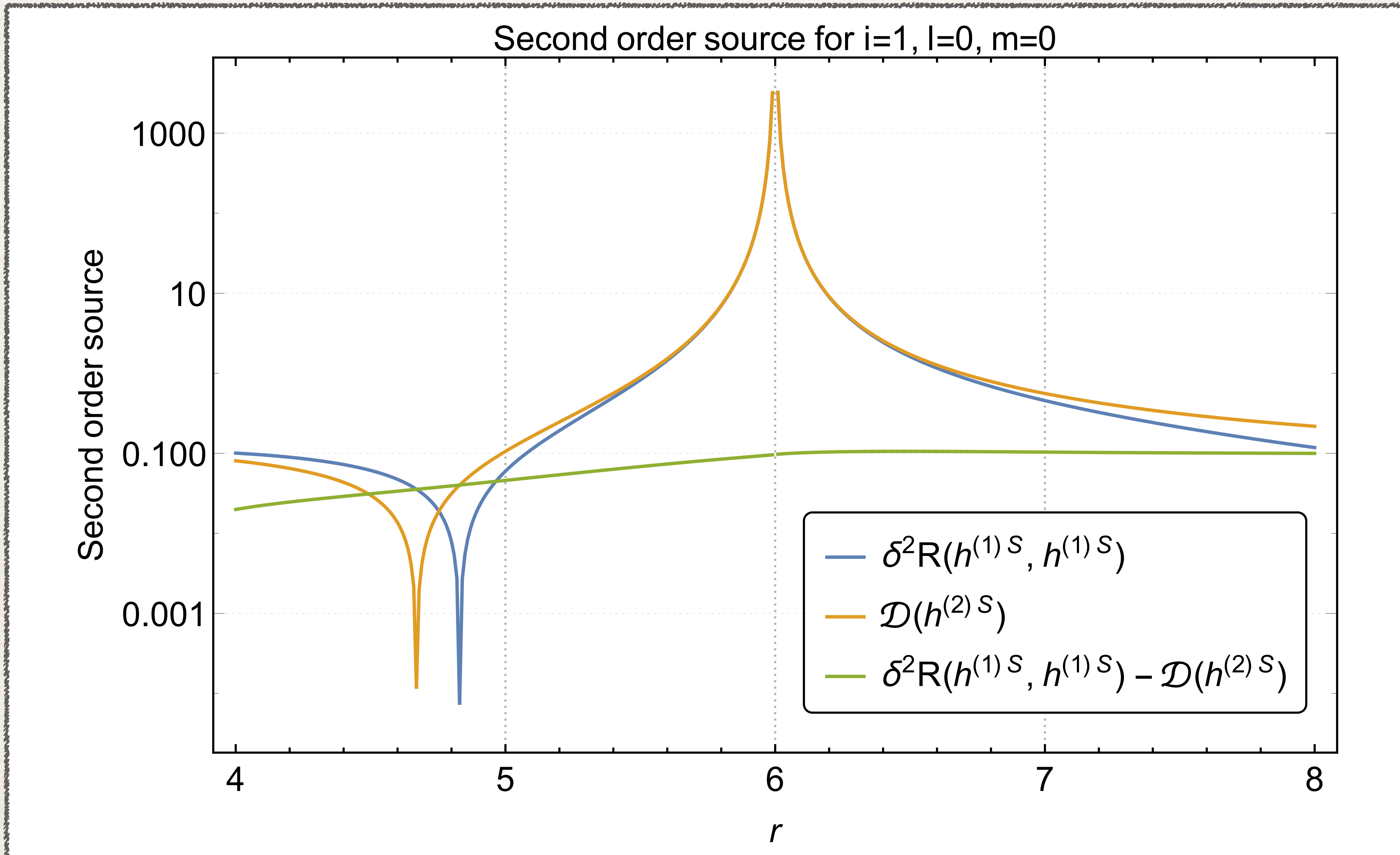
Mode coupling



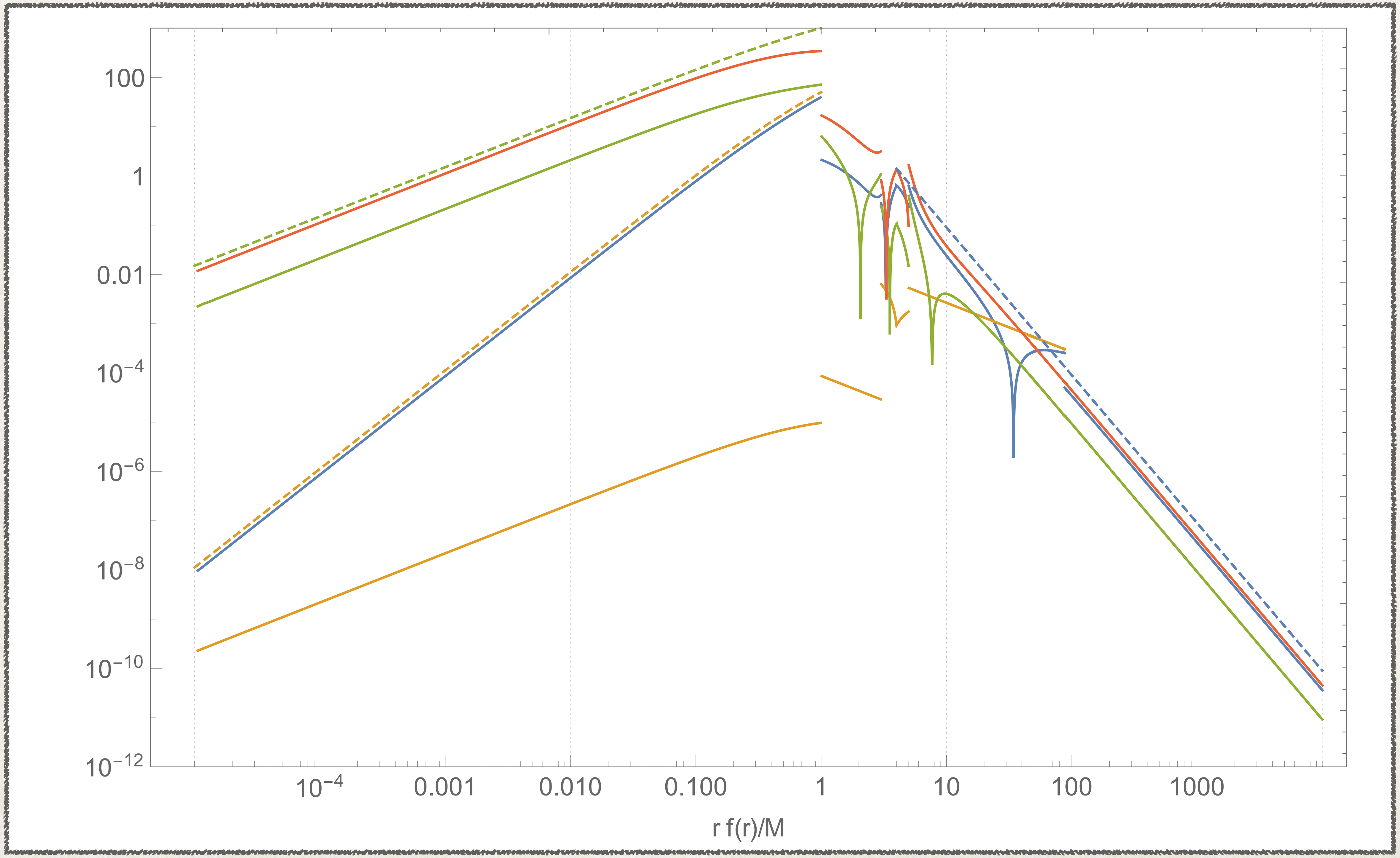
Second order Ricci tensor

$$
\begin{aligned}
\delta^2 R_{\alpha\beta}[h^{1\mathrm{ret}}, h^{1\mathrm{ret}}] = {}& \\
& \delta^2 R_{\alpha\beta}[h^{1\mathrm{R}}, h^{1\mathrm{R}}] \Longleftarrow \quad \text{mode coupling} \\
& + 2\delta^2 R_{\alpha\beta}[h^{1\mathrm{R}}, h^{1\mathrm{S}}] \Longleftarrow \quad \text{mode coupling} \\
& + \quad \delta^2 R_{\alpha\beta}[h^{1\mathrm{S}}, h^{1\mathrm{S}}] \Longleftarrow \quad \text{mode decomposition (c.f. } h^{\mathrm{S}2}\text{)}
\end{aligned}
$$

Second Order Effective Source



Second Order Effective Source



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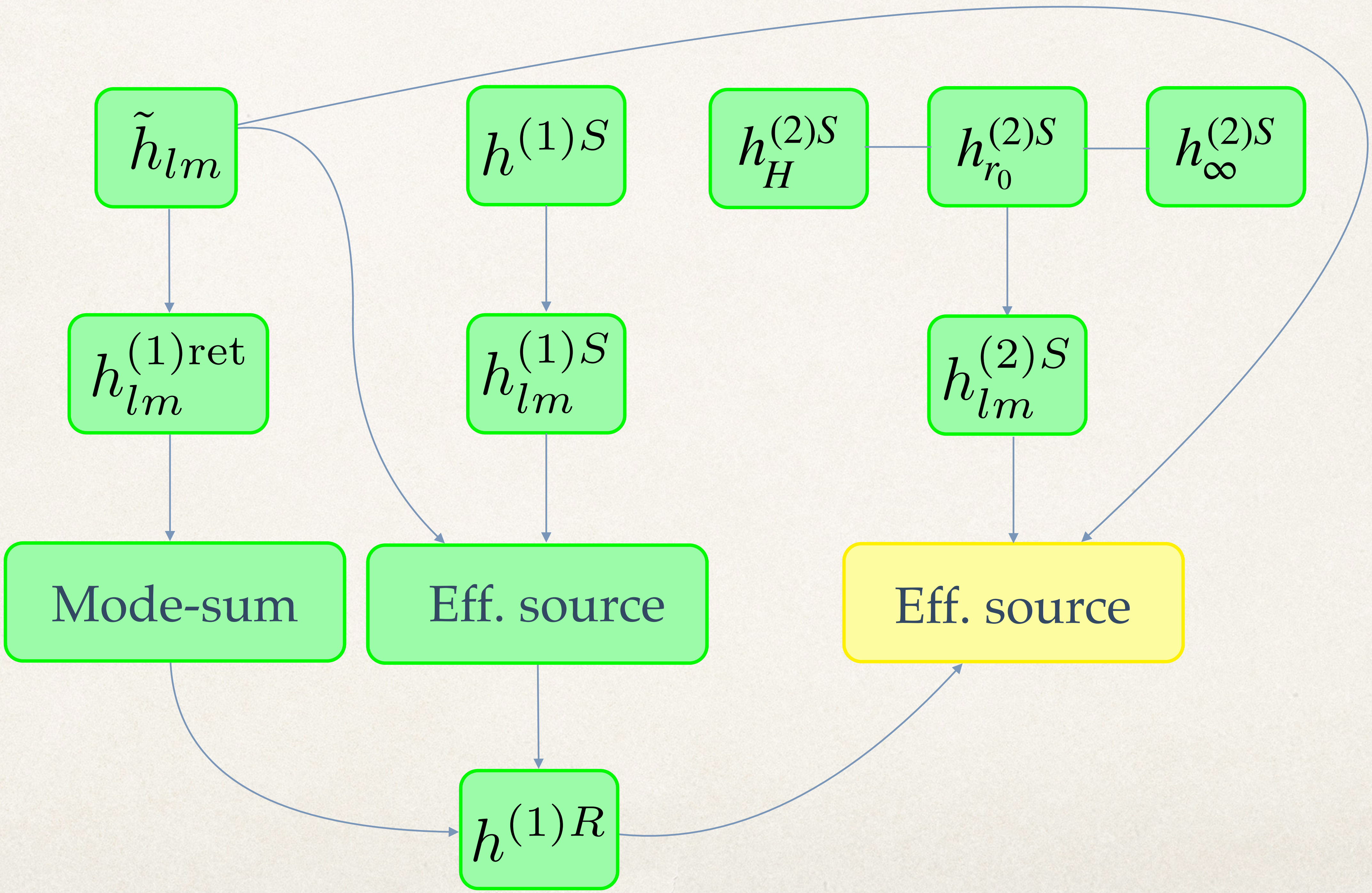
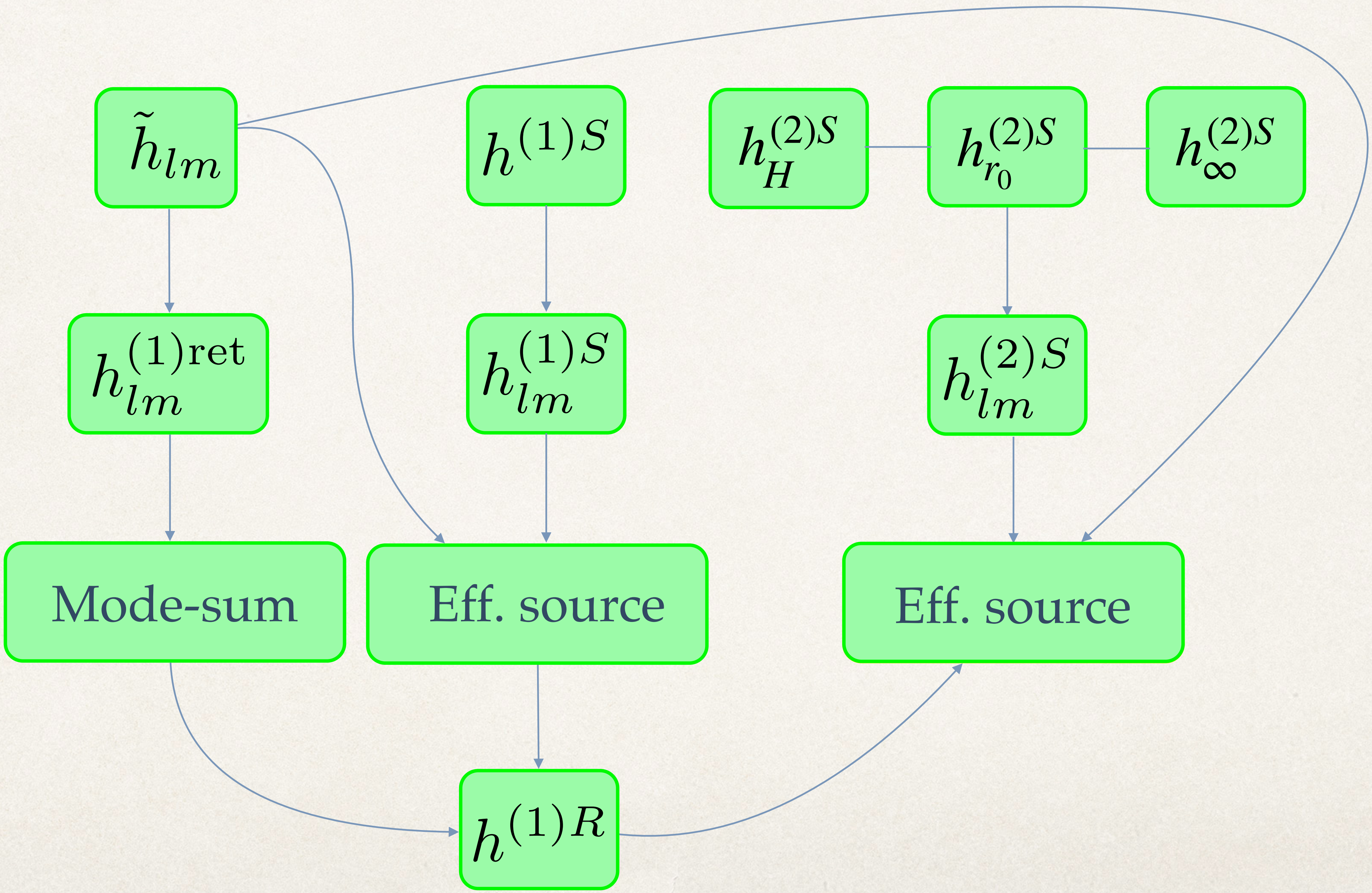
& + \tfrac{1}{4} h^{\mu\nu}{}_{;\alpha} h_{\mu\nu;\beta} + \tfrac{1}{2} h^{\mu}{}_{\beta}{}^{;\nu}\left(h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}\right) \\
& - \tfrac{1}{2} \bar{h}^{\mu\nu}{}_{;\nu}\left(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}\right)
\end{aligned}$$

Towards second order self-force



Towards second order self-force



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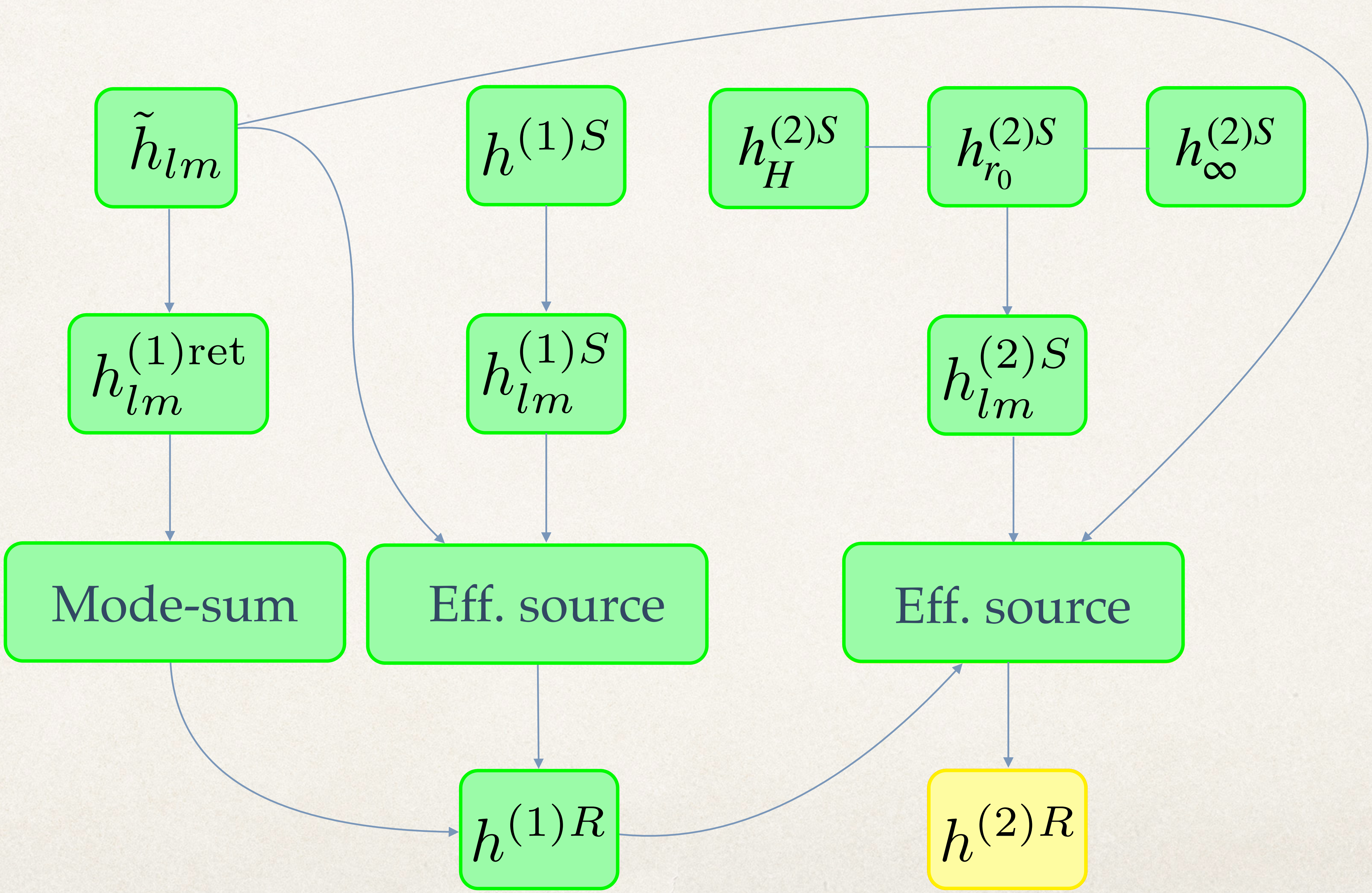
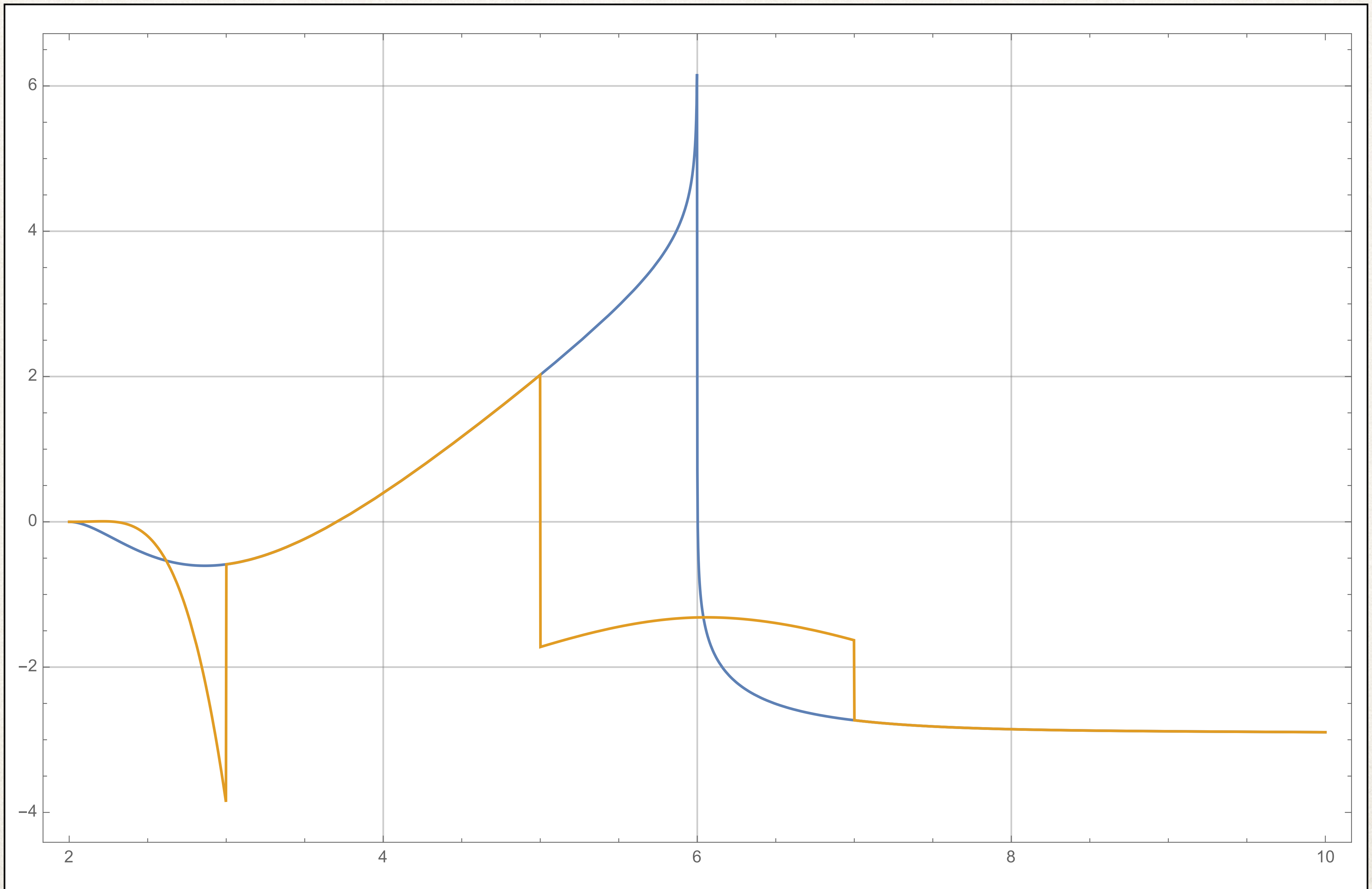


Second order regular field (l=0, i=3)



Validation Checks: Source

- ✓ Coupling formula for $\boldsymbol{\delta^2 R}$ agrees with direct integration for any two input modes with generic time dependence
- ✓ Summing up modes of coupling formula to get 4D $\boldsymbol{\delta^2 R}$ agrees with first summing up modes of $\boldsymbol{h^1}$ and then directly computing 4D $\boldsymbol{\delta^2 R}$, using numerical data for $\bar{h}^1_{\rm ret}$
- ✓ Near-particle calculation of $\boldsymbol{\delta^2 R}$ agrees with result of coupling formula
- ✓ Coupling formula for $\boldsymbol{\delta^2 R}$ satisfies Bianchi identity
- ✓ Numerical coupling results for $\boldsymbol{\delta^2 R}$ have correct falloff at large $\boldsymbol{r}$
- ✓ Numerical coupling results for $\boldsymbol{\delta^2 R}$ agree with near-horizon expansion

Validation Checks: Punctures at Worldline

Wave equation

- ✓ h^{SS} cancels $2\delta^2R[h^S,h^S]$ to correct order (~ $\Delta r^2 \ln\Delta r$ difference, from term ~ $\varrho^2 \ln \varrho$ in h^{SS})
- ✓ h^{SR} cancels $2\delta^2R[h^S,h^R]$ to correct order (C^0 with first two terms in h^{SR})
- ✓ $h^{\delta m}$ contains the correct δ function
- ✓ $h^{\delta z}$ contains the correct δ' function
- ✓ $h^{\delta z}$ contains the correct δ function
- ✓ Ignoring distributions, $h^{\delta m}$ is smooth to correct order (C^0 with first two terms in $h^{\delta m}$)
- ✓ Ignoring distributions, $h^{\delta z}$ is smooth to correct order (C^0 with first three terms in $h^{\delta z}$)

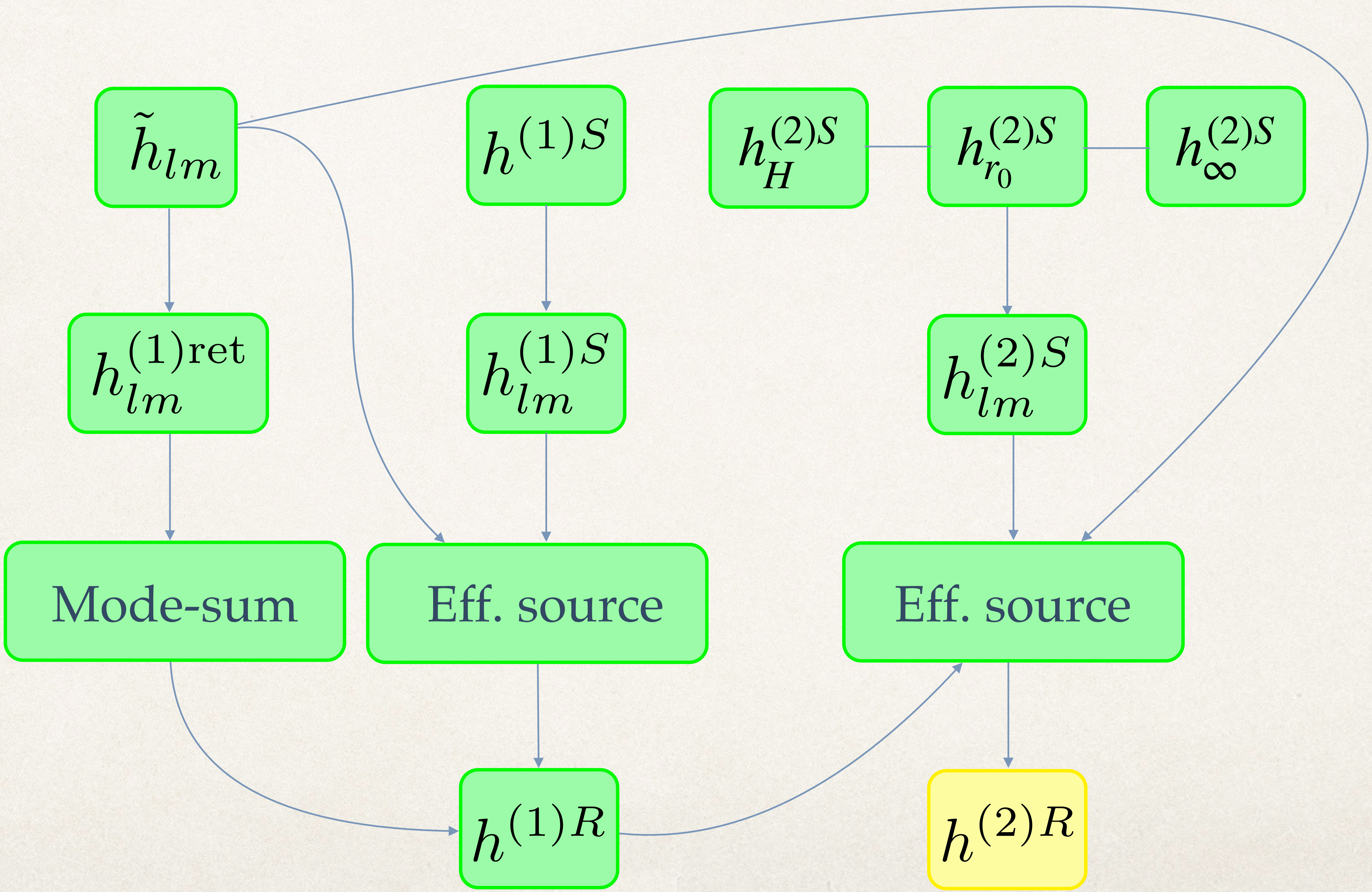
Gauge condition

- ✓ h^{SS} satisfies gauge condition to correct order (~ $\Delta r \ln \Delta r$ difference, from term ~ $\varrho^2 \ln \varrho$ in)
- ✓ $h^{SR} + h^{\delta m} + h^{\delta z}$ satisfies gauge condition to correct order (C^1 if punctures valid through ϵ^0)

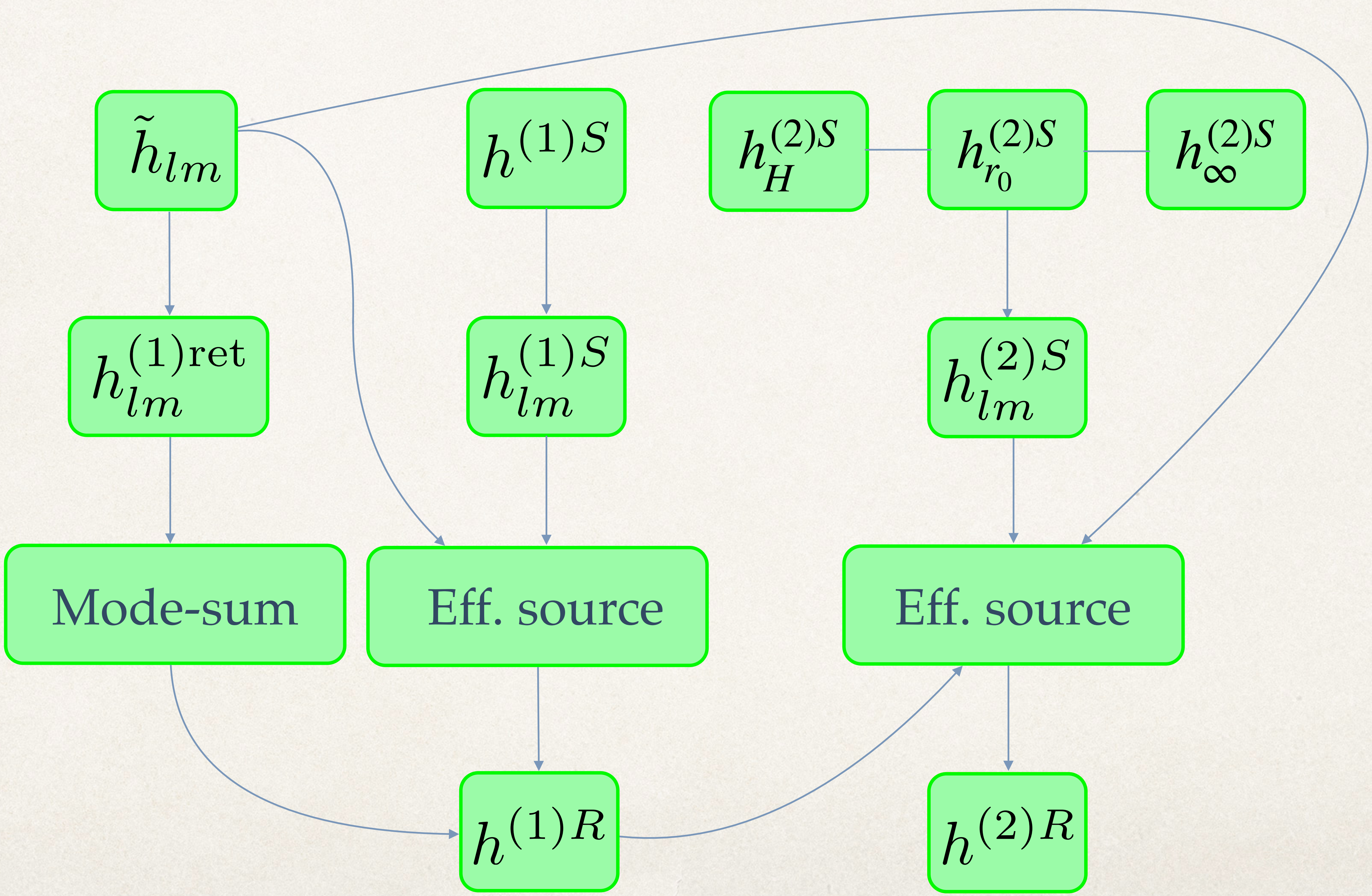
Validation Checks: Numerics and boundary conditions

- ✓ Correct dependence on parameter κ.
- ✓ Worldtube and window function computations agree with analytical $i = 2$ result.
- ✓ $\Box h^{\infty}$ and $\Box h^{H}$ cancel $\delta^2 R$ to correct order in f(r) and 1/r.
- ✓ Agreement between worldtube and window function computations.
- ✓ Agreement between results using different combinations of $i = 1, 3, 6$ equations.
- ✓ Residual field is regular at the horizon (i.e., $i = 1$ and $i = 2$ agree through order f(r)).

Towards second order self-force



Towards second order self-force



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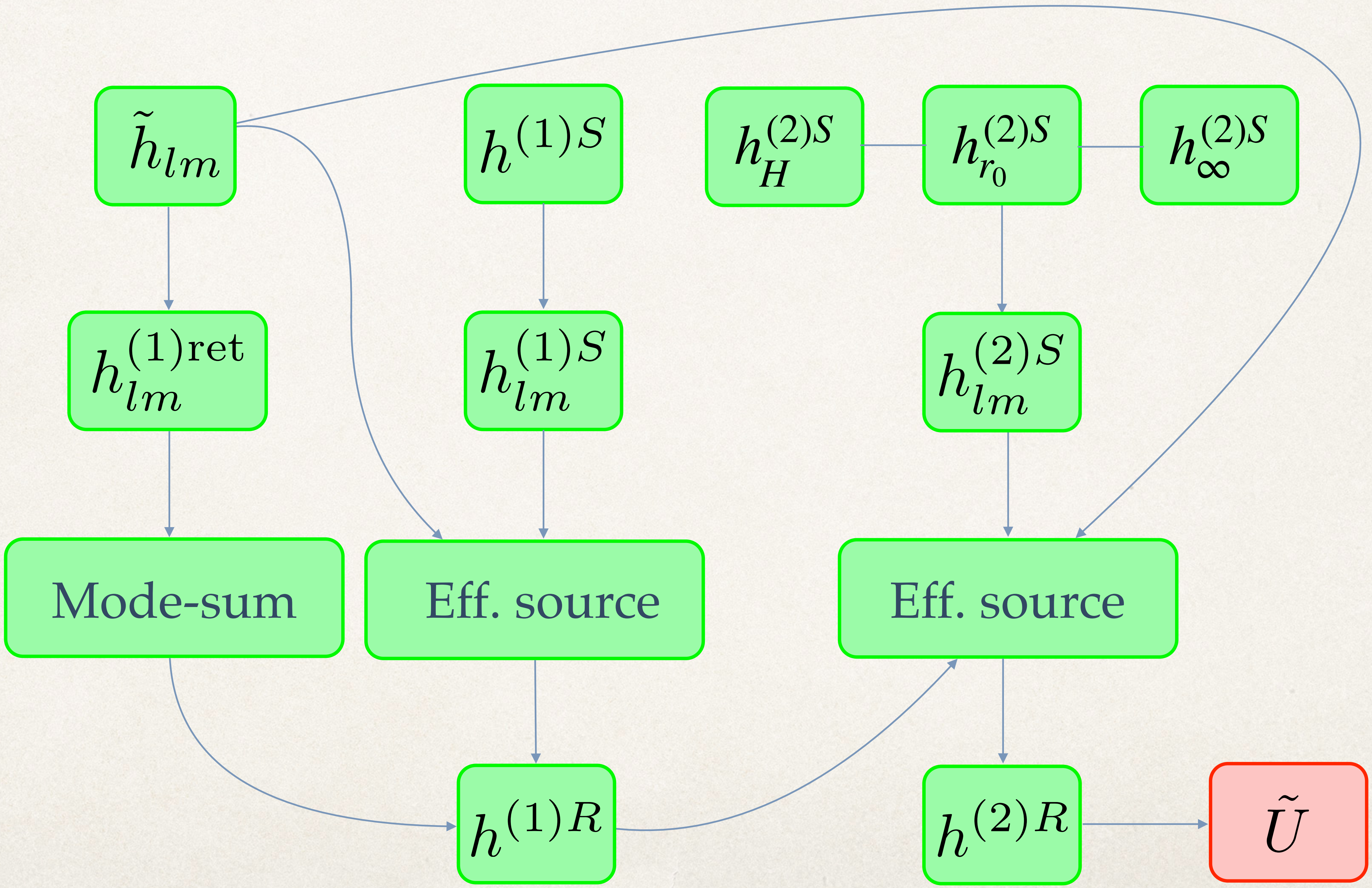


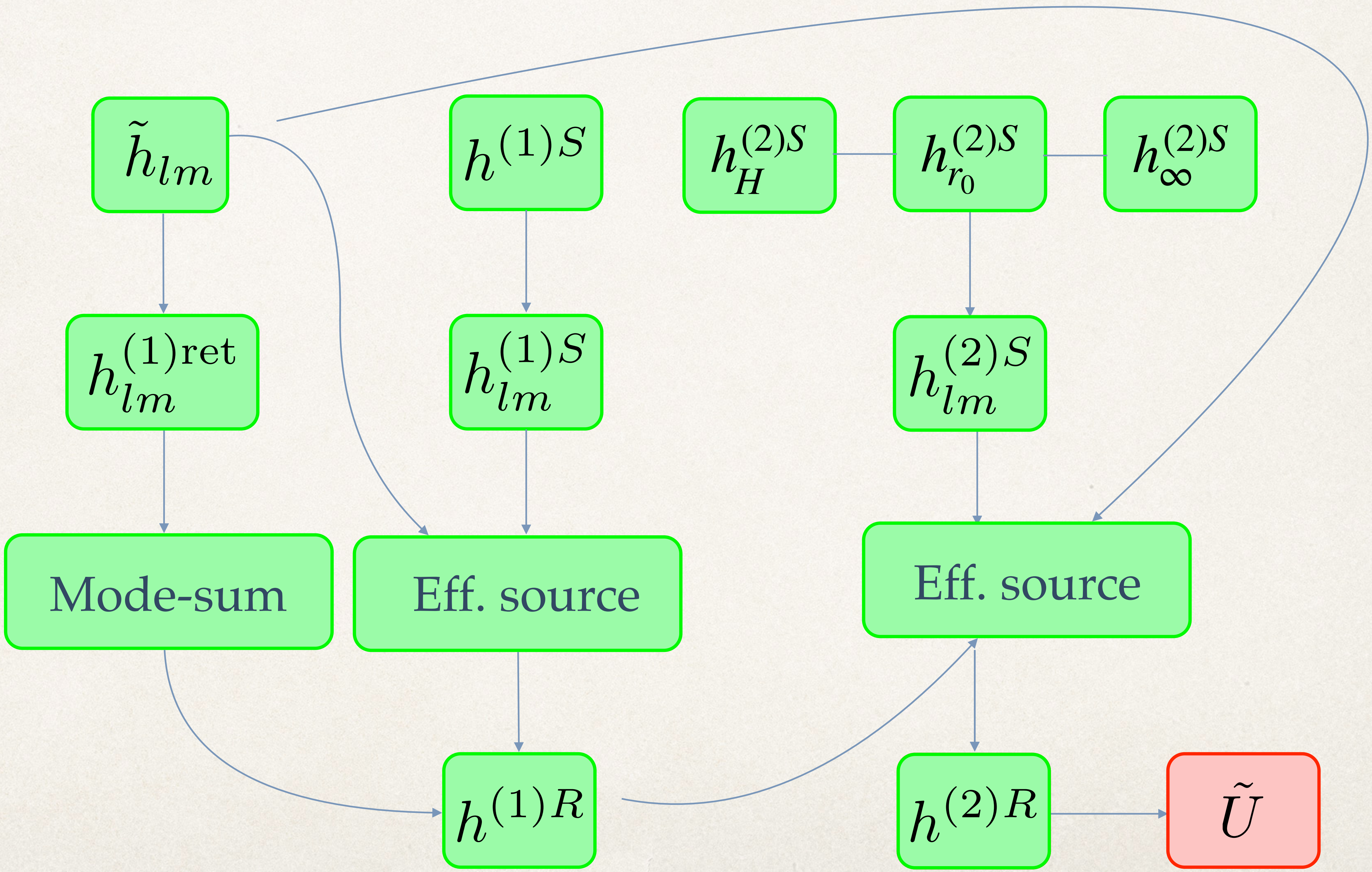
Second order conservative effects

Generalised redshift invariant for circular orbits
[Pound, Phys. Rev. D90, 084039]

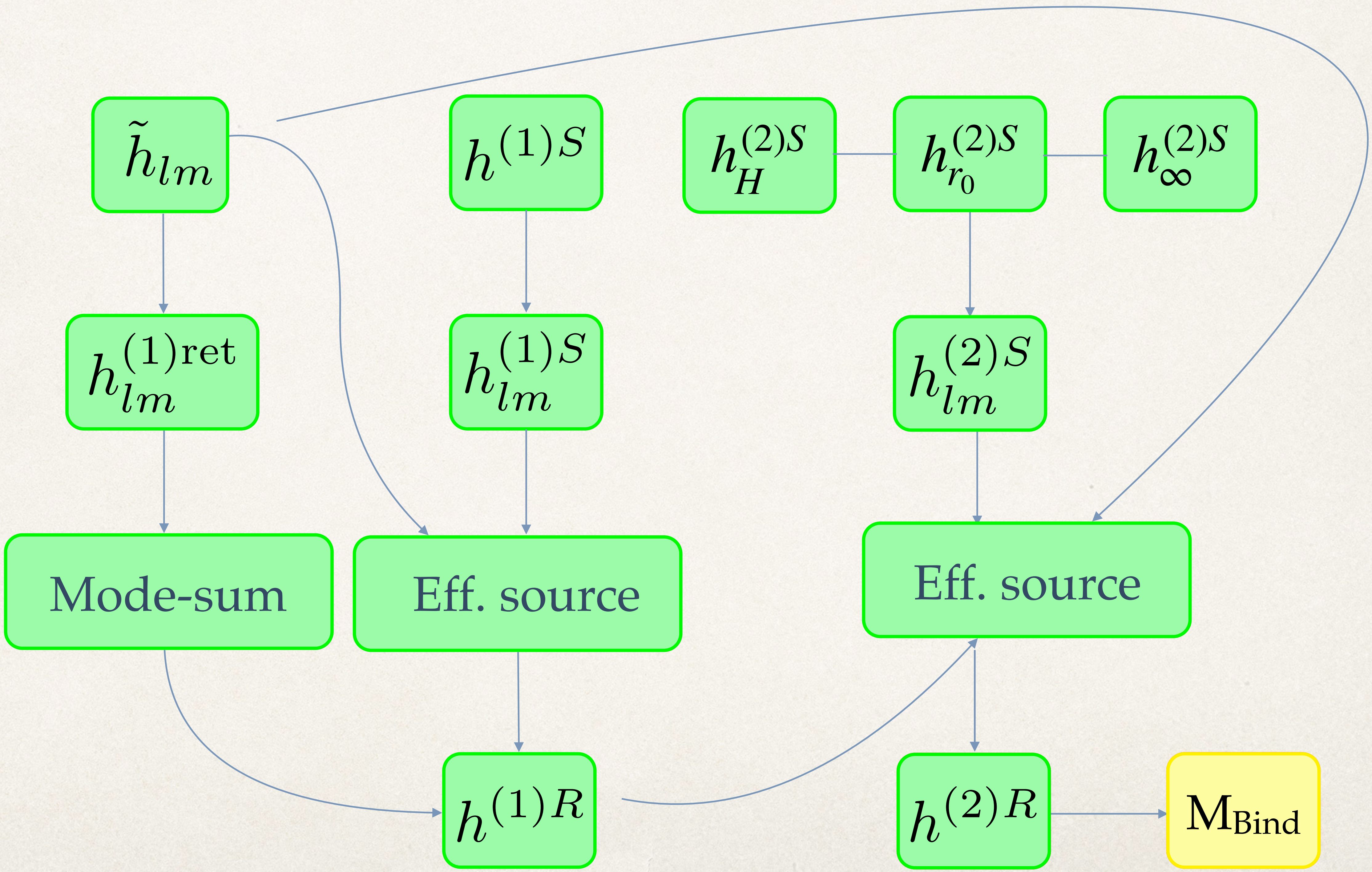
$$\boxed{U_0(\Omega) = \left(1 - 3M/r_\Omega\right)^{1/2} \qquad \Omega = \sqrt{\frac{M}{r_\Omega^3}}}$$

$$\boxed{\begin{aligned}\tilde{U} =& U_0(\Omega)\Big[1 + \tfrac{1}{2}\epsilon h^{\mathrm{R1}}_{u_0u_0} + \epsilon^2\big(\tfrac{1}{2}h^{\mathrm{R2}}_{u_0u_0} + \tfrac{3}{8}(h^{\mathrm{R1}}_{u_0u_0})^2 \\ &- \tfrac{r_\Omega^3}{6M^2}(F_{1r})^2\big(1 - 3M/r_\Omega\big)\big)\Big]\end{aligned}}$$

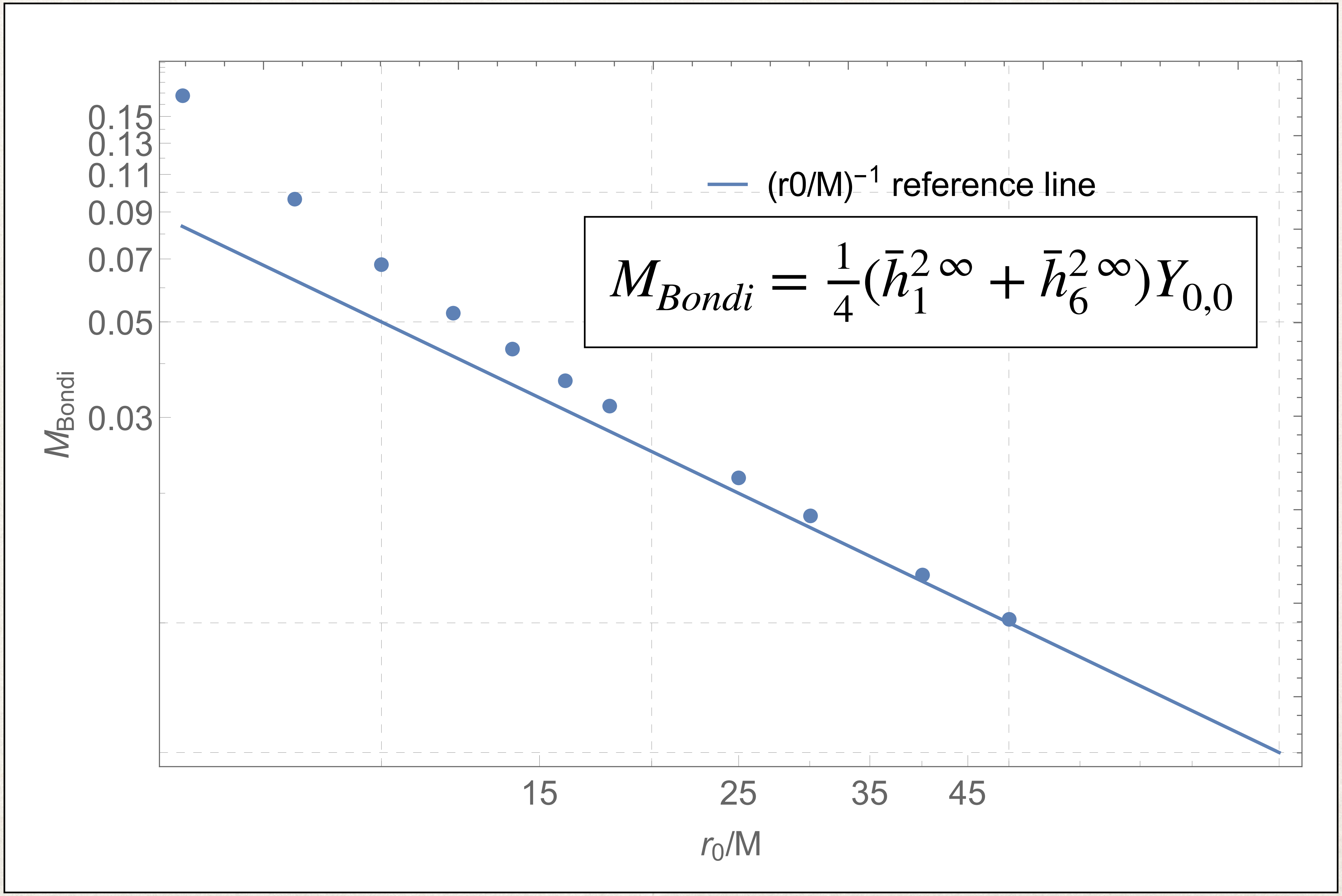
Towards second order self-force



Towards second order self-force

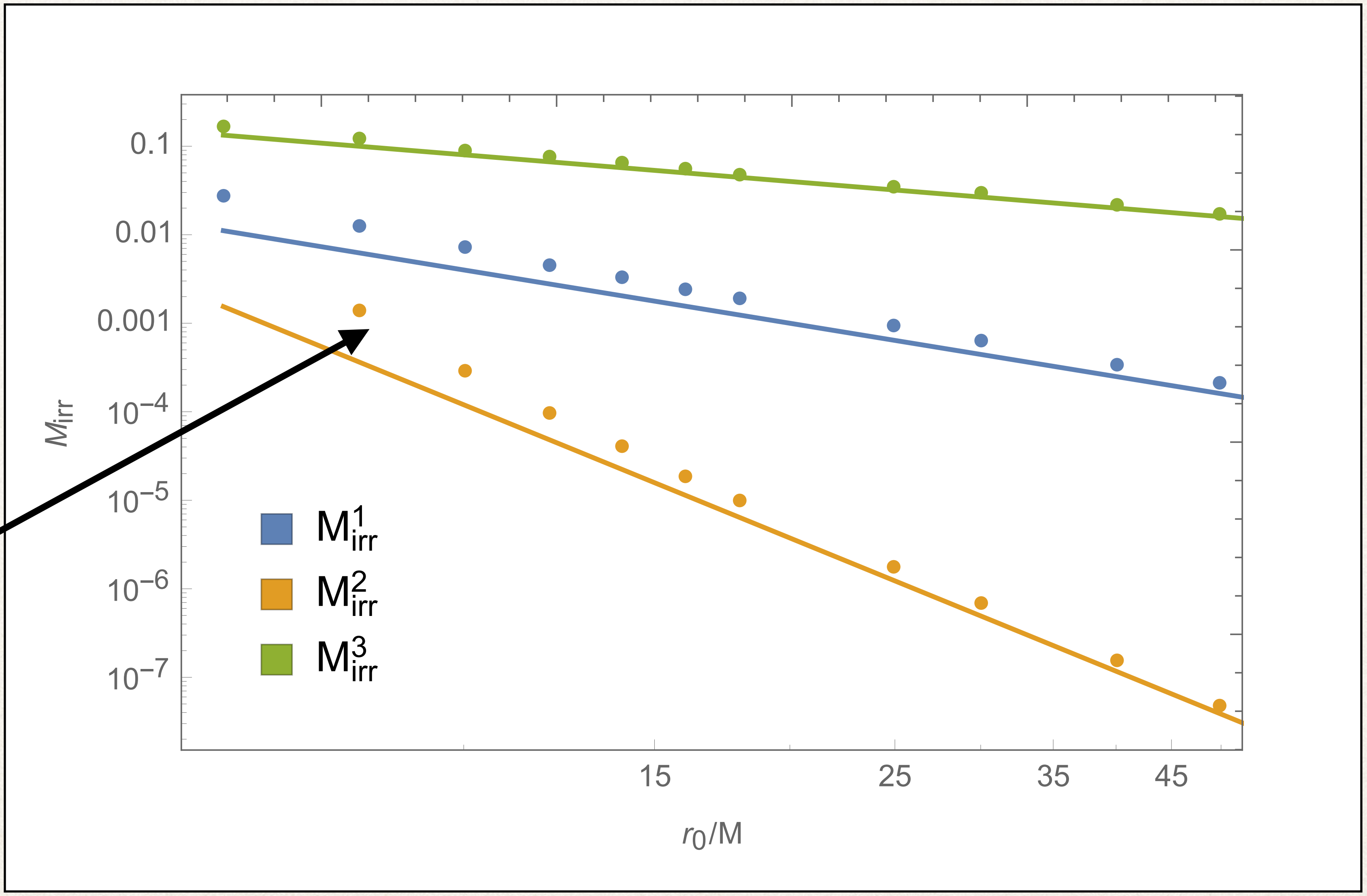


Second Order Bondi Mass



Second Order Irreducible Mass

Diverges at ISCO



$$M_{irr} = M^1_{irr}[h^1, h^1] + M^2_{irr}[h^1, \dot{r}_0] + M^3_{irr}[h^{2\,H}]$$

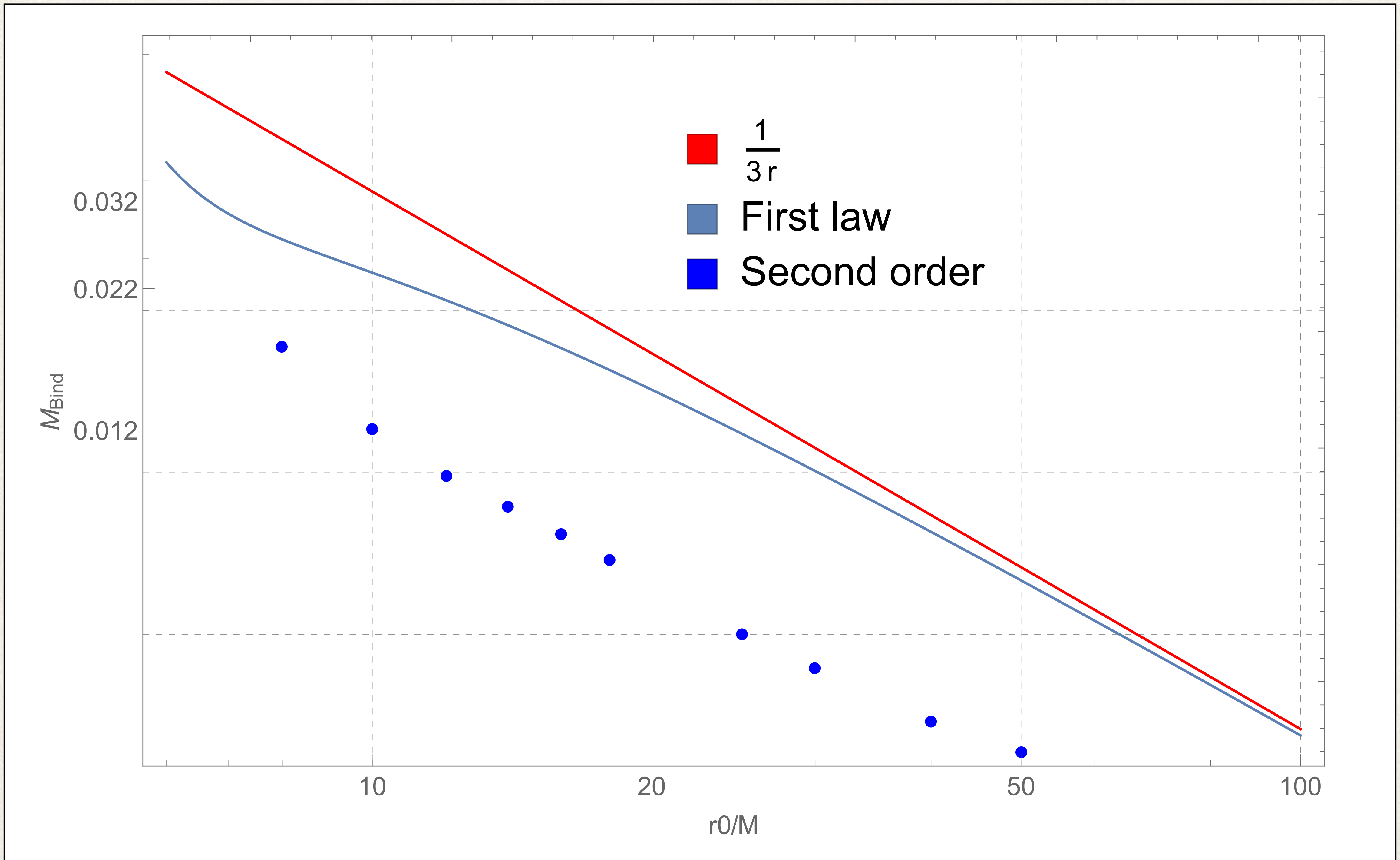
Second Order Binding Energy

Second order binding energy can be written as a sum of contributions from the irreducible mass, Bondi mass, and relation between energy and specific energy.

$$\boxed{M_{\text{Bind}} = M_{\text{Bondi}} - M_{\text{irr}} + \left[\left(1 - \frac{2M}{r_0}\right)\left(1 - \frac{3M}{r_0}\right)^{-1/2} - 1\right]}$$

Energy vs Specific Energy

Second Order Binding Energy



Validation Checks: Irreducible mass and formalism

- ✓ M_{irr} formula works for expansion of Vaidya around Schwarzschild.
- ✓ M_{irr} formula works for expansion of Kerr around Schwarzschild.
- ◉ M_{irr} formula works for expansion of moving Vaidya.
- ◉ M_{irr} is invariant under suitable class of first order gauge transformations.
- ✓ M_{irr} is invariant under suitable class of second order gauge transformations.
- ✓ $i = 2$ solution recovers balance law.

Why don't we agree with First Law?

- We now have a **finite** answer that we **trust**, but it **doesn't agree** with the prediction from the first law.

- Two possibilities:
  - M_{irr} is **not** invariant under suitable class of first order gauge transformations?
  - Puncture at horizon has not been derived from a fundamental principle. Constructed to cancel divergence and nothing more. A more careful prescription may be needed (see talk by Kei Yamada).

Outstanding issues and questions

- Higher modes not a problem. Tools are all set up for all modes, have some parts but we have yet to finish the calculation

- Horizon punctures
  Is there a physically motivated choice?

- Are there other things we could compute in our current setup?
  So far, we know how to compute binding energy, ΔU and dissipative self-force. Any other useful quantities?

- Is there a PN or NR prediction for Delta U as we define it, or can we adjust what we're doing so that we use the same definitions?
  We can directly use the prediction through 3PN from older papers, but at higher orders we run into ambiguity in the definition of conservative dynamics. (We have tried using Damour's definition without success.)

- Dissipative effects appear even in seemingly "conservative" quantities like the instantaneous binding energy.

- Can this be made to work in other gauges or with other variables (Teukolsky, RWZ)?

- How should long-term evolution be tackled?