Near-horizon Expansion of Second-order BH Perturbations

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Summary of today's talk

- The vacuum field equations to the second order are

$$\delta G^{\mu}{}_{\nu}[\varepsilon h^{(1)} + \varepsilon^2 h^{(2)}] = -\delta^2 G^{\mu}{}_{\nu}[\varepsilon h^{(1)}, \varepsilon h^{(1)}],$$

  where $\delta G_{\mu\nu}[h]$ & $\delta^2 G_{\mu\nu}[h,\ h]$ are linear & quadratic in h.

  - Solutions in FD should diverge near the horizon of BH.

- Once we identify the secularly growing piece $h^{(1)\mathrm{sec}}$, solutions to the following equations do NOT diverge:

$$\delta G^{\mu}{}_{\nu}[\varepsilon \tilde{h}^{(1)} + \varepsilon^2 h^{(2)}] = -\delta^2 G^{\mu}{}_{\nu}[\varepsilon h^{(1)}] - \delta G^{\mu}{}_{\nu}[\varepsilon h^{(1)\mathrm{sec}}].$$

Outlines

- Introduction
- Near-horizon expansion
- Static/secularly growing pieces
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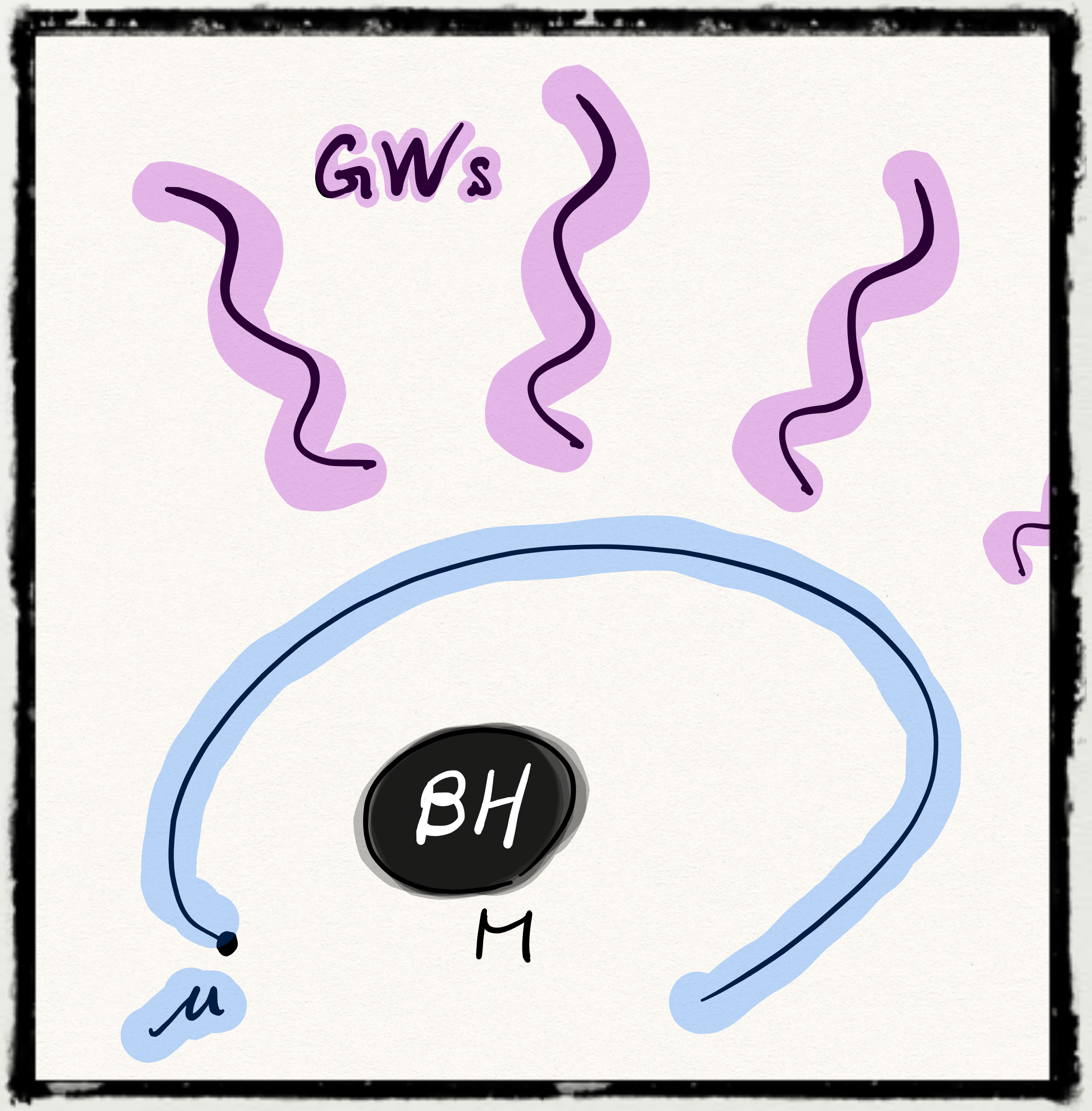
Extreme Mass Ratio Inspirals

- Expand equations in the mass ratio:

$$\varepsilon \equiv \mu/M \ll 1,$$

- Consider the two time-scale expansion.
  - Orbital "fast time": v
  - Inspiral "slow time": $\tilde{v}$



IR divergence around boundaries

- We expand equations in the mass ratio:

$$g_{\mu\nu} = g^{\mathrm{BG}}_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} + \varepsilon^2 h^{(2)}_{\mu\nu} + \cdots .$$

- The field equations to the second-order are

$$\delta G^{\mu}{}_{\nu}[\varepsilon h^{(1)} + \varepsilon^2 h^{(2)}] = -\delta^2 G^{\mu}{}_{\nu}[\varepsilon h^{(1)}, \varepsilon h^{(1)}],$$

where $\delta G^{\mu}{}_{\nu}[h]$ & $\delta^2 G^{\mu}{}_{\nu}[h, h]$ are linear & quadratic in h .

IR divergence around boundaries

- The field equations to the second-order are

$$\delta G^{\mu}{}_{\nu}[\varepsilon h^{(1)} + \varepsilon^2 h^{(2)}] = -\delta^2 G^{\mu}{}_{\nu}[\varepsilon h^{(1)}, \varepsilon h^{(1)}].$$

- Schematically, the following integral diverges around boundaries.

$$h^{(2)}_{\omega\ell m} = \int_{r_{\rm h}}^{\infty} G_{\omega\ell m}(r, r')\delta^2 G(r')dr',$$

✓ @infinity ➡ use the PN/PM results (cf. [Pound 2015]).

- We discuss the near-horizon expansion.

Decomposition of $h^{(i)}$

- The origins of secular growth are the "stationary" parts of $-\delta^2 G^{\mu}{}_{\nu}$.
  - Almost all of the "stationary" parts should be balanced with "stationary" solutions $h^{(i)\mathrm{sta}}$.
- Let us decompose $h^{(i)}$ as

$$h^{(i)}_{\mu\nu} = h^{(i)\,v-\mathrm{dep}}_{\mu\nu}(\tilde{v}, v, r, \theta, \phi) + h^{(i)\,\mathrm{sta}}_{\mu\nu}(\tilde{v}, r, \theta).$$

Oscillatory piece — "Stationary" piece

- Focus on the "stationary" piece, because $h^{(i)\,v-\mathrm{dep}}_{\mu\nu}$ does not cause the divergence.

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The Eddington-Finkelstein coordinates

- The Schwarzschild background metric is

$$ds^2 = -f dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

in the ingoing Eddington-Finkelstein coordinates, where $f = 1 - 2M/r$.

✓ Regularity of metric perturbations on the horizon.

✓ On the horizon $g^{rr} = 0$:

➡ d^2/dr^2 does not appear in the Lorenz gauge.

Near-horizon expansion

- The metric perturbations are

$$g_{\mu\nu} = g^{\mathrm{BG}}_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} + \varepsilon^2 h^{(2)}_{\mu\nu} + \cdots .$$

- Expand the perturbations near the horizon

$$h^{(i)\mathrm{sta}}_{\mu\nu}(\tilde{v}, r, \theta) = h^{(i)\mathrm{sta}}_{0\,\mu\nu}(\tilde{v}, \theta) + f h^{(i)\mathrm{sta}}_{1\,\mu\nu}(\tilde{v}, \theta) + \mathrm{O}(f^2),$$

where $\tilde{v} \equiv \varepsilon v$ is a slow-time variable.

✓ $\mathrm{O}(f^2)$ in $h^{(i)\mathrm{sta}}_{\mu\nu}(\tilde{v}, r, \theta)$ is not necessary because of the absence of d^2/dr^2.

The Lorenz gauge conditions

- The Lorenz gauge conditions $\bar{h}^{(i);\nu}_{\mu\nu} = 0$ are formally

$$h^{(i)}_{1vv} = F^{(i)}_v[h_0],\ h^{(i)}_{1I}{}^{I} = F^{(i)}_r[h_0],$$
$$h^{(i)}_{1v\theta} = F^{(i)}_\theta[h_0],\ h^{(i)}_{1v\phi} = F^{(i)}_\phi[h_0],$$

in the near-horizon limit $f \to 0$, where $I = \theta, \phi$.

✓ Under the Lorenz gauge conditions,
we can eliminate $h^{(i)}_1$ for 4 components of $\delta G^{\mu}{}_{\nu}$.

First order in ε

- At the first order in ε,

  4 components of $\delta G^{\mu}{}_{\nu}$, NOT containing $h_1^{(1)}$ are

$$\sin\theta\delta G^{(1)\mathrm{sta}\,r}{}_{v} = -\partial_\theta \left[\frac{\sin\theta}{8M^2} \left(h^{(1)\mathrm{sta}}_{0v\theta} + \partial_\theta h^{(1)\mathrm{sta}}_{0vv} \right) \right],$$

$$\vdots$$

- Although these do not vanish in general,

$$\int \sin\theta\, \delta G^{(1)\mathrm{sta}\,r}{}_{v}\, d\theta\, d\phi = \int \sin\theta\, \delta G^{(1)\mathrm{sta}\,r}{}_{\phi}\, d\theta\, d\phi = 0.$$

  - Non-trivial solutions for the zero modes.

First order in ε

$$\int \sin\theta\, \delta G^{(1)\mathrm{sta}\, r}{}_{v}\, d\theta\, d\phi = \int \sin\theta\, \delta G^{(1)\mathrm{sta}\, r}{}_{\phi}\, d\theta\, d\phi = 0.$$

- The first-order field equations do NOT determine the zero modes of $h^{(1)\mathrm{sta}}$.
- The second order does,
  because $\partial_{\tilde{v}} h^{(1)\mathrm{sta}}$ appears in the second order.
  - The solutions are secularly growing.

Energy & angular momentum fluxes

- Ingoing GW's $\dot{E}$ & $\dot{L}$ across the horizon are

$$\dot{E} \equiv \frac{1}{8\pi} \int \sqrt{-g}\left(-\delta^2 G^r{}_v\right) d\theta\, d\phi,$$

$$\dot{L} \equiv \frac{1}{8\pi} \int \sqrt{-g}\left(-\delta^2 G^r{}_\phi\right) d\theta\, d\phi,$$

where $\dot{} \equiv \partial/\partial\tilde{v}$.

- These components of the second-order source terms for zero modes of $h^{(1)\mathrm{sta}}$ correspond to $\dot{E}$ & $\dot{L}$ of first-order ingoing GWs.

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Second-order Einstein tensor

- At the second order in ε,
  4 components of $\delta G^{\mu}{}_{\nu}$ NOT containing $h_{1}^{(2)}$ are

$$\sin\theta\,\delta G^{(2)\mathrm{sta}\,r}{}_{v} = \frac{\sin\theta}{8M}\partial_{\tilde{v}}\left(4h^{(1)\mathrm{sta}}_{0vv} + h^{(1)\mathrm{sta}}_{0\theta\theta} + h^{(1)\mathrm{sta}}_{0\phi\phi}\right) - \partial_\theta\left[\frac{\sin\theta}{8M^2}\left(h^{(2)\mathrm{sta}}_{0v\theta} + \partial_\theta h^{(2)\mathrm{sta}}_{0vv} - 4M\partial_{\tilde{v}}h^{(1)\mathrm{sta}}_{0v\theta}\right)\right],$$

$$\sin\theta\,\delta G^{(2)\mathrm{sta}\,r}{}_{\phi} = \frac{\sin^2\theta}{2}\partial_{\tilde{v}}\left(4h^{(1)\mathrm{sta}}_{0v\phi} + h^{(1)\mathrm{sta}}_{0r\phi}\right) + \partial_\theta\left[-\frac{\sin^3\theta}{4M}\partial_\theta\left(\frac{1}{\sin\theta}h^{(2)\mathrm{sta}}_{0v\phi}\right) + \sin^2\theta\,\partial_{\tilde{v}}h^{(1)\mathrm{sta}}_{0\theta\phi}\right],$$

$$\vdots$$

Abbott & Deser's quantities

- We find

$$\frac{1}{8\pi}\int \sqrt{-g}\,\sin\theta\,\delta G^{(2)\mathrm{sta}\,r}{}_{v}\,d\theta\,d\phi\bigg|_{r=r_{\mathrm{h}}} = -\partial_{\tilde{v}}\left(M^{\mathrm{AD}}\right)\Big|_{r=r_{\mathrm{h}}},$$

$$\frac{1}{8\pi}\int \sqrt{-g}\,\sin\theta\,\delta G^{(2)\mathrm{sta}\,r}{}_{\phi}\,d\theta\,d\phi\bigg|_{r=r_{\mathrm{h}}} = -\partial_{\tilde{v}}\left(L^{\mathrm{AD}}\right)\Big|_{r=r_{\mathrm{h}}},$$

where

$$M^{\mathrm{AD}} = \frac{1}{2}\int F^{(v)\alpha\beta}\,d\Sigma_{\alpha\beta}, \quad L^{\mathrm{AD}} = \frac{1}{2}\int F^{(\phi)\alpha\beta}\,d\Sigma_{\alpha\beta},$$

$$F^{(v/\phi)}_{\mu\nu} \equiv -\frac{1}{8\pi}\left[\xi^{(v/\phi)\alpha}\bar{h}^{(1)\mathrm{sta}}_{\alpha[\mu;\nu]} + \xi^{(v/\phi)\alpha}{}_{;[\mu}\bar{h}^{(1)\mathrm{sta}}_{\nu]\alpha} + \xi^{(v/\phi)}_{\mu}\bar{h}^{(1)\mathrm{sta};\alpha}_{\nu]\alpha}\right].$$

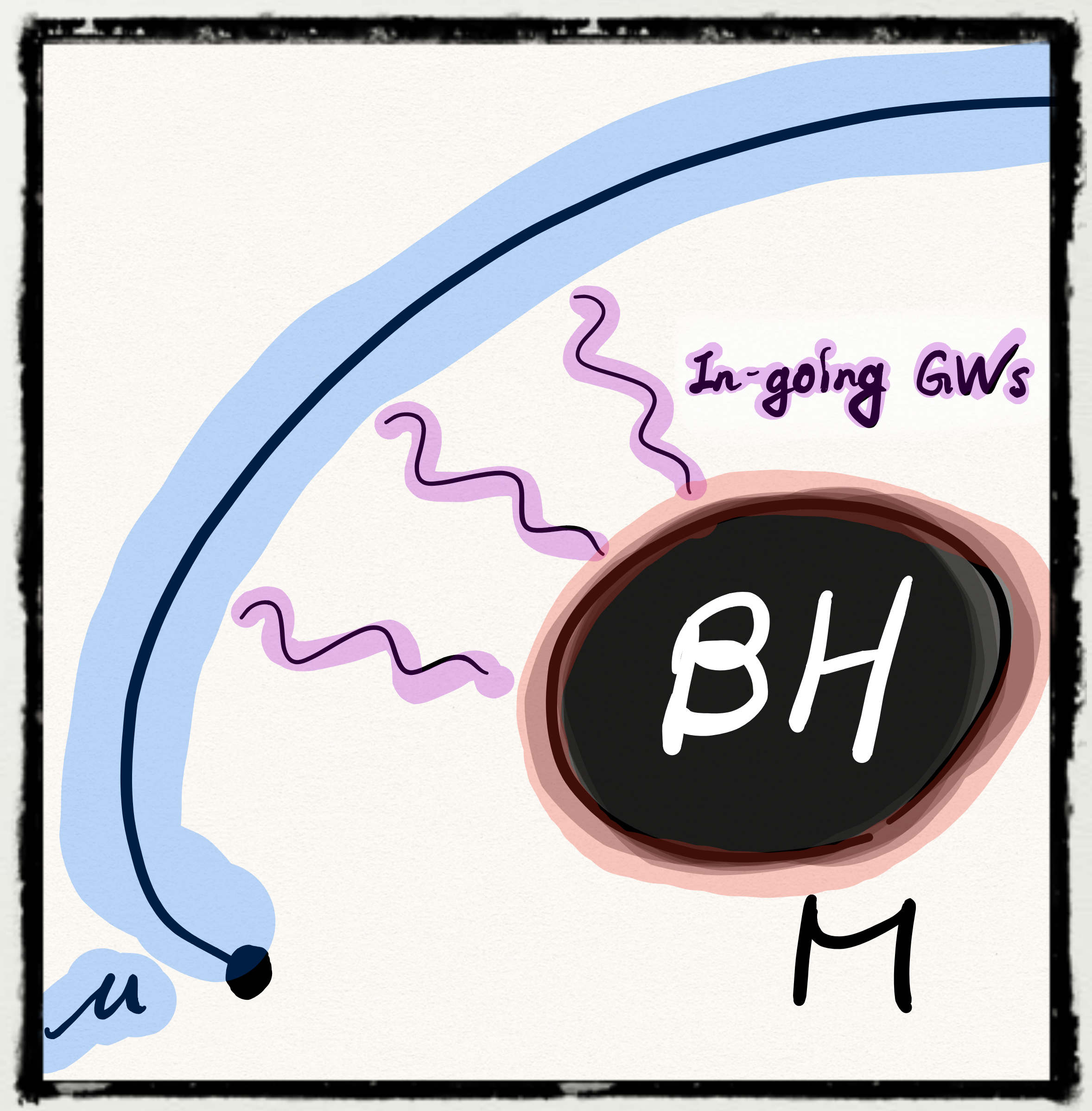
Physical secular growth

- $\{rv\}$ & $\{r\phi\}$ components of

$$\int \sqrt{-g}\,\delta G^{\mu}{}_{\nu}\,d\theta\,d\phi = \int \sqrt{-g}\left(-\delta^2 G^{\mu}{}_{\nu}\right)\,d\theta\,d\phi,$$

determine the secular growth of zero modes.

- The secular growth ➡ the secular change of the BH's mass/spin $\delta M = \dot{M}\tilde{v}$ & $\delta a = \dot{a}\tilde{v}$.



Identification of the secular pieces

- Calculating deviations due to δM, δa, we obtain

$$\delta G_{\mu\nu}[h^{(1)\mathrm{sec}}] = \begin{pmatrix} \frac{2}{r^2}\partial_{\tilde{v}}\,\delta M(\tilde{v}) & 0 & 0 & -\frac{(M+r)\sin^2\theta}{r^2}\partial_{\tilde{v}}\,\delta a(\tilde{v}) \\ 0 & 0 & 0 & \frac{\sin^2\theta}{r}\partial_{\tilde{v}}\,\delta a(\tilde{v}) \\ 0 & 0 & 0 & 0 \\ -\frac{(M+r)\sin^2\theta}{r^2}\partial_{\tilde{v}}\,\delta a(\tilde{v}) & \frac{\sin^2\theta}{r}\partial_{\tilde{v}}\,\delta a(\tilde{v}) & 0 & 0 \end{pmatrix},$$

  which reproduces the zero modes.

- Therefore, the effective source term,

$$-\delta^2 G^{\mu}{}_{\nu}[\varepsilon h^{(1)}, \varepsilon h^{(1)}] - \delta G^{\mu}{}_{\nu}[\varepsilon h^{(1)\mathrm{sec}}],$$

  can be integrated without any divergences.

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Summary

- Need the second-order metric perturbations for EMRI observations by LISA.
  - IR divergences appear around boundaries.
- By the near-horizon expansion, we found
  - the 2nd-order eqs. determine the zero modes of $h^{(1)\mathrm{sta}}$.
  - the secular growth ➡ $\partial_{\tilde{v}} M^{\mathrm{AD}}$ & $\partial_{\tilde{v}} L^{\mathrm{AD}}$ due to $\dot{E}$ & $\dot{L}$.
- Extension to the Kerr case is straightforward.
- How to tame secular growth of pure gauge modes…?

THANK YOU FOR YOUR ATTENTION