General Relativistic Dynamics of an Extreme Mass Ratio Binary with an External Body

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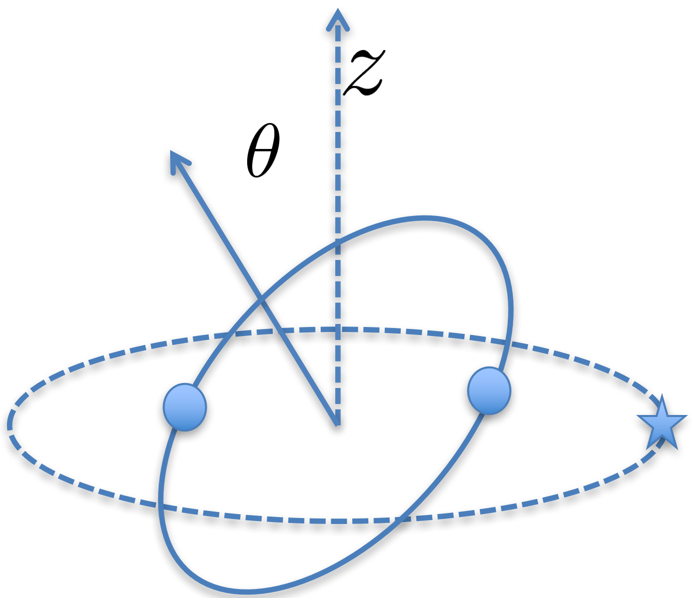
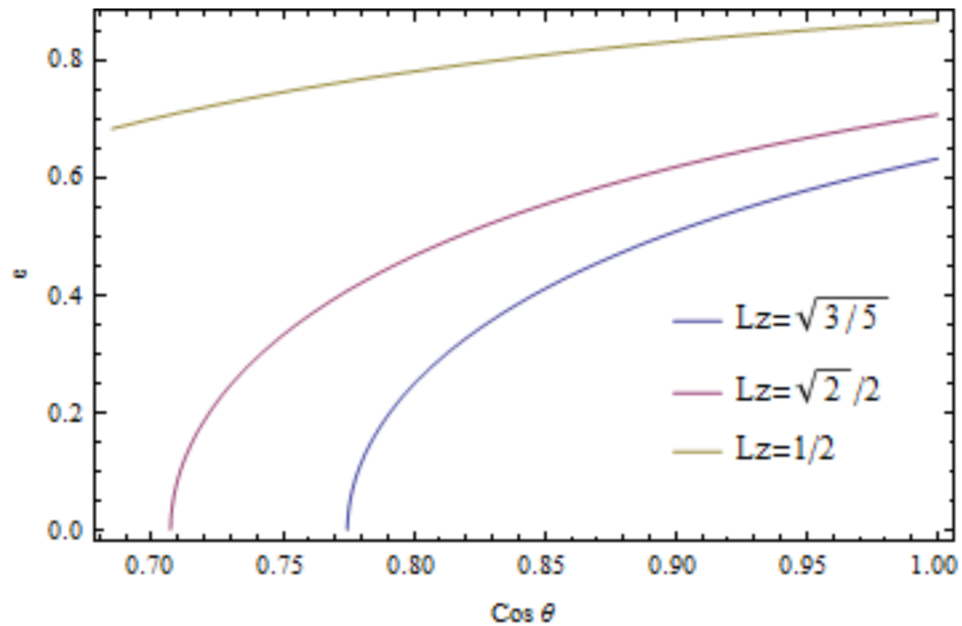
Triple systems

- Planetary triple systems: Kozai-Lidov mechanism.

Lidov 1961;Kozai 1962
Tremaine 2016





$$L_z = \sqrt{1-e^2}\cos\theta : \text{conserved}$$

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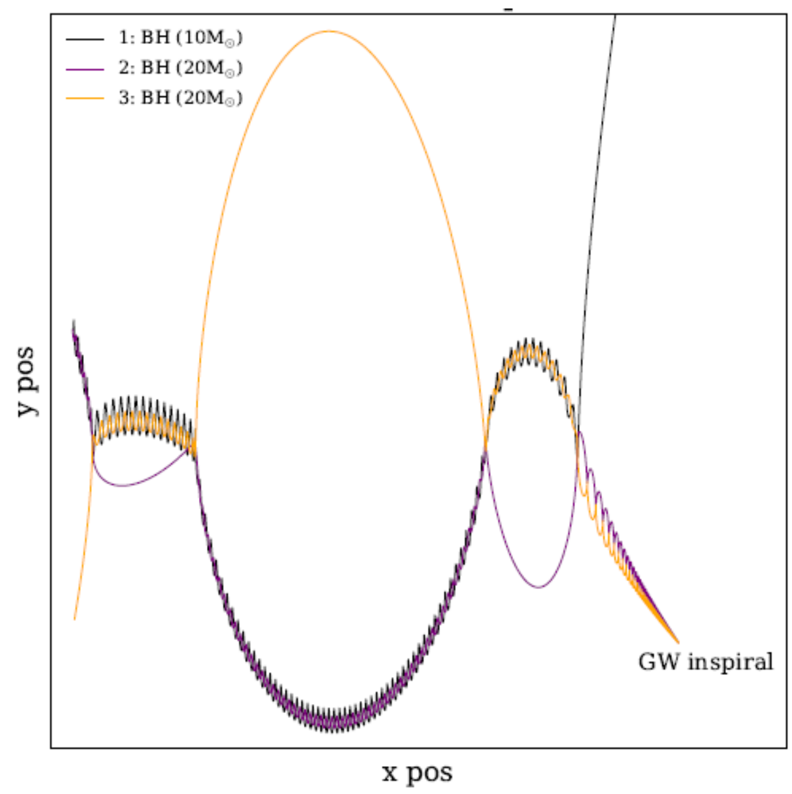
Samsing et al. 2017



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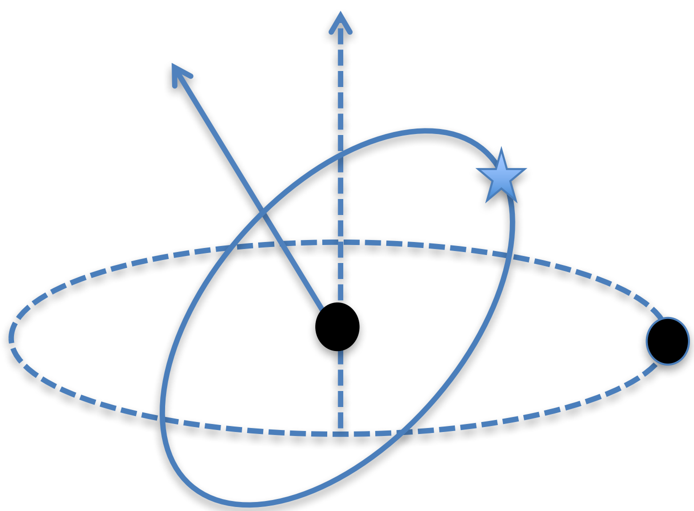
- EMRI with a SMBH perturber. Yunes et al. 2011



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- Many more…

Content

- Formalism: particle moving in a tidally perturbed Schwarzschild spacetime
- Interesting relativistic effects:
  - Precession due to coupling between tidal field and quadrupole moment of the orbit.
  - Relativistic Kozai-Lidov effect for transient resonances.
  - Shift on ISCO.
- Future directions.

A tidally deformed Schwarzschild black hole

- The external tidal field can be decomposed into multipoles: Poisson 2005

Electric part: $\mathcal{E}_{ab}$ Magnetic part: $\mathcal{B}_{ab}$

- In the quasi-stationary limit, neglect the contribution from magnetic part of the tidal field. A Schwarzschild black hole metric is perturbed as

$$h_{tt} = -r^2 f^2 \mathcal{E}, \quad h_{rr} = -r^2 \mathcal{E}, \quad h_{tr} = -r^2 f \mathcal{E},$$
$$h_{tA} = -\frac{2}{3} r^3 f \mathcal{E}_A, \quad h_{rA} = -\frac{2}{3} r^3 \mathcal{E}_A$$
$$h_{AB} = -\frac{1}{3} r^4 \left(1 - \frac{2M^2}{r^2}\right) \mathcal{E}_{AB}$$

- A perturbed Kerr metric is solved by Yunes et al in 2006.

A tidally deformed Schwarzschild black hole II

- Keep only the quadrupole piece of tidal field

Poisson 2010

Tidal potential: $$U_{\rm ext} = \frac{M_* r^2 (1 - 3\cos^2\theta)}{2d^3}$$

- The corresponding tidal tensor is

$$\mathcal{E}_{\theta\theta} = -\frac{3M_* \sin^2\theta}{d^3}, \quad \mathcal{E}_{\phi\phi} = \frac{3M_* \sin^4\theta}{d^3},$$
$$\mathcal{E}_{\theta} = \frac{3M_* \sin\theta\cos\theta}{d^3}, \quad \mathcal{E} = \frac{M_*(1 - 3\cos^2\theta)}{d^3},$$
$$\mathcal{E}_{\theta\phi} = \mathcal{E}_{\phi\theta} = \mathcal{E}_{\phi} = 0$$

- The perturbed metric can be obtained analytically.

A particle moving in deformed Schwarzschild metric

- First viewpoint: accelerated particle in Schwarzschild spacetime

$$a^\mu = -\frac{1}{2}(g^{\mu\nu} + u^\mu u^\nu)(2h_{\nu\lambda;\rho} - h_{\lambda\rho;\nu})u^\lambda u^\rho$$

- Second viewpoint: particle moving on geodesic of perturbed spacetime

$$\frac{d\tilde{q}^\nu}{d\tilde{\tau}} = \frac{\partial H}{\partial \tilde{p}_\nu}, \quad \frac{d\tilde{p}_\nu}{d\tilde{\tau}} = -\frac{\partial H}{\partial \tilde{q}^\nu}$$

With Hamiltonian given by:

$$H = \frac{1}{2}\tilde{p}^\mu \tilde{p}^\nu \tilde{g}_{\mu\nu} = \frac{1}{2}\tilde{p}^\mu \tilde{p}^\nu (g_{\mu\nu} + h_{\mu\nu})$$

Orbital-averaged change of conserved quantities

- Change rate of conserved quantities in Schwarzschild spacetime

$$\frac{d\mathcal{C}}{d\tau} = \frac{\partial \mathcal{C}}{\partial p^\nu} a^\nu$$

- Conserved quantities: energy, vector angular momentum

- Geodesic motion in Schwarzschild is planar. Radial and angular motions are independent (separable), indexed by Mino time.

$$\left(\frac{dr}{d\lambda}\right)^2 = r^4 \left(E^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L_z^2 + Q}{r^2}\right)\right)$$

$$\left(\frac{d\theta}{d\lambda}\right)^2 = Q - L_z^2 \cot^2\theta$$

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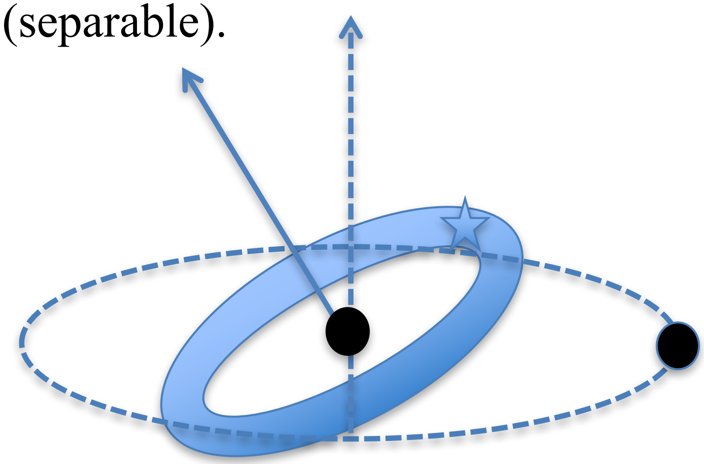

$$\left\langle \frac{d\mathcal{C}}{d\lambda} \right\rangle = \frac{1}{\Lambda_r \Lambda_\theta} \int_0^{\Lambda_r} d\lambda_r \int_0^{\Lambda_\theta} d\lambda_\theta \frac{d\mathcal{C}}{d\tau} r^2$$

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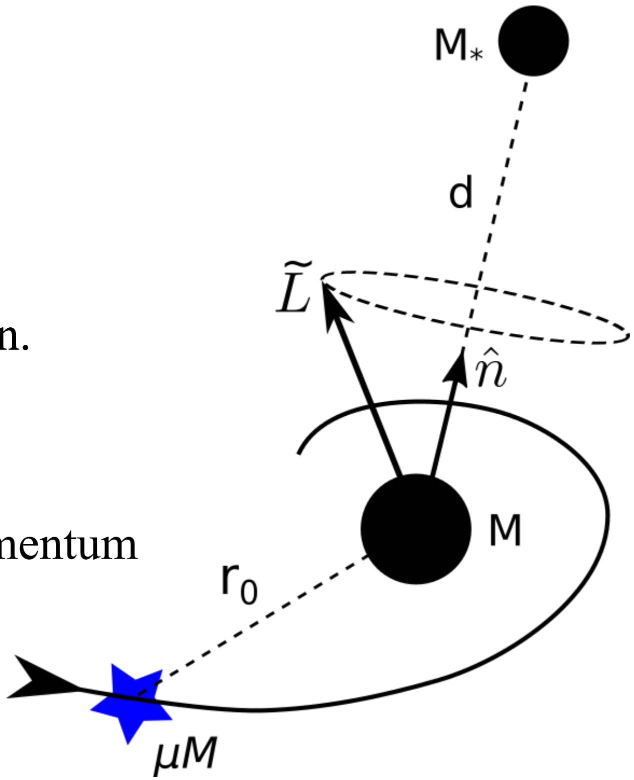
Precession

- Stationarity: $\tilde{E}$ must be conserved
- Axis symmetry: $\tilde{L}_{\hat{n}}$ must be conserved
- The difference between $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are $\sim \mathcal{O}(h)$, interchangeable for long-term secular evolution.
- Time reversal symmetry + 2 D ergodic orbit implies that the magnitude of total angular momentum must be conserved.



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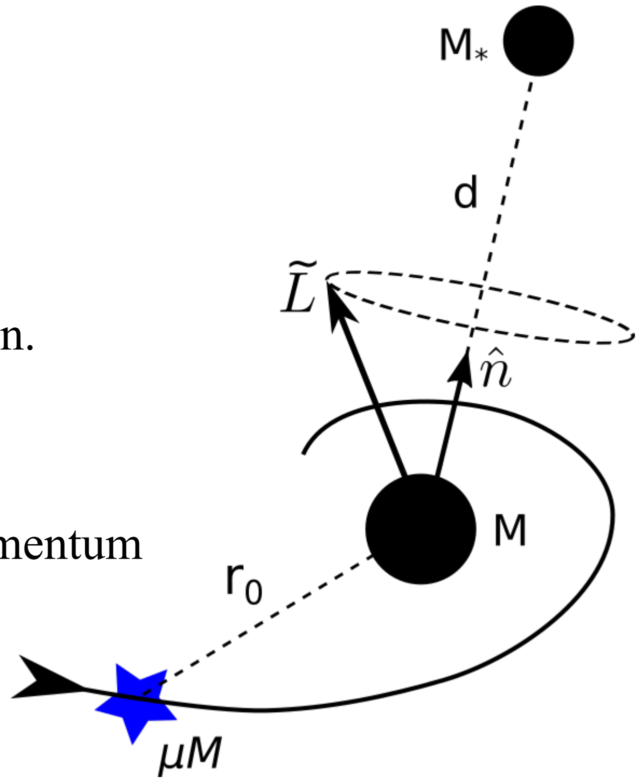

Precession around $\hat{n}$

Precession

- The precession frequency is given by, (align $\hat{z}$ with $\hat{n}$):

$$\Omega_{\rm prec} = \frac{1}{\Gamma_t \Lambda_r \Lambda_\theta} \int_0^{\Lambda_r} d\lambda_r \int_0^{\Lambda_\theta} d\lambda_\theta \left(\frac{L_x dL_y}{d\tau} - \frac{L_y dL_x}{d\tau} \right) \frac{r^2}{Q}$$

With average lapse rate:

$$\Gamma_t = \left\langle \frac{dt}{d\lambda} \right\rangle = \frac{E}{\Lambda_r} \int_0^{\Lambda_r} d\lambda \frac{r^2(\lambda)}{1 - 2M/r(\lambda)}$$

$$\frac{1}{\mu M} \frac{dL_x}{d\tau} = -a^\theta r^2 \sin\phi, \quad \ldots$$

$$\Omega_{\rm prec} \propto \frac{M M_*}{d^3}$$

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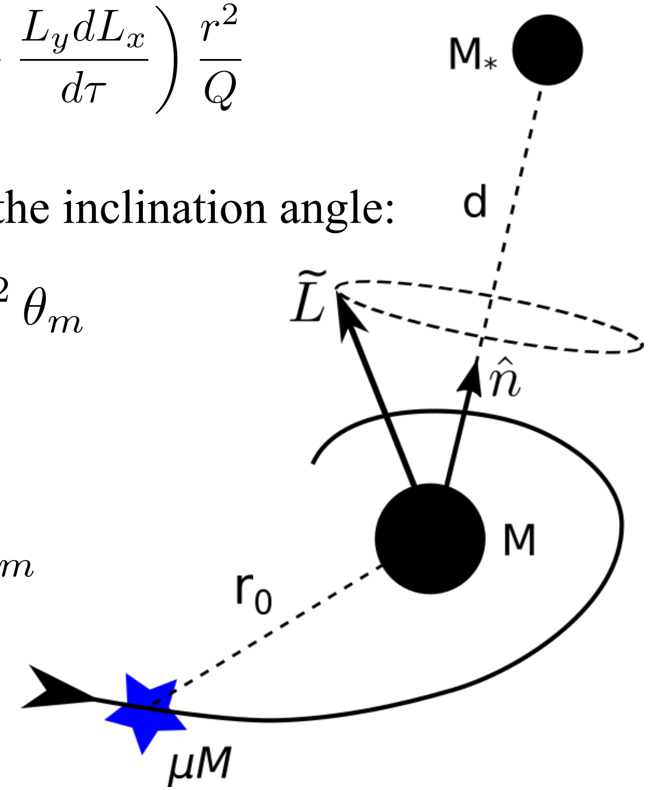
- The orbit-averaged interaction energy depends on the inclination angle:

$$E_{\rm int} \propto 1 - 3(\hat{n} \cdot \hat{L})^2 = 1 - 3\sin^2\theta_m$$

- Torque is $dE_{\rm int}/d\theta_m \propto \sin\theta_m \cos\theta_m$
- Orthogonal piece of angular momentum $\propto \cos\theta_m$

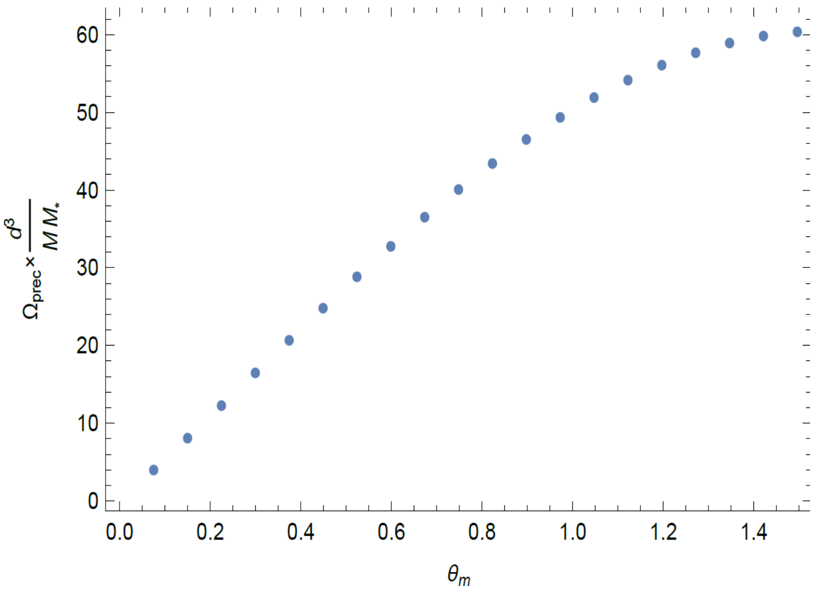


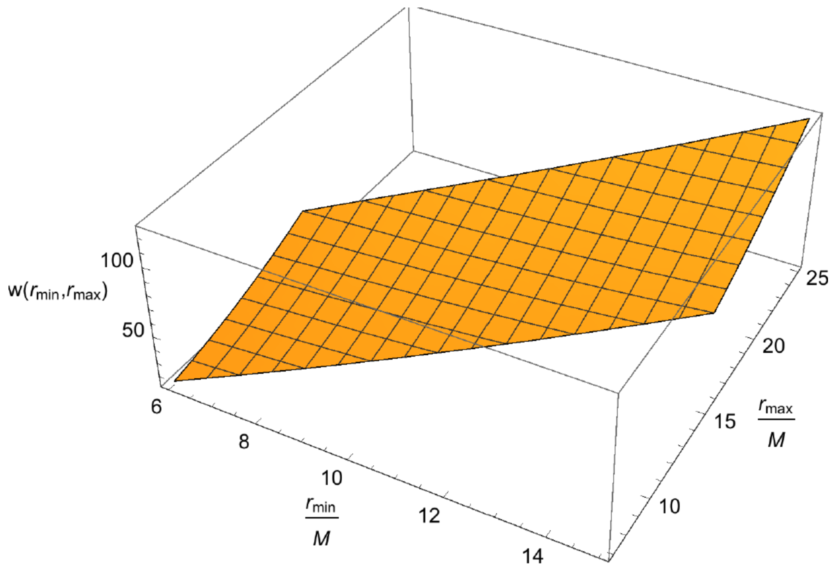
$$\Omega_{\rm prec} \propto \sin\theta_m$$

Precession

- The precession frequency can be expressed by:

$$\Omega_{\rm prec} = w(r_{\rm min}, r_{\rm max})\,\frac{M M_*}{d^3}\,\sin\theta_m$$





Precession

- The precession frequency can be expressed by:

$$\Omega_{\text{prec}} = w(r_{\min}, r_{\max}) \frac{M M_*}{d^3} \sin\theta_m$$

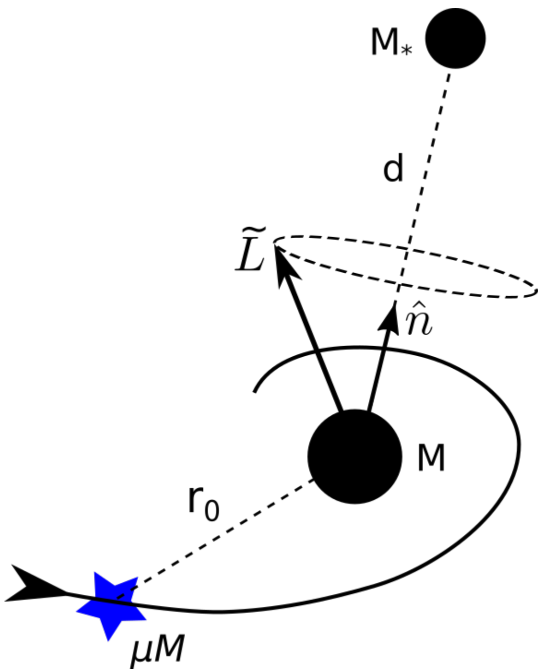
- A reasonably accurate fit:

$$w \approx 1.3 \left(\frac{r_{\max} + r_{\min}}{2M} \right)^{3/2}$$

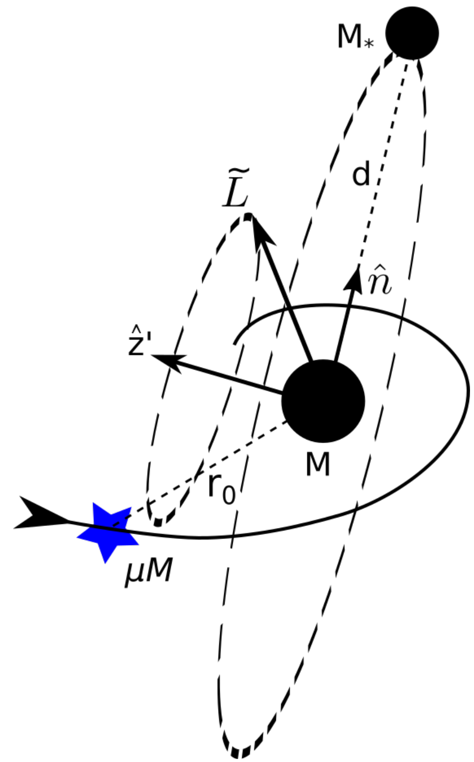
- Precession for stellar mass (10 Msun) black hole binary near a SMBH

$$\frac{2\pi}{\Omega_{\text{prec}}} \sim 2.6\, days \left(\frac{d}{30 M_*} \right)^3 \left(\frac{M_*}{M_{\text{SgA}^*}} \right)^2 \left(\frac{f_{GW}}{1\text{mHz}} \right)^{2/3}$$

Precession



Average over outer orbit



$$\frac{d\mathbf{L}}{dt} = w(r_{\min}, r_{\max})\frac{MM_*}{d^3}(\hat{n}\cdot\hat{L})\hat{n}\times\mathbf{L}$$

$$\left\langle\frac{d\mathbf{L}}{dt}\right\rangle = -w(r_{\min}, r_{\max})\frac{MM_*}{2d^3}(\hat{z}'\cdot\hat{L})\hat{z}'\times\mathbf{L}$$

Relativistic Kozai mechanism

- The argument about conservation of |L| is invalid for closed orbit.
- Such orbits correspond to transient resonances:

$$\Lambda = p\Lambda_r = q\Lambda_\theta$$

- Instead of averaging over the 2-D ring, now average over 1-D trajectory.

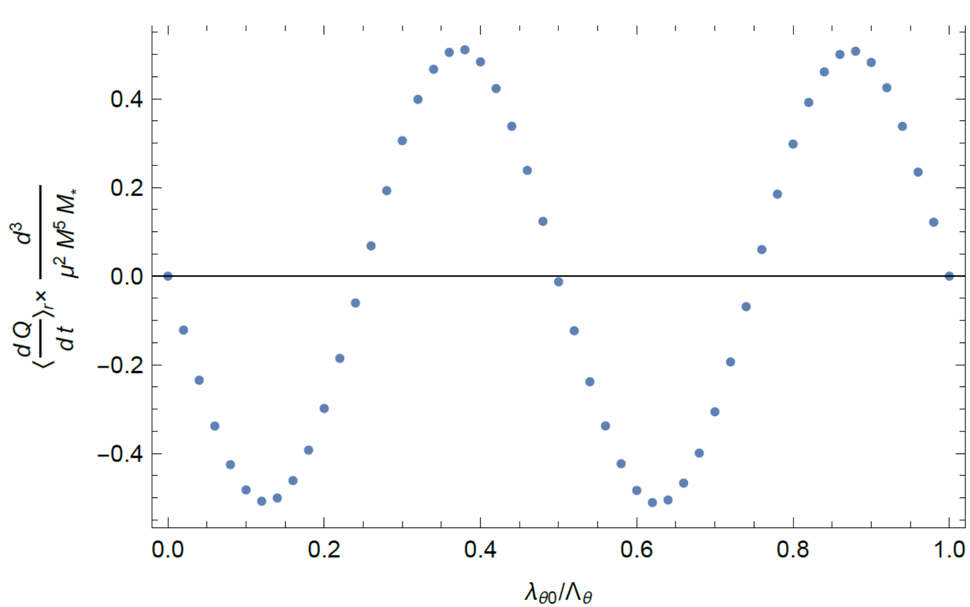
$$\left\langle \frac{d\mathcal{C}}{d\lambda} \right\rangle = \int_0^{\Lambda} d\lambda \, r^2 \, \frac{d\mathcal{C}}{d\tau}$$

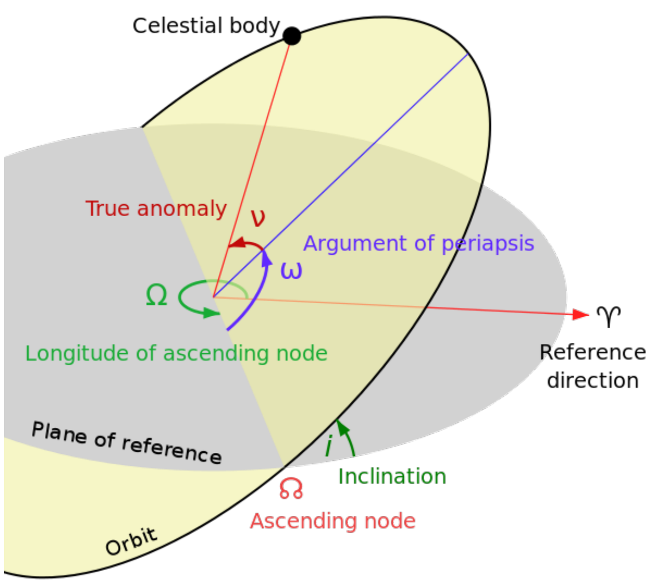
Relativistic Kozai mechanism

- Pick a 1:2 resonance, with $r_{\min} = 7M, \quad r_{\max} = 9.391M$
- Conservation of energy and angular momentum around symmetry axis

$$\langle dE/d\lambda \rangle = \langle dL_z/d\lambda \rangle = 0$$

- The change rate of total angular momentum may be nonzero!





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- The change rate of total angular momentum may be nonzero!
- A whole new set of Kozai-type configurations around the relativistic transient resonance points. Newtonian limit = 1:1 resonance!
- An unique signature for tidally-perturbed Kerr/Schwarzschild metric: test gravity.

Astrophysical relevance

- The scaling of dissipative part of self force and the tidal force:

$$a_s \sim \mu v^9/M \sim (M/r_0)^{9/2}\mu/M \qquad a_{\rm tide} \sim M_* r_0/d^3$$

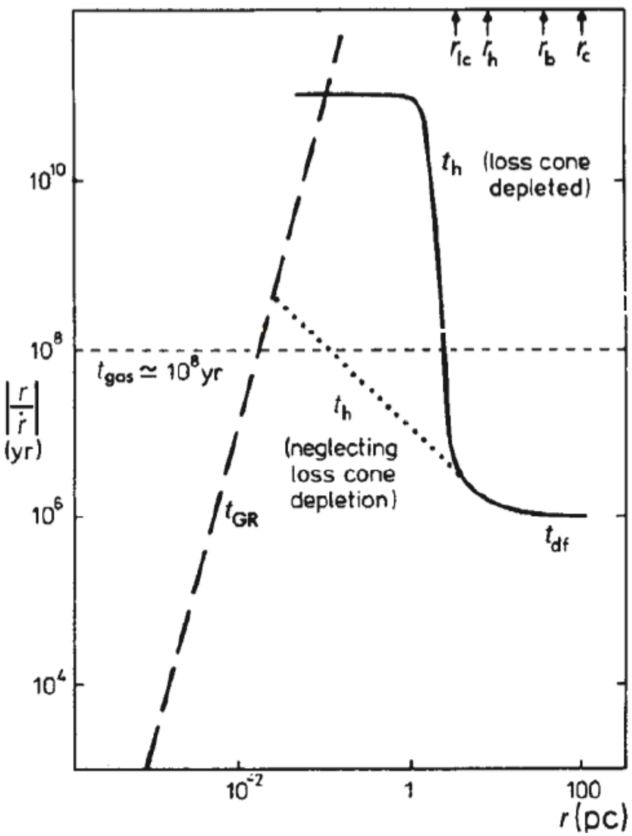
- Phase correction due to self force contribution during resonance $\sim \mu^{-1/2}$ cycles
- Phase correction due to tidal force during resonance $\sim \mu^{-1/2} a_{\rm tide}/a_s$ cycles
- Phase resolution of LISA: $\delta\phi \sim \frac{2\pi}{\rm SNR}$
- The tidal effect is visible if

$$d \le 0.1{\rm pc}\left(\frac{\mu}{10^{-6}}\right)^{-1/2}\left(\frac{M_*}{M}\right)^{1/3}\left(\frac{\rm SNR}{40}\right)^{1/3}\left(\frac{r_0}{15M}\right)^{11/6}\frac{M}{M_{SgA_*}}$$

Astrophysical relevance

- Tens of percent of Milky-way alike galaxies experience a MBH merger within the past 10 Gyrs. Bell et al. 2006

- Time spent for distance less than sub-parsec is uncertain: final parsec problem.



Begelman, Rees, and Blandford 1980

Astrophysical relevance

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- Time spent for distance less than sub-parsec is uncertain: final parsec problem.
- Take the merger time of MBH binary to be several Gyrs [Kelley et al. 2017], the time spent from sub-parsec scale might be several 10^8 yrs.
- The optimal EMRI rate > 10^3 yr^-1; the average rate >10^2 yr^-1 Gair et al. 2017
- The optimal rate of detection for tidal effect ~ a few yr^-1; more possible rate ~ 1 per several years.

$$\mathcal{O}(10\%) \times \frac{\text{several} \quad 10^8\text{yrs}}{10\,\text{Gyrs}} \times [> 10^3 yr^{-1}, \mathcal{O}(10^2) yr^{-1}]$$

ISCO shift

- A topic of mainly theoretical interest.
- Detweiler-type of gauge-invariant quantities relies on the assumption of helical symmetry.
- An angular averaged version of helical symmetry:

$$\int_0^{2\pi} d\phi \, \mathcal{L}_k \, h_{\alpha\beta} = \mathcal{O}(\mu^2), \quad k \equiv \partial_t + \Omega \partial \phi$$

- Gauge invariance of angular-averaged frequency:

$$\int_0^{2\pi} d\phi \, \Omega$$

ISCO shift

- Up to linear order in strength of tidal field, it suffices to consider mean-motion of ISCO.

- Hamiltonian of the mean motion:

$$\frac{H_I}{\mu^2 M^2} = -\frac{E^2}{2(1-2M/r)} + \frac{L^2}{2r^2} - \frac{M_*(1-3(\hat{n}\cdot\hat{L})^2)}{4d^3}\left(E^2 r^2 + \left(1 - \frac{2M^2}{r^2}\right)\right)$$

- ISCO condition:

$$H_I = -\frac{\mu^2 M^2}{2}\,, \quad \frac{\partial H_I}{\partial r} = \frac{\partial^2 H_I}{\partial r^2} = 0$$

- ISCO shift:

$$\Omega M = \frac{1}{6\sqrt{6}} - \frac{277}{54}\frac{M^2 M_*}{4d^3}\left(1 - 3(\hat{n}\cdot\hat{L})^2\right)$$

Future work

- Extend the study to Kerr.

- Joint evolution with radiation reaction.

- An EOB-type construction for cases with comparable mass-ratio inner binary.

- Stellar-mass triple systems in the PN regime, a self-consistent description of orbit including PN correction, tidal force, self force: Multiscale analysis / RG method.