Time-domain evolutions of Lorenz-gauge metric perturbations: taming the $\ell = 1$ gauge instability

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in collaboration with

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Metric perturbations

assume background metric g_{ab} is Ricci-flat, vacuum,
satisfies Einstein eqns (e.g., Schwarzschild, Kerr)

small metric perturbation: $g_{ab} \to g_{ab} + h_{ab}$, and perhaps also $T_{ab} \neq 0$

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Lorenz gauge condition $\bar{h}_{ab}{}^{|b} = 0$ zeros the red terms,
leaving a wave equation (hyperbolic, uncoupled in principal part):

$$\Box \bar{h}_{ab} + 2R^{c}{}_{a}{}^{d}{}_{b}\bar{h}_{cd} = -16\pi T_{ab}$$

$\Rightarrow$ nice for time-domain evolutions

Lorenz gauge properties

Point particles

Lorenz gauge is well-behaved in the presence of point-particle perturbations: the metric perturbation from a point particle is

- localized ($\sim 1/r$ falloff away from the particle)
- nonsingular everywhere away from the particle
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In general these nice properties do **not** hold for other gauges, e.g., Regge-Wheeler or radiation gauge (infinite-string gauge singularities).

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(if a suitable constraint-damping scheme is used)

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But they were unable to stabilize the **dipole mode**

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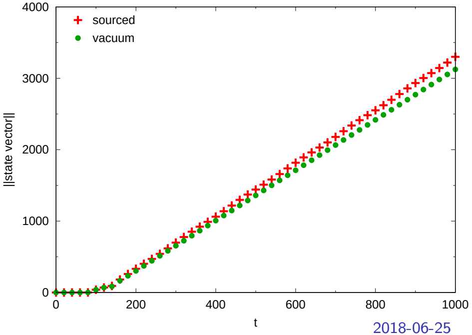
$\ell = m = 1$ Cauchy evolution

same (random-Gaussians) initial data for both evolutions
$\Rightarrow$ constraint-violating, but constraint violations damp away

sourced evolution: point particle in circular orbit at $r = 10.1M$



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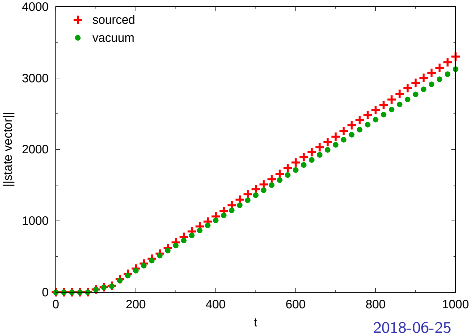

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Subtracting (orthogonalizing) the homogeneous mode

Observe that our main interest is in sourced evolutions
(either with a point particle or with an effective source). So …

- define (choose) an inner product (u_1, u_2) on state vectors;
  this implicitly also defines a norm $\|u\| = \sqrt{(u, u)}$

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$\Rightarrow$ since u_{hom} is a homogeneous solution and (in between updates) λ is just a fixed complex scalar, u_{diff} is also a solution of the sourced evolution eqns (the hope is that u_{diff} will not have the growing mode)

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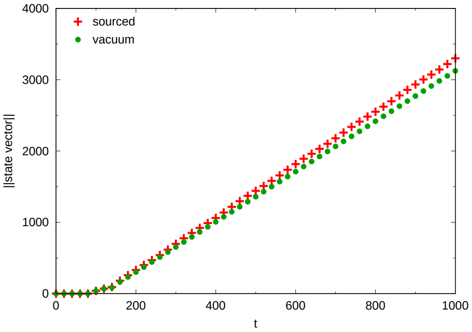

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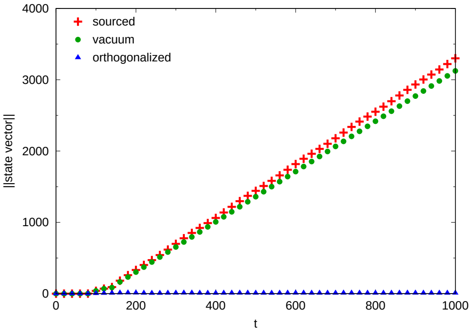

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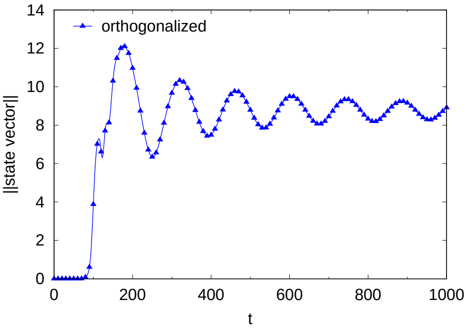

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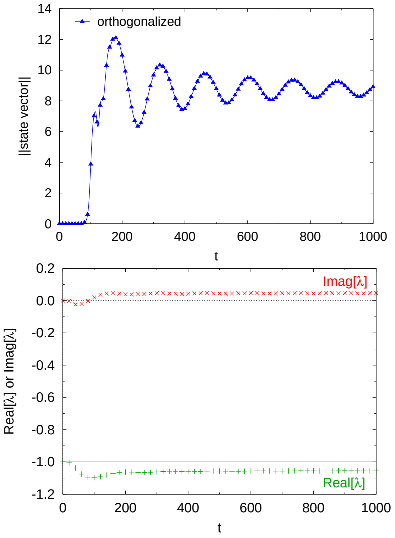

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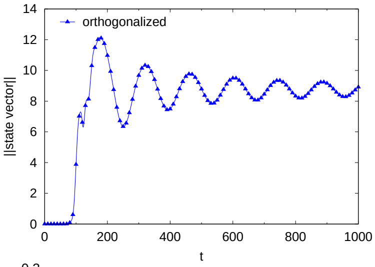

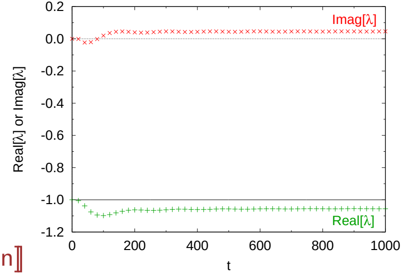

⟦movie of orthogonalized evolution⟧
(note greatly expanded vertical scale compared to first movie!)

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