A spinning test black hole in curved spacetime

Justin Vines

AEI Potsdam

Capra 2018, AEI Potsdam

Mathisson-Papapetrou-Dixon dynamics

- Dynamics of an "extended test body", worldline $z(\lambda)$ in background g_{ab},

$$\frac{Dp_a}{d\lambda} + \frac{1}{2} R_{abcd} \dot{z}^b S^{cd} = F_a = -\frac{1}{6} R_{bcde;a} J_2^{bcde} - \frac{1}{12} R_{bcde;fa} J_3^{fbcde} + \ldots$$

$$\frac{DS^{ab}}{d\lambda} - 2p^{[a} \dot{z}^{b]} = N^{ab} = \frac{4}{3} R^{[a}{}_{cde} J_2^{b]cde} + \ldots$$

- Multipoles $J_n^{abcd\cdots}$ depend on body's internal structure and dynamics
- What should they be for a **"spinning test black hole"** in vacuum**?**

  —an infinitely-small-mass, ultra-super-extremal Kerr
  naked ring singularity of finite radius $a = S/m$? (string worldsheet?)
- Assume only d.o.f.s are z, p, S (others "integrated out") —"minimal MPD"
- Constrain $p_a S^{ab} = 0$, solve for $\dot{z}^a$
- $\Rightarrow$ $J_n^{abcd\cdots}$ built covariantly from only p^a, S^{ab} and $R_{abcd;\cdots}(z)$

Couplings in an effective action

- Action approach to minimal MPD: $J_n^{abcd\cdots}(p,S,R)$ all determined by one scalar function $\mathcal{M}^2(u,S,R)$, where $u^a = \dfrac{p^a}{\sqrt{-p^2}}$, such that $p^2 + \mathcal{M}^2 = 0$: dynamical mass shell condition

$$F_a = \frac{p\cdot\dot z}{2}\nabla_a^{\text{horizontal}}\log\mathcal{M}^2,$$
$$N^{ab} = p\cdot\dot z\left(p^{[a}\frac{\partial}{\partial p_{b]}} + 2S^{[a}{}_c\frac{\partial}{\partial S_{b]c}}\right)\log\mathcal{M}^2.$$

- From matching to (unperturbed) (linearized) Kerr, we know

$$\mathcal{M}^2 = m^2 + 2m^2\left(-\frac{R_{uaua}}{2!} + \frac{R^*_{uaua;a}}{3!} + \frac{R_{uaua;aa}}{4!} - \frac{R^*_{uaua;aaa}}{5!} + \ldots\right)$$

$$\boxed{+O(R^2)} \qquad \leftarrow \text{ : tidal effects?} \qquad\qquad \uparrow \text{ : } J_n \sim m\,a^n$$

Couplings in an effective action

- Action approach to minimal MPD: $J_n^{abcd\cdots}(p, S, R)$ all determined by one scalar function $\mathcal{M}^2(u, S, R)$, where $u^a = \dfrac{p^a}{\sqrt{-p^2}}$,
  such that $p^2 + \mathcal{M}^2 = 0$: dynamical mass shell condition
- (Rescaled) spin vector a^a, $S_{ab} = m\,\epsilon_{abcd}u^c a^d$, $u \cdot a = 0$,
  constant bare rest mass m, $\mathcal{M}^2 = m^2 + O(R)$
- From matching to (unperturbed) (linearized) Kerr, we know

$$\mathcal{M}^2 = m^2 + 2m^2\left(-\frac{R_{uaua}}{2!} + \frac{R^*_{uaua;a}}{3!} + \frac{R_{uaua;aa}}{4!} - \frac{R^*_{uaua;aaa}}{5!} + \dots\right)$$

$\boxed{+O(R^2)}$ $\leftarrow$: tidal effects? $\uparrow$: $J_n \sim m\,a^n$

Relevant couplings for a spinning test black hole

- Suppressing indices, $*$'s, u's, dimensionless coefficients,

$$\frac{\mathcal{M}^2}{m^2} = 1 + R\,a^2 + \nabla R\,a^3 + \nabla^2 R\,a^4 + \nabla^3 R\,a^5 + \nabla^4 R\,a^6 + \dots$$
$$\oplus\ R^2 a^4 \quad \oplus\ \nabla R^2 a^5 \ \oplus\ \nabla^2 R^2 a^6 \ \oplus \dots$$
$$\oplus\ R^3 a^6 \quad \oplus \dots$$

plus many other $R^{\geq 2}$ terms (with powers of m),

$$\oplus\ (m\nabla)^k (a\nabla)^l \left(\frac{a}{m}\right)^n \Big(m^4 R^2 \oplus m^6 R^3 \oplus \dots\Big),$$

e.g.: $m^4 R^2$ terms, $k, l, n = 0$: leading adiabatic quadrupolar tidal effects

- Reasonable conjecture?:
If a spinning test black hole limit exists, it should have only m^0 terms.

Curvature^2-spin^4 and -spin^5 terms; scattering angle

$$\frac{\mathcal{M}^2}{m^2} = 1 + R\,a^2 + \frac{2}{3!}\nabla R\,a^3 + \frac{2}{4!}\nabla^2 R\,a^4 + \frac{2}{5!}\nabla^3 R\,a^5 + \ldots$$

$$\oplus\; C_4\,R^2 a^4 \quad \oplus\; C_5\,\nabla R^2 a^5 \;\oplus \ldots$$

- A simple gauge-invariant observable: the post-Minkowskian expansion of the scattering angle χ for aligned-spin scattering in Schwarzschild, $\gamma = \frac{1}{\sqrt{1-v^2}}$, at infinity, b : impact parameter,

$$\chi = \frac{2GM}{\gamma^2 v^2 b}\,\frac{2\gamma^2 - 1 - 2\gamma^2 va/b}{1 - a^2/b^2}$$

$$+ \frac{3\pi G^2 M^2}{v^2 b^2}\left[\frac{5\gamma^2-1}{4\gamma^2} - \frac{5\gamma^2-3}{2\gamma^2 v}\,\frac{a}{b} + \frac{15\gamma^4 v^2 + 2}{4\gamma^4 v^2}\,\frac{a^2}{b^2} - \frac{5\gamma^2-2}{\gamma^2 v}\,\frac{a^3}{b^3}\right.$$

$$\left. + \left(5\frac{5\gamma^2-4}{4\gamma^2 v^2} + C_4\frac{\gamma^4\ldots}{\gamma^2\ldots}\right)\frac{a^4}{b^4} + \left(\frac{\gamma^4\ldots}{\gamma^2\ldots} + C_5\frac{\gamma^4\ldots}{\gamma^2\ldots}\right)\frac{a^5}{b^5} + \ldots\right] + O(G^3)$$

The high-energy limit

$$\chi = \frac{2GM}{\gamma^2 v^2 b}\,\frac{2\gamma^2 - 1 - 2\gamma^2 va/b}{1 - a^2/b^2}$$

$$+\frac{3\pi G^2M^2}{v^2b^2}\left[\frac{5\gamma^2-1}{4\gamma^2} - \frac{5\gamma^2-3}{2\gamma^2 v}\,\frac{a}{b} + \frac{15\gamma^4v^2+2}{4\gamma^4v^2}\,\frac{a^2}{b^2} - \frac{5\gamma^2-2}{\gamma^2 v}\,\frac{a^3}{b^3}\right.$$

$$\left.+\left(5\frac{5\gamma^2-4}{4\gamma^2v^2} + C_4\frac{\gamma^4\cdots}{\gamma^2\cdots}\right)\frac{a^4}{b^4} + \left(\frac{\gamma^4\cdots}{\gamma^2\cdots} + C_5\frac{\gamma^4\cdots}{\gamma^2\cdots}\right)\frac{a^5}{b^5} + \ldots\right] + O(G^3)$$

- Ultra-relativistic/high-energy/null limit: $v \to 1, \qquad \frac{E}{m} = \gamma \to \infty$
- If we demand that χ is finite as $\gamma \to \infty$, then C_4's $= 0$, and there is a one-param. family of solutions for the C_5's (preliminary)

$$\chi|_{\gamma\to\infty} = \frac{4GM}{b+a} + \frac{15\pi}{4}\frac{G^2M^2}{b^2}\left[1 - \frac{2a}{b} + \frac{3a^2}{b^2} - \frac{4a^3}{b^3} + \frac{5a^4}{b^4} + \ldots = \frac{b^2}{(b+a)^2}\,?\right]$$

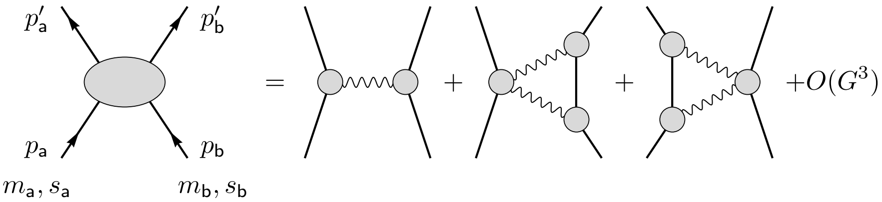
Matching calculations

- How to determine C_5 , ... ?

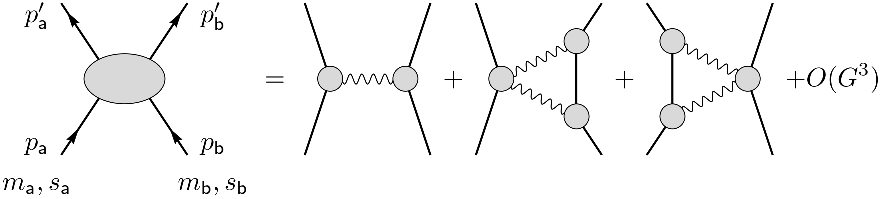
$$\frac{\mathcal{M}^2}{m^2} = 1 + R\,a^2 + \frac{2}{3!}\nabla R\,a^3 + \frac{2}{4!}\nabla^2 R\,a^4 + \frac{2}{5!}\nabla^3 R\,a^5 + \ldots$$
$$+ \;0\,R^2 a^4 \qquad + \;C_5\,\nabla R^2 a^5 \;\oplus \ldots$$

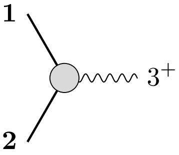
- BH perturbation theory + self-force-type analysis ... ?
- Or is it enough to match to unperturbed Kerr beyond linear order?
- Match to classical limits of relativistic quantum scattering amplitudes for minimally coupled massive spin-s particles exchanging gravitons?

($s \to \infty$?) (for arbitrary mass ratios)

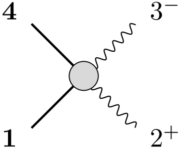


Amplitudes for massive spin-s particles





$$= \frac{1}{m_{\rm P}} \left(\frac{\langle \zeta | p_1 | 3]}{\langle \zeta 3 \rangle} \right)^2 \left(\frac{\langle \mathbf{12} \rangle}{m} \right)^{2s}, \qquad \text{for any } s,$$



$$= \frac{-\langle 3 | p_1 | 2]^4}{m_{\rm P}^2 \mathsf{t} (\mathsf{s} - m^2)(\mathsf{u} - m^2)} \left(\frac{\langle \mathbf{4}3 \rangle [\mathbf{12}] + \langle \mathbf{1}3 \rangle [\mathbf{4}2]}{\langle 3 | p_1 | 2]} \right)^{2s},$$

for $s \leq 2$.

[Arkani-Hamed+ '17, Guevara '17]