Fast Self-Forced Inspirals

Niels Warburton
Royal Society - Science Foundation Ireland University Research Fellow
University College Dublin

In collaboration with Maarten van de Meent
Albert Einstein Institute

21st Capra Meeting
Albert Einstein Institute
June 25-29, 2018

Motivation

LISA Data Analysis Work Packages:
LISA-LCST-SGS-WPD-001
Section 1.2:

"Design and implement a framework for incorporating self-force-based numerical calculations, as they become available, into a flexible semi-analytical Kludge model that enables fast production of waveform templates"

EMRI waveform generation

3 Steps:

- 1. Phase space trajectory

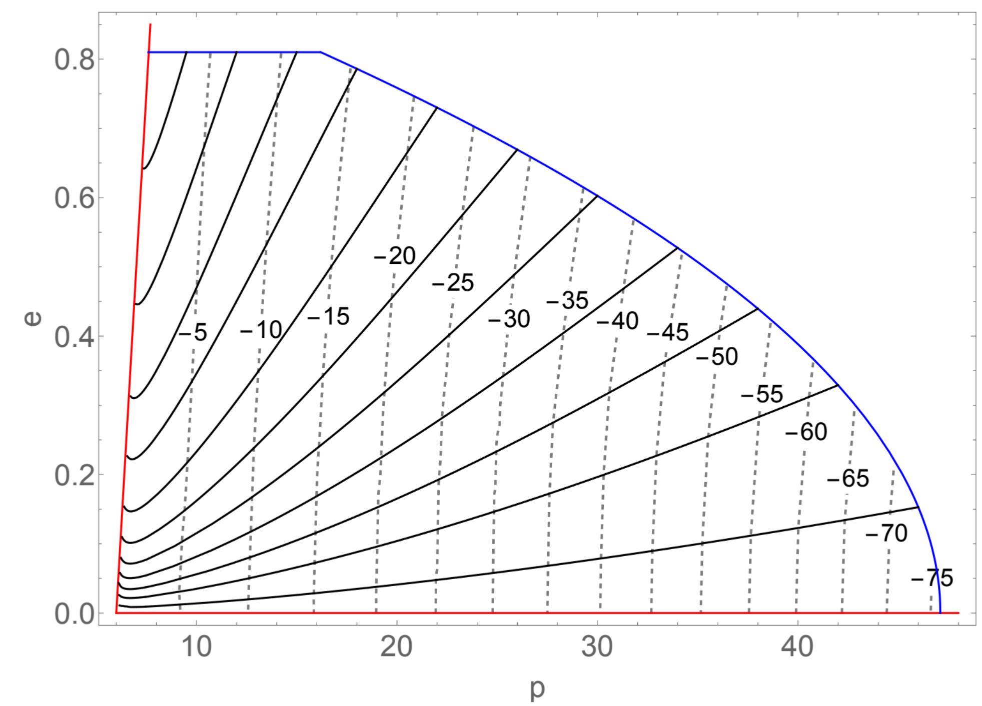
  $\rightarrow \{\mathscr{E}(\lambda), \mathscr{L}(\lambda), \mathscr{Q}(\lambda)\}$

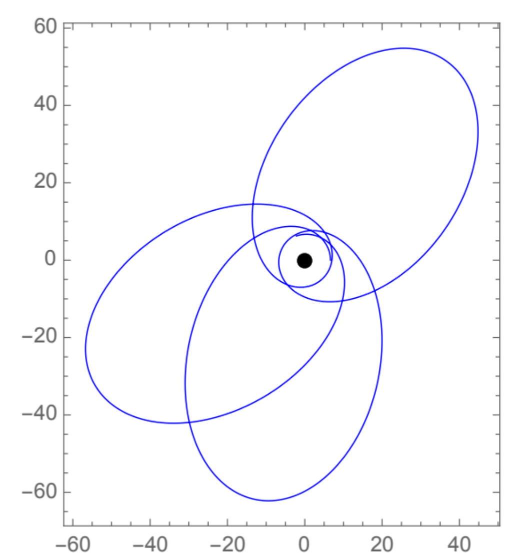
- 2. Physical trajectory

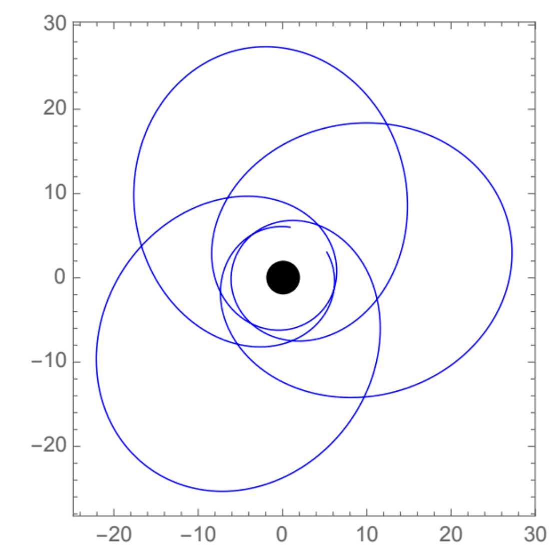
  $\rightarrow \{t(\lambda), r(\lambda), \varphi(\lambda), \theta(\lambda)\}$

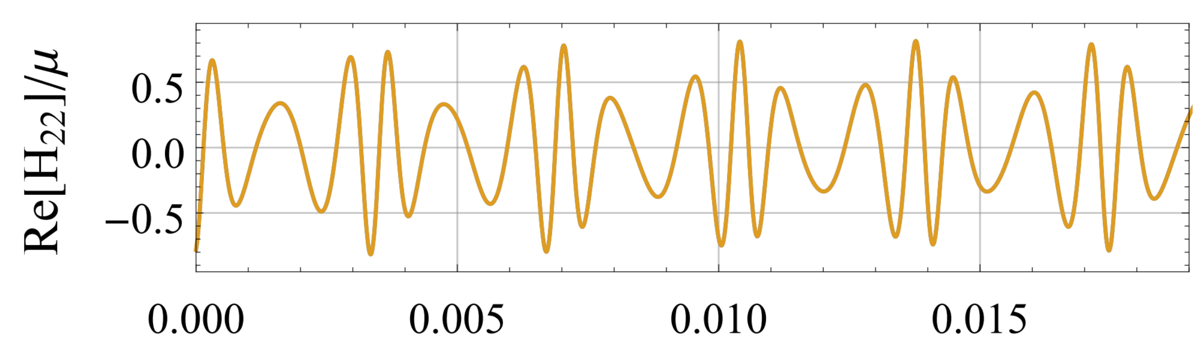
  Focus of this

- 3. Waveform
  - Quadrupole/Octupole
  - Teukolsky









State of the art

Method	Description	Fast?	Accurate?
Flux balance	Balance local changes in (E,L) with radiated fluxes	Yes	$O(q^0)$ error in phase
Kludge	Combine approximate description of the motion with pN and flux data	Yes	~$O(q^0)$ error in phase
Self-force	Compute the local force acting on the particle	No, inclusion of local self-force adds (short) orbital timescale	Yes, once second-order SF included: error is $O(q)$ in the phase

Self-force equations of motion

$$\dot{P}_j = 0 + \epsilon F_j^{(1)}(\vec{P}, \vec{q}) + \epsilon^2 F_j^{(2)}(\vec{P}, \vec{q}) + \mathcal{O}(\epsilon^3)$$

$$\dot{q}_i = \Omega_i(\vec{P}) + \epsilon f_i^{(1)}(\vec{P}, \vec{q}) + \epsilon^2 f_i^{(2)}(\vec{P}, \vec{q}) + \mathcal{O}(\epsilon^3)$$

$$\dot{S}_k = s_k^{(0)}(\vec{P}, \vec{q}) + \epsilon s_k^{(1)}(\vec{P}, \vec{q}) + \epsilon^2 s_k^{(2)}(\vec{P}, \vec{q}) + \mathcal{O}(\epsilon^3)$$

where

$\vec{P} = \{P_1, \ldots, P_{j_{\max}}\}$ is some set of (geodesic) constants of motion which together specify a zeroth-order trajectory in phase space

$\vec{q} = \{q_1, \ldots, q_{i_{\max}}\}$ are some set of "phases" that specify where along a zeroth-order trajectory the system currently is

$\vec{S} = \{S_1, \ldots, S_{k_{\max}}\}$ are a set of quantities that are extrinsic to the EMRI's dynamics in the sense that the RHS functions in the evolution equations do not depend on them.

over dots represent differentiation with respect to some "time" parameter used for the evolution of the inspiral

Why is the current self-force approach slow?

EMRIs have at least two, disparate timescales

$$t_{orbit} \ll t_{RR}$$

RHS of equations of motion depend explicitly on orbital phases

$$\dot{P}_j = 0 + \epsilon F_j^{(1)}(\vec{P}, \underline{\vec{q}}) + \epsilon^2 F_j^{(2)}(\vec{P}, \underline{\vec{q}}) + \mathcal{O}(\epsilon^3)$$

$$\dot{q}_i = \Omega_i(\vec{P}) + \epsilon f_i^{(1)}(\vec{P}, \underline{\vec{q}}) + \epsilon^2 f_i^{(2)}(\vec{P}, \underline{\vec{q}}) + \mathcal{O}(\epsilon^3)$$

$$\dot{S}_k = s_k^{(0)}(\vec{P}, \underline{\vec{q}}) + \epsilon s_k^{(1)}(\vec{P}, \underline{\vec{q}}) + \epsilon^2 s_k^{(2)}(\vec{P}, \underline{\vec{q}}) + \mathcal{O}(\epsilon^3)$$

Numerically resolving all the associated small oscillations takes a long time

Near-identity (averaging) transformations

$$\tilde{P}_j = P_j + \epsilon Y_j^{(1)}(\overrightarrow{P}, \overrightarrow{q}) + \epsilon^2 Y_j^{(2)}(\overrightarrow{P}, \overrightarrow{q}) + \mathcal{O}(\epsilon^3)$$

$$\tilde{q}_i = q_i + \epsilon X_i^{(1)}(\overrightarrow{P}, \overrightarrow{q}) + \epsilon^2 X_i^{(2)}(\overrightarrow{P}, \overrightarrow{q}) + \mathcal{O}(\epsilon^3)$$

This transformation has two important properties:

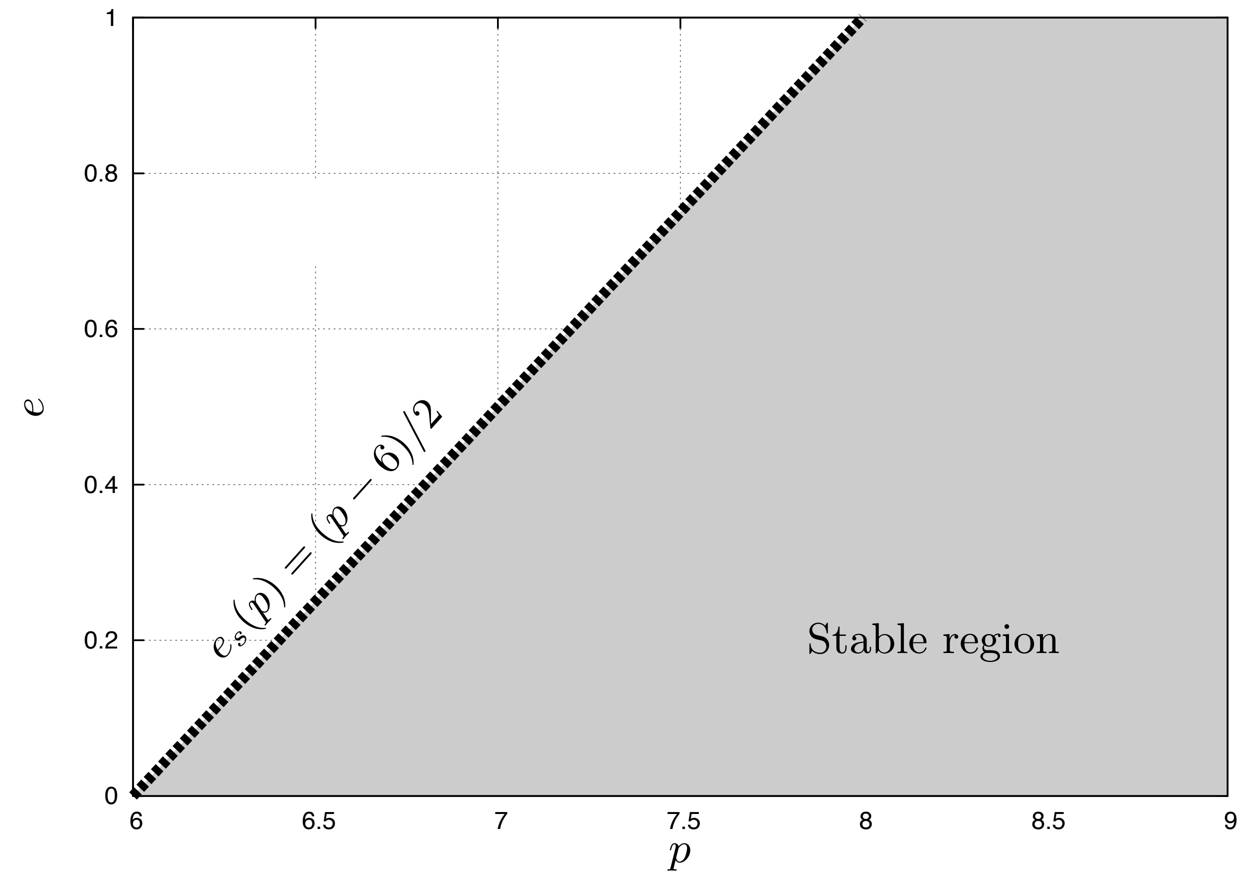
- The resulting equations of motion do not depend explicitly on the orbital phase

- The transformation is small (hence 'near-identity') such that the solution to the transformed equations of motion remains always close to the solution to the original equations of motion.

Schwarzschild EMRI parameterization

$$p = \frac{r_{min} r_{max}}{(r_{min} + r_{max})M} \qquad e = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}$$

$$r = \frac{pM}{1 + e\cos\xi} \qquad \xi = \chi - \chi_0$$



$$\overrightarrow{P} = \{p, e\}$$

$$\overrightarrow{q} = \{\xi\}$$

over dot is differentiation w.r.t. χ

$$\overrightarrow{S} = \{t, \varphi\}$$

Near identity transformations (Schwarzschild)

$$\dot{p} = qF_p(p, e, \xi)$$
$$\dot{e} = qF_e(p, e, \xi)$$
$$\dot{\xi} = 1 + qf_\xi(p, e, \xi)$$

Introduce Near-identity transformations (NITs):

$$\tilde{p} = p + qY_p^{(1)}(p, e, \xi) + q^2Y_p^{(2)}(p, e, \xi) + \mathcal{O}(q^3)$$
$$\tilde{e} = e + qY_e^{(1)}(p, e, \xi) + q^2Y_e^{(2)}(p, e, \xi) + \mathcal{O}(q^3)$$
$$\tilde{\xi} = \xi + qX_\xi^{(1)}(p, e, \xi) + \mathcal{O}(q^2)$$

Near identity transformations, O(q) example

NIT and Inverse NIT

$$\tilde{p} = p + qY_p^{(1)}(p, e, \xi) \qquad p = \tilde{p} - qY_p^{(1)}(p, e, \xi)$$

$$\tilde{e} = e + qY_e^{(1)}(p, e, \xi) \qquad e = \tilde{e} - qY_e^{(1)}(p, e, \xi)$$

$$\tilde{\xi} = \xi + qX_\xi^{(1)}(p, e, \xi) \qquad \xi = \tilde{\xi} - qX_\xi^{(1)}(p, e, \xi)$$

Derivative of NIT

$$\dot{\tilde{\xi}} = \dot{\xi} + q\left(\frac{\partial X}{\partial p}\frac{dp}{d\chi} + \frac{\partial X}{\partial e}\frac{de}{d\chi} + \frac{\partial X}{\partial \xi}\frac{d\xi}{d\chi}\right)$$

Substitute EoM and Inverse NIT

$$\dot{\tilde{\xi}} = 1 + q\left(f_\xi(\tilde{p}, \tilde{e}, \tilde{\xi}) + \frac{\partial X}{\partial \xi}(\tilde{p}, \tilde{e}, \tilde{\xi})\right)$$

$$\dot{p} = qF_p(p, e, \xi)$$

$$\dot{e} = qF_e(p, e, \xi)$$

$$\dot{\xi} = 1 + qf_\xi(p, e, \xi)$$

Split functions into const and oscillatory pieces

$$h(\tilde{p}, \tilde{e}, \tilde{\xi}) = \langle h \rangle(\tilde{p}, \tilde{e}) + \breve{h}(\tilde{p}, \tilde{e}, \tilde{\xi})$$

$$\dot{\tilde{\xi}} = 1 + q\left(\langle f_\xi \rangle + \breve{f}_\xi + \frac{\partial \breve{X}}{\partial \xi}\right)$$

Choose X to cancel $\breve{f}_\xi$

$$\breve{X} = -\int \breve{f}_\xi \, d\xi$$

Near identity transformations (Schwarzschild)

$$\dot{\tilde{\xi}} = 1 + q\tilde{f}_{\xi}^{(1)}(\tilde{p},\tilde{e}) + \mathcal{O}(q^3),$$

$$\dot{\tilde{p}} = q\tilde{F}_p^{(1)}(\tilde{p},\tilde{e}) + q^2\tilde{F}_p^{(2)}(\tilde{p},\tilde{e}) + \mathcal{O}(q^3),$$

$$\dot{\tilde{e}} = q\tilde{F}_e^{(1)}(\tilde{p},\tilde{e}) + q^2\tilde{F}_e^{(2)}(\tilde{p},\tilde{e}) + \mathcal{O}(q^3)$$

Equations of motion no longer depend on orbital phase

$$\tilde{f}_{\xi}^{(1)} = \langle f_{\xi}\rangle, \qquad \tilde{F}_p^{(1)} = \langle F_p\rangle, \qquad \tilde{F}_e^{(1)} = \langle F_e\rangle,$$

$$\tilde{F}_p^{(2)} = -\langle \breve{F}_p \int \frac{\partial \breve{F}_p}{\partial p} d\xi\rangle - \langle \breve{F}_e \int \frac{\partial \breve{F}_p}{\partial e} d\xi\rangle - \langle \breve{F}_p \breve{f}_{\xi}\rangle,$$

$$\tilde{F}_e^{(2)} = -\langle \breve{F}_e \int \frac{\partial \breve{F}_e}{\partial p} d\xi\rangle - \langle \breve{F}_e \int \frac{\partial \breve{F}_e}{\partial e} d\xi\rangle - \langle \breve{F}_e \breve{f}_{\xi}\rangle,$$

RHS functions computed from the self-force and its derivatives w.r.t. (p,e)

Near identity transformations, explicit variables

$$\dot{P}_j = 0 + \epsilon F_j^{(1)}(\vec{P}, \vec{q}) + \epsilon^2 F_j^{(2)}(\vec{P}, \vec{q}) + \mathcal{O}(\epsilon^3)$$

$$\dot{q}_i = \Omega_i(\vec{P}) + \epsilon f_i^{(1)}(\vec{P}, \vec{q}) + \epsilon^2 f_i^{(2)}(\vec{P}, \vec{q}) + \mathcal{O}(\epsilon^3)$$

$$\dot{S}_k = s_k^{(0)}(\vec{P}, \vec{q}) + \epsilon s_k^{(1)}(\vec{P}, \vec{q}) + \epsilon^2 s_k^{(2)}(\vec{P}, \vec{q}) + \mathcal{O}(\epsilon^3)$$

$$\vec{S} = \{t, \varphi\}$$

Not a NIT due to these terms

$$\dot{t} = \omega_t(p, e, \xi), \qquad \tilde{t} = t + Z_t^{(0)}(p, e, \xi) + q Z_t^{(1)}(p, e, \xi) + \mathcal{O}(q^2),$$

$$\dot{\phi} = \omega_\phi(p, e, \xi) \qquad \tilde{\phi} = \phi + Z_\phi^{(0)}(p, e, \xi) + q Z_\phi^{(1)}(p, e, \xi) + \mathcal{O}(q^2)\,.$$

$$\dot{\tilde{t}} = \frac{T_r(\tilde{p}, \tilde{e})}{2\pi} + q \tilde{f}_t^{(1)}(\tilde{p}, \tilde{e}) + \mathcal{O}(q^2),$$

$$\dot{\tilde{\phi}} = \frac{\Phi_r(\tilde{p}, \tilde{e})}{2\pi} + q \tilde{f}_\phi^{(1)}(\tilde{p}, \tilde{e}) + \mathcal{O}(q^2),$$

$$\tilde{f}_t^{(1)} = \langle \frac{\partial \breve{Z}_t^{(0)}}{\partial p} \breve{F}_p \rangle + \langle \frac{\partial \breve{Z}_t^{(0)}}{\partial e} \breve{F}_e \rangle,$$

$$\tilde{f}_\phi^{(1)} = \langle \frac{\partial \breve{Z}_\phi^{(0)}}{\partial p} \breve{F}_p \rangle + \langle \frac{\partial \breve{Z}_\phi^{(0)}}{\partial e} \breve{F}_e \rangle\,.$$

Explicit Schwarzschild example

To construct NIT equations of motion need SF and its derivatives w.r.t. (p,e)

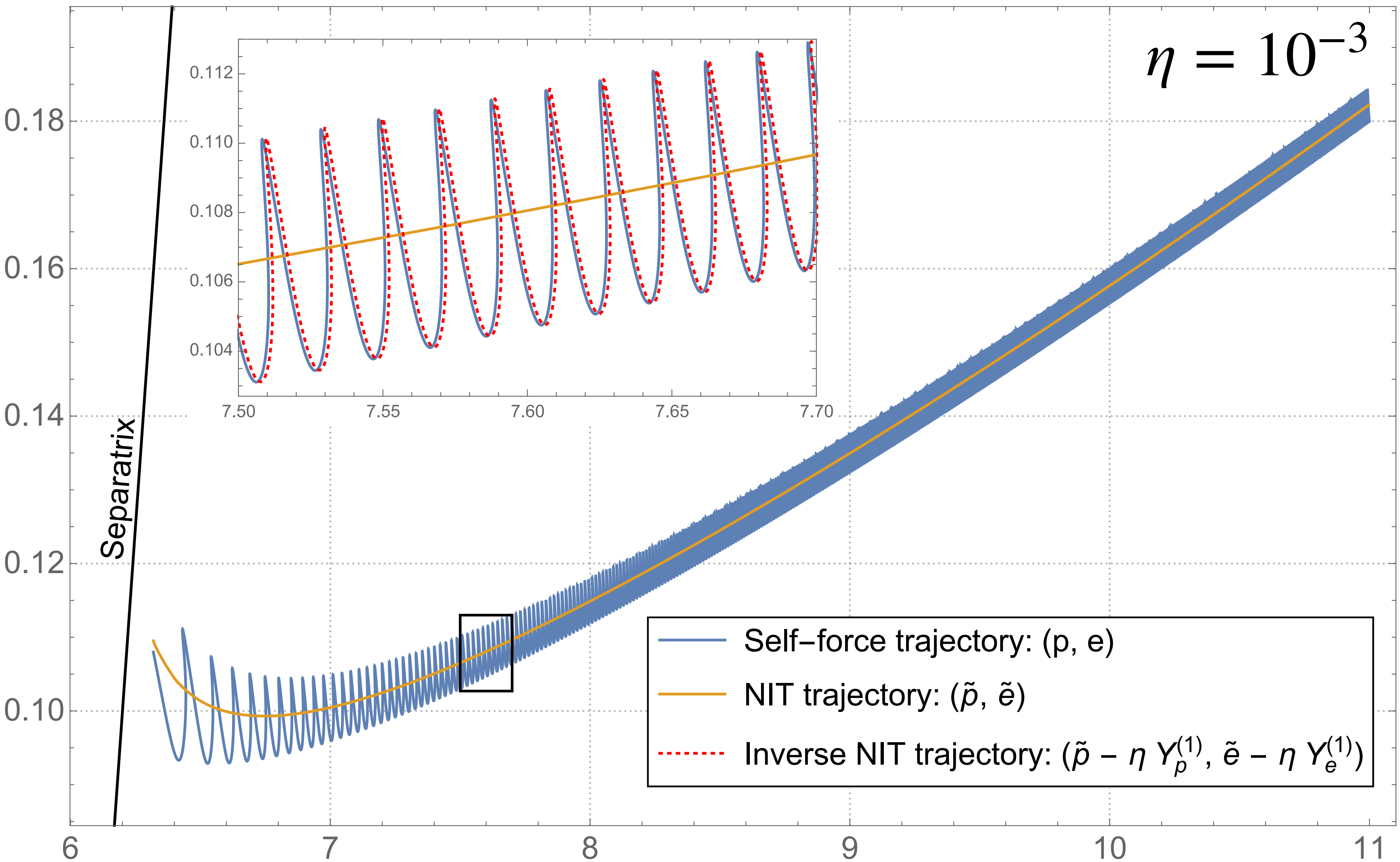
We used the analytically fitted model of
NW+, Phys. Rev. D 85, 061501(R) (2012), arXiv:1111.6908

This model is valid for 0 < e < 0.2, 6+2e < p < 12

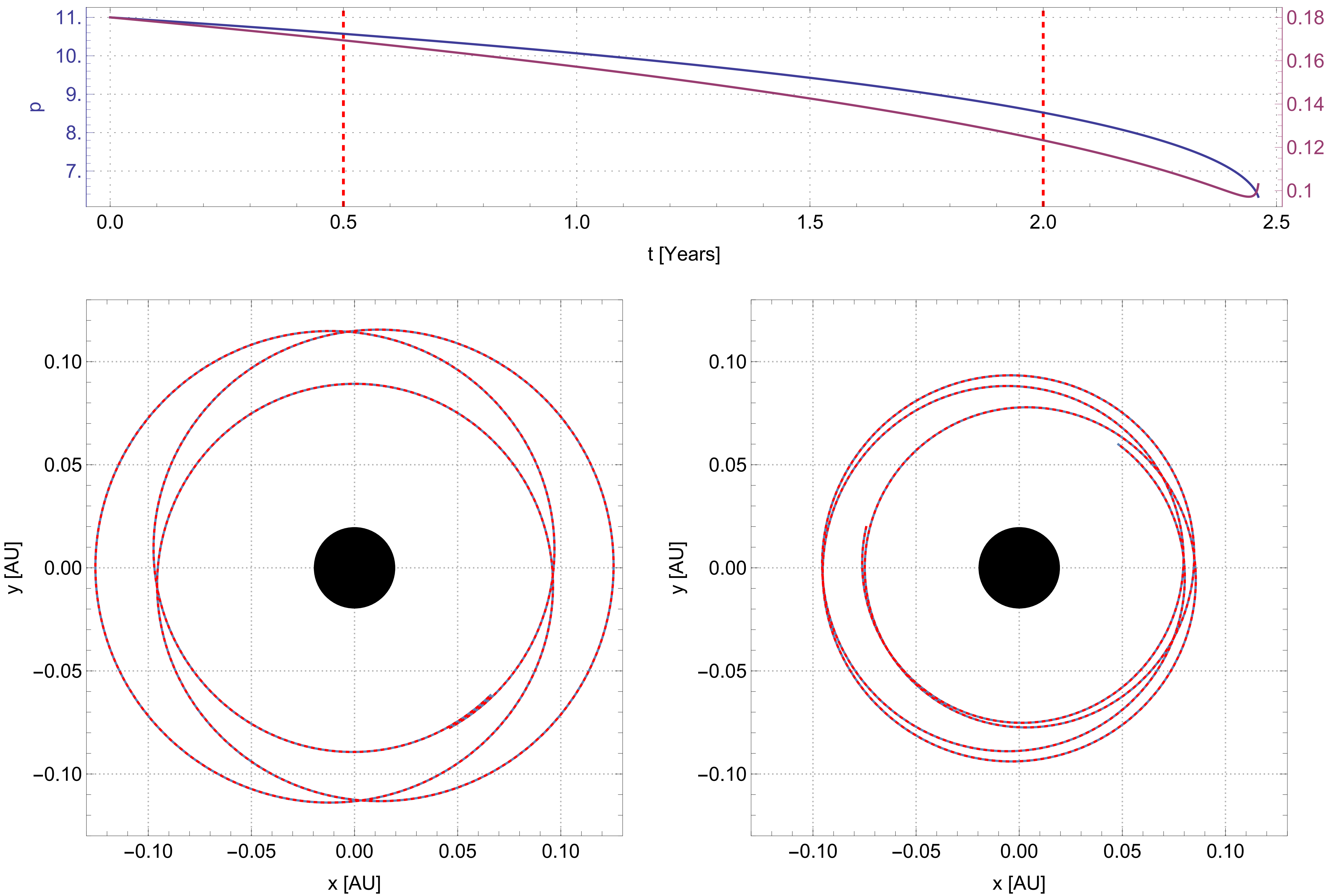
Steps:

- Combine SF model with Pound and Poisson osculating element equations to get $\{F_p, F_e, f_\xi\}$
- Decompose these into Fourier modes on a regular grid in parameter space
- Construct $\{\tilde{F}_p^{(1)}, \tilde{F}_p^{(2)}, \tilde{F}_e^{(1)}, \tilde{F}_e^{(2)}, \tilde{f}_\xi^{(1)}, \tilde{f}_t^{(1)}, \tilde{f}_\varphi^{(1)}\}$
- Interpolate these across parameter space
- Evolve inspiral using NIT'd equations of motion $\{\dot{\tilde{p}}, \dot{\tilde{e}}, \dot{\tilde{\xi}}, \dot{\tilde{t}}, \dot{\tilde{\varphi}}\}$
- Waveform from Kludgy quadrupole method

Results: inspiral track in phase space

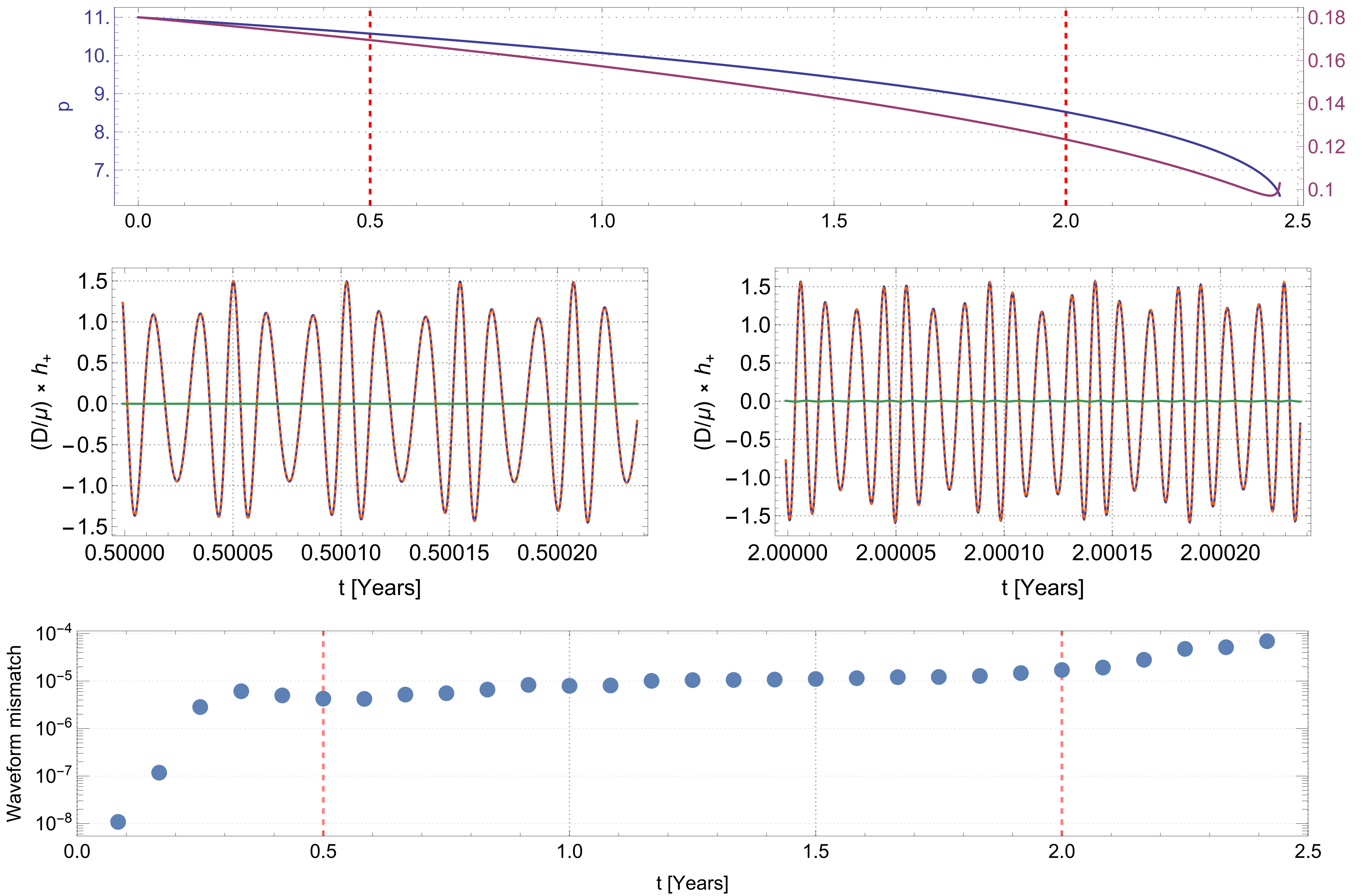


Results: inspiral and quadrupole waveform



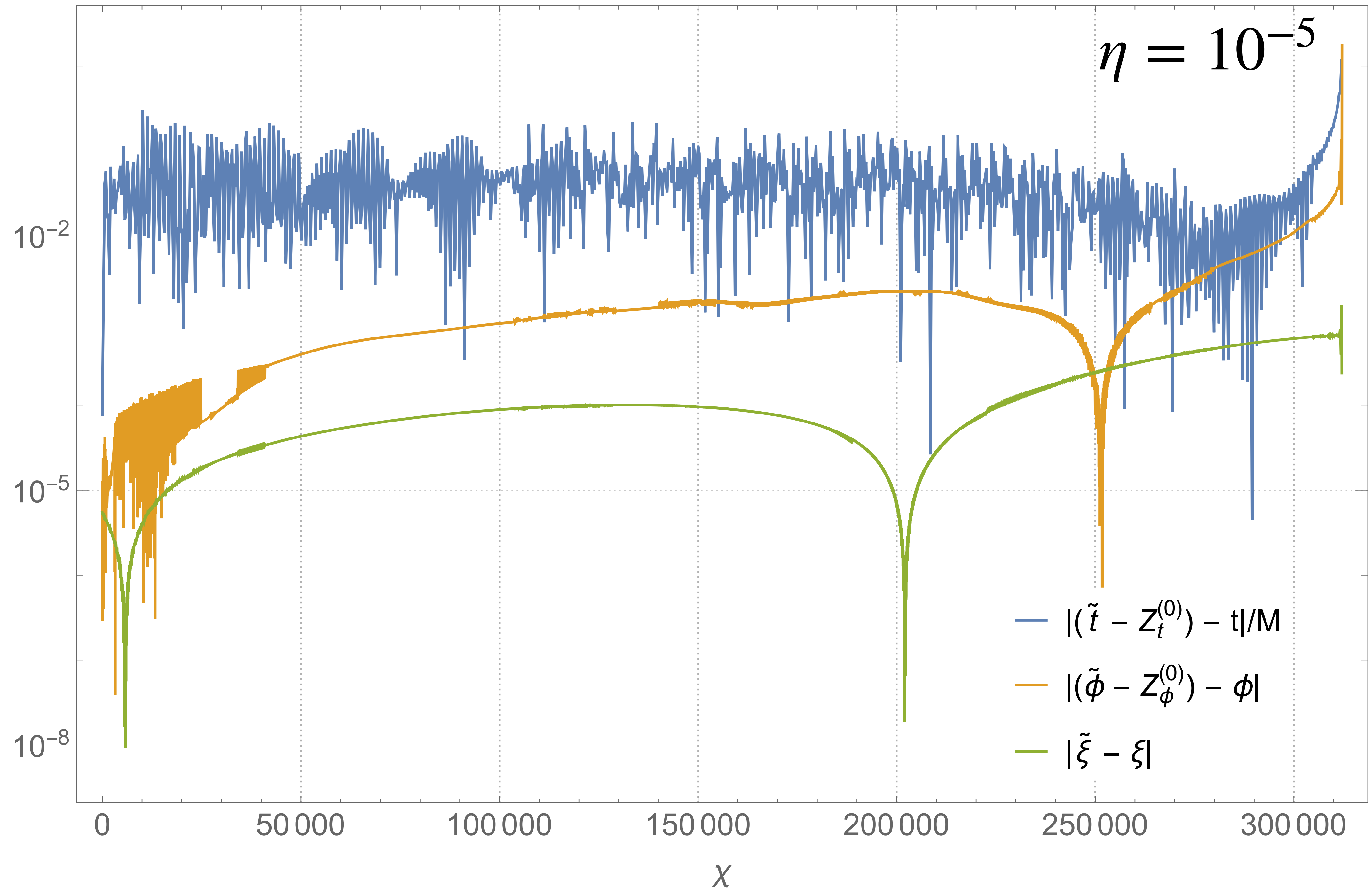
$$p_0 = 11, \quad e_0 = 0.18, \quad \eta = 10^{-5}, \quad M = 10^{-6} M_\odot$$

Results: inspiral and quadrupole waveform



$$p_0 = 11, \quad e_0 = 0.18, \quad \eta = 10^{-5}, \quad M = 10^6 M_\odot$$

Results: phase difference



Full inspiral and rapidly computed NIT inspiral remain in phase over almost all of the inspiral

Results: speed up

Time to calculate phase space inspiral

Mass Ratio	Time (Full)	Time (NIT)	Speed up
10^{-3}	6.2s	~0.008s	~700
10^{-4}	43s	~0.008s	~5,000
10^{-5}	5m40s	~0.008s	~40,000
10^{-6}	42m20s	~0.008s	~300,000
	C++	C++	

RHS of EoM does not depend on orbital phases
so no longer need to resolve the (short) orbital timescale

Approaches overview

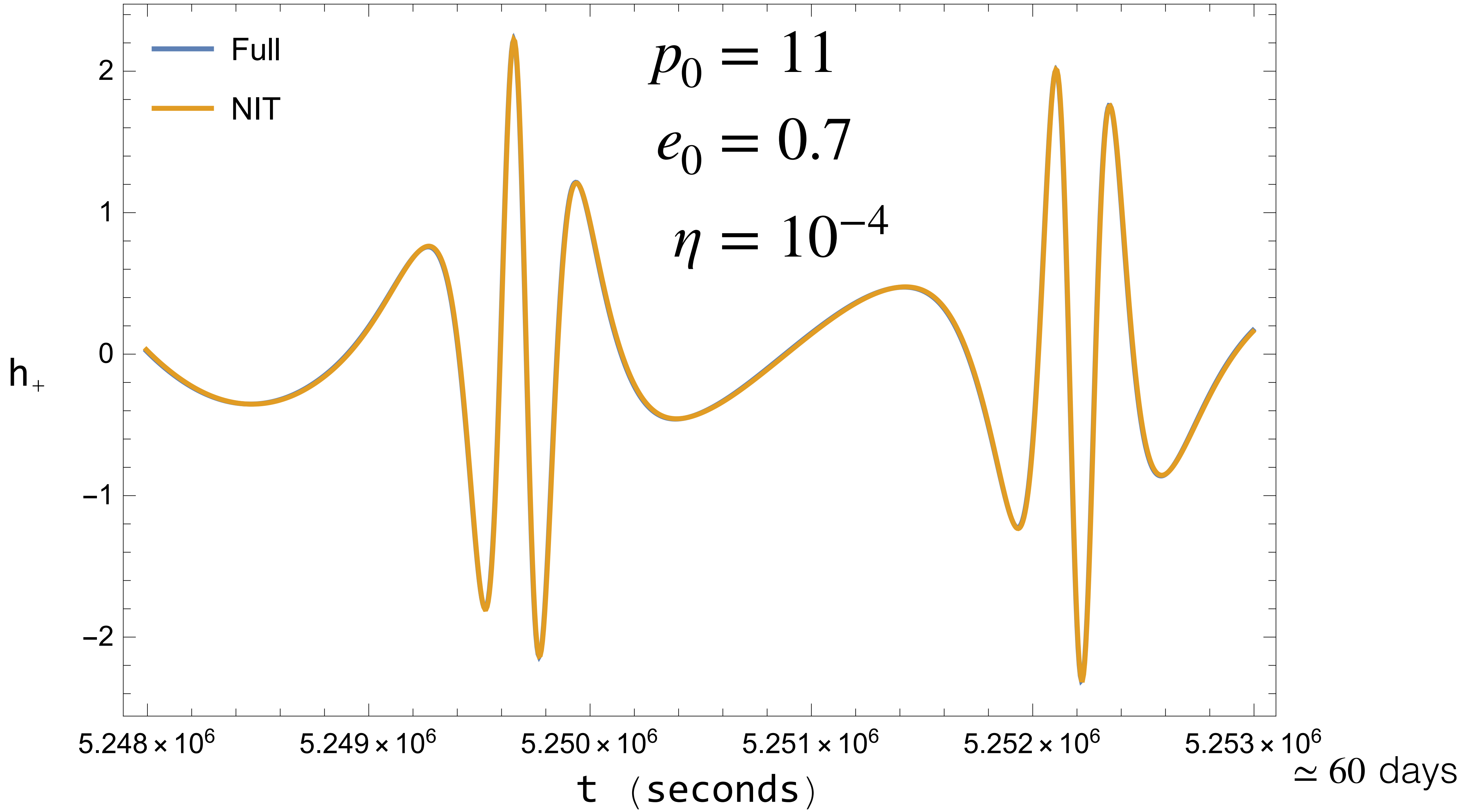
Method	Description	Fast?	Accurate?
Flux balance	Balance local changes in (E,L) with radiated fluxes	Yes ✓	$O(q^0)$ error in phase ✗
Kludge	Combine approximate description of the motion with pN and flux data	Yes ✓	~$O(q^0)$ error in phase ✗
Self-force	Compute the local force acting on the particle	No, inclusion of local self-force adds (short) orbital timescale ✗	Yes, once second-order SF included error is $O(q)$ in the phase ✓
Self-force (NIT)	Preprocess SF data for EoM that do not depend on orbital phase	Yes ✓	Yes, once second-order SF included: error is $O(q)$ in the phase ✓

Recap and future directions

"Design and implement a framework for incorporating self-force-based numerical calculations, as they become available, into a flexible semi-analytical Kludge model that enables fast production of waveform templates"

- Inspiral
  - Cover full Schwarzschild parameter space
  - Kerr
  - Direct calculation of SF derivatives w.r.t. $\vec{P}$
- Rapid waveform generation
  - Can we do better than Kludge methods?

Code available on Black Hole Perturbation Toolkit (bhptoolkit.org)



Already extended by T. Osburn to $6 + 2e < p < 24, \quad e < 0.8$