# Spin dissipation force in EMRIs

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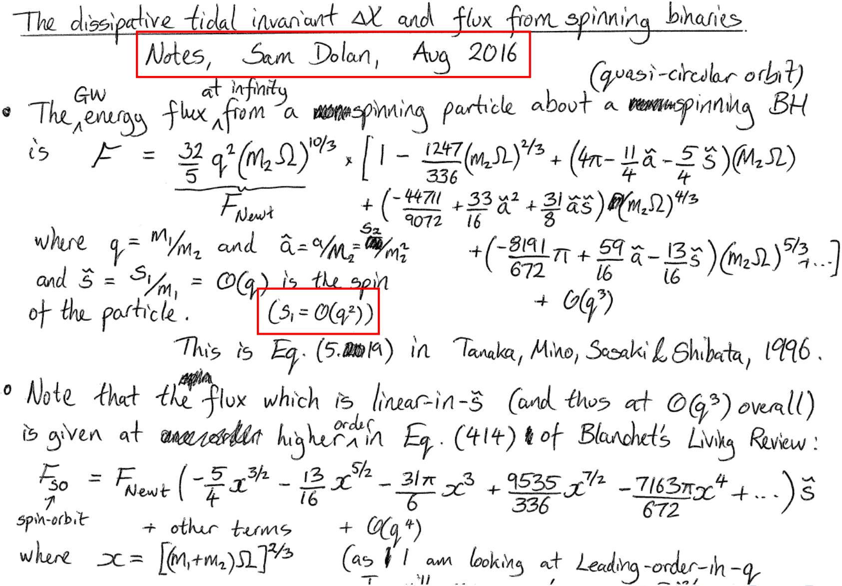
# Motivation: an invitation from Sam



The dissipative tidal invariant $\Delta\chi$ and flux from spinning binaries

Notes, Sam Dolan, Aug 2016

- The GW energy flux at infinity from a spinning particle about a spinning BH (quasi-circular orbit) is

$$\mathcal{F} = \underbrace{\frac{32}{5} q^2 (M_2\Omega)^{10/3}}_{F_{Newt}} \times \Big[ 1 - \frac{1247}{336}(M_2\Omega)^{2/3} + \left(4\pi - \frac{11}{4}\hat{a} - \frac{5}{4}\hat{s}\right)(M_2\Omega) + \left(-\frac{44711}{9072} + \frac{33}{16}\hat{a}^2 + \frac{31}{8}\hat{a}\hat{s}\right)(M_2\Omega)^{4/3} + \left(-\frac{8191}{672}\pi + \frac{59}{16}\hat{a} - \frac{13}{16}\hat{s}\right)(M_2\Omega)^{5/3} + \dots \Big] + \mathcal{O}(q^3)$$

where $q = m_1/M_2$ and $\hat{a} = a/M_2 = S_2/M_2^2$ and $\hat{s} = S_1/m_1 = \mathcal{O}(q)$ is the spin of the particle. $(S_1 = \mathcal{O}(q^2))$

This is Eq. (5.19) in Tanaka, Mino, Sasaki & Shibata, 1996.

- Note that the flux which is linear-in-$\hat{s}$ (and thus at $\mathcal{O}(q^3)$ overall) is given at higher order in Eq. (414) of Blanchet's Living Review:

$$\underset{\substack{\uparrow \\ \text{spin-orbit}}}{F_{SO}} = F_{Newt}\left(-\frac{5}{4}x^{3/2} - \frac{13}{16}x^{5/2} - \frac{31\pi}{6}x^3 + \frac{9535}{336}x^{7/2} - \frac{7163\pi}{672}x^4 + \dots\right)\hat{s}$$

$$+ \text{ other terms } + \mathcal{O}(q^4)$$

where $x = [(m_1+M_2)\Omega]^{2/3}$ (as I am looking at Leading-order-in-$q$

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Force-flux balance law with spins?

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Deficit in the $\mathcal{O}(qS_1) \sim \mathcal{O}(q^3)$ spin-orbit sector

$$F^{\text{spin}}_{t,\text{diss}} = \frac{\mathcal{F}^{\infty}_{22}}{2}\, qS^2_1 \left(-\frac{5}{4}y^{3/2} - \frac{11}{16}y^{5/2} - \frac{31\pi}{6}y^3 + \frac{1261}{56}y^{7/2} + \ldots\right)$$

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$$T^{\alpha\beta} = \int d\tau\, p^{(\alpha}u^{\beta)}\frac{\delta^{(4)}(x^\mu - z^\mu)}{\sqrt{-g}} - \nabla_\gamma \int d\tau\, S^{\gamma(\alpha}u^{\beta)}\frac{\delta^{(4)}(x^\mu - z^\mu)}{\sqrt{-g}}$$

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Kinematics: Mathisson-Papapetrou-Dixon EoM

$$\begin{aligned} u^\gamma \nabla_\gamma S^{\alpha\beta} &\equiv p^\alpha u^\beta - p^\beta u^\alpha \\ u^\gamma \nabla_\gamma p^\alpha &= -\frac{1}{2} S^{\beta\mu} u^\delta R^\alpha{}_{\delta\beta\mu}, \end{aligned}$$

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Fix unphysical DoF: spin-supplementary condition [Tulczyjew covariant]

$$S^{\alpha\beta}p_\beta = 0$$

# Simplifications

SSC gives us

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$$p^\alpha = m_1 u^\alpha + \mathcal{O}(S_1^2) \Rightarrow \ m_1 \frac{Du^\alpha}{d\tau} = F^\alpha_{q^2} + F^\alpha_{q^3} + \mathcal{O}(S_1^2)$$

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Circular, equatorial orbits in Kerr.

$$r(\tau) = m_2 r_0,$$

$$u^\alpha = \left(u^t, 0, 0, \Omega u^t\right)^T$$

Let $S_1^\alpha = S_1 \delta^\alpha_\theta \quad \Rightarrow \quad \mathbf{S}_1 /\!/ \mathbf{L} /\!/ \mathbf{S}_2$ and $p^\theta = u^\theta = 0$

$$S^{\alpha\beta} = \begin{cases} S^{tr} = -S^{rt} \neq 0 \\ S^{t\phi} = -S^{\phi t} \neq 0 \\ S^{r\phi} = -S^{\phi r} \neq 0 \end{cases}$$

# Monopole-Dipole Source

Expand to $\mathcal{O}(S_1)$

$$T^{\alpha\beta} = \; T^{\alpha\beta}_{\mathsf{NS},(0)} \; + \; T^{\alpha\beta}_{\mathsf{NS},(1)} \; + \; T^{\alpha\beta}_{\mathsf{S},(0)}$$

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$$T^{\alpha\beta}_{\mathsf{NS}} = \left\{ \frac{1}{\sqrt{-g}} \, \frac{1}{u^t} \, u^\alpha u^\beta \right\}\Bigg|_{\mathbf{x}=\mathbf{z}} \delta^{(3)}(\mathbf{x}-\mathbf{z}),$$

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$$T^{\mu\nu}_{S} = -\frac{1}{\sqrt{-g}} \left[ \partial_\rho \delta^{(3)}(\mathbf{x}-\mathbf{z}) \left\{ \frac{1}{u^t} S^{\rho(\alpha} u^{\beta)} \right\}\Bigg|_{\mathbf{x}=\mathbf{z}} + \left\{ S^{\rho(\alpha} \Gamma^{\beta)}_{\rho\sigma} \frac{u^\sigma}{u^t} \right\}\Bigg|_{\mathbf{x}=\mathbf{z}} \delta^{(3)}(\mathbf{x}-\mathbf{z}) \right],$$

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$$u^t = \frac{1}{\sqrt{-(g_{tt}+2g_{t\phi}\Omega+g_{\phi\phi}\Omega^2)}}, \qquad u^\phi = \Omega u^t,$$

$$\Omega = \Omega_0\left[1-\frac{3}{2}\sigma\Omega_0\left(1-\frac{a}{\sqrt{r_0}}\right)\right]+\mathcal{O}(\sigma^2),$$

$$\Omega_0 = \left(r_0^{3/2}+a\right)^{-1}, \quad \sigma \equiv \tfrac{qS_1}{Gm_1^2} = q\chi_1$$

# Road map

Divide and conquer

1. Lorenz gauge

2. Radiation gauge

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   - Work with $h_{\alpha\beta}$ directly.
   - Regularization straightforward (not needed for $F^{\alpha}_{\text{diss}}$).
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2. Radiation gauge
    - One field equation $\Rightarrow$ 2 new sources.
    - Numerical or analytic (MST) approaches.
    - Kerr.
    - Reconstructing $h_{\alpha\beta}$ involved, but understood.
    - Computing $F^{\alpha}$ . . .

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In progress: Self-force: $F^{\alpha} = P^{\alpha\beta\gamma\delta}\nabla_{\beta}h_{\gamma\delta}$

# Teukolsky $s = -2$ source

- Project $T_{\mathsf{S}}^{\alpha\beta}$ along $n^\mu, \bar{m}^\mu$: $T_{nn}, T_{n\bar{m}}, T_{\bar{m}\bar{m}}$.
- Hit these with 2nd-order radial/angular derivatives: $B', B'^*$.
- Fourier transform

$$ {}_{-2}\mathcal{T}_{\ell m\omega} = 4\int d\Omega dt\, \frac{\Sigma}{\rho^4}(B'_I + B'^*_I)\ {}_{-2}S^{a\omega}_{\ell m}(\theta)\mathrm{e}^{i\omega t}\mathrm{e}^{-im\phi} $$

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- Integration for flux amplitudes via $\int dr\, f(r)\, \delta^{(n)}(r - r_0) = (-1)^n\, \frac{d^n f}{dr^n}\Big|_{r=r_0}$

$$ Z^{\pm} \sim \int dr' R^{\mp}_{\ell m\omega}(r')\; {}_{-2}\mathcal{T}_{\ell m\omega}/\Delta(r')^2, $$

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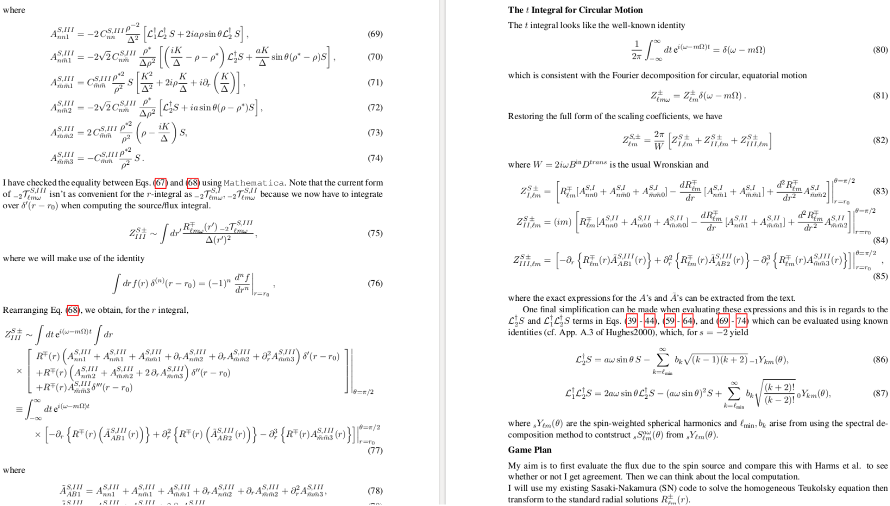
- $t$-integral using $\frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, \mathrm{e}^{i(\omega - m\Omega)t} = \delta(\omega - m\Omega)$

# Teukolsky $s = -2$ source



where

$$A^{S,III}_{nn1} = -2\,C^{S,III}_{nn}\frac{\rho^{-2}}{\Delta^2}\left[\mathcal{L}^\dagger_1\mathcal{L}^\dagger_2\,S + 2ia\rho\sin\theta\mathcal{L}^\dagger_2\,S\right], \quad (69)$$

$$A^{S,III}_{n\bar m1} = -2\sqrt{2}\,C^{S,III}_{n\bar m}\,\frac{\rho^*}{\Delta\rho^2}\left[\left(\frac{iK}{\Delta} - \rho - \rho^*\right)\mathcal{L}^\dagger_2 S + \frac{aK}{\Delta}\sin\theta(\rho^* - \rho)S\right], \quad (70)$$

$$A^{S,III}_{\bar m\bar m1} = C^{S,III}_{\bar m\bar m}\frac{\rho^{*2}}{\rho^2}\,S\left[\frac{K^2}{\Delta^2} + 2i\rho\frac{K}{\Delta} + i\partial_r\left(\frac{K}{\Delta}\right)\right], \quad (71)$$

$$A^{S,III}_{n\bar m2} = -2\sqrt{2}\,C^{S,III}_{n\bar m}\,\frac{\rho^*}{\Delta\rho^2}\left[\mathcal{L}^\dagger_2 S + ia\sin\theta(\rho - \rho^*)S\right], \quad (72)$$

$$A^{S,III}_{\bar m\bar m2} = 2\,C^{S,III}_{\bar m\bar m}\,\frac{\rho^{*2}}{\rho^2}\left(\rho - \frac{iK}{\Delta}\right)S, \quad (73)$$

$$A^{S,III}_{\bar m\bar m3} = -C^{S,III}_{\bar m\bar m}\frac{\rho^{*2}}{\rho^2}\,S\,. \quad (74)$$

I have checked the equality between Eqs. (67) and (68) using `Mathematica`. Note that the current form of ${}_{-2}\mathcal{T}^{S,III}_{\ell m\omega}$ isn't as convenient for the $r$-integral as ${}_{-2}\mathcal{T}^{S,I}_{\ell m\omega}$, ${}_{-2}\mathcal{T}^{S,II}_{\ell m\omega}$ because we now have to integrate over $\delta'(r - r_0)$ when computing the source/flux integral.

$$Z^{S\pm}_{III} \sim \int dr'\,\frac{R^\mp_{\ell m\omega}(r')\,{}_{-2}\mathcal{T}^{S,III}_{\ell m\omega}}{\Delta(r')^2}, \quad (75)$$

where we will make use of the identity

$$\int dr\,f(r)\,\delta^{(n)}(r - r_0) = (-1)^n\,\frac{d^n f}{dr^n}\bigg|_{r=r_0}, \quad (76)$$

Rearranging Eq. (68), we obtain, for the $r$ integral,

$$\begin{aligned} Z^{S\pm}_{III} &\sim \int dt\,\mathrm{e}^{i(\omega - m\Omega)t}\int dr \\ &\times\left[\begin{array}{l} R^\mp(r)\left(A^{S,III}_{nn1} + A^{S,III}_{n\bar m1} + A^{S,III}_{\bar m\bar m1} + \partial_r A^{S,III}_{n\bar m2} + \partial_r A^{S,III}_{\bar m\bar m2} + \partial^2_r A^{S,III}_{\bar m\bar m3}\right)\delta'(r - r_0) \\ + R^\mp(r)\left(A^{S,III}_{n\bar m2} + A^{S,III}_{\bar m\bar m2} + 2\,\partial_r A^{S,III}_{\bar m\bar m3}\right)\delta''(r - r_0) \\ + R^\mp(r)A^{S,III}_{\bar m\bar m3}\delta'''(r - r_0) \end{array}\right]\Bigg|_{\theta=\pi/2} \\ &\equiv \int_{-\infty}^{\infty} dt\,\mathrm{e}^{i(\omega - m\Omega)t} \\ &\times\left[-\partial_r\left\{R^\mp(r)\left(\tilde A^{S,III}_{AB1}(r)\right)\right\} + \partial^2_r\left\{R^\mp(r)\left(\tilde A^{S,III}_{AB2}(r)\right)\right\} - \partial^3_r\left\{R^\mp(r)A^{S,III}_{\bar m\bar m3}(r)\right\}\right]\Bigg|^{\theta=\pi/2}_{r=r_0} \end{aligned} \quad (77)$$

where

$$\tilde A^{S,III}_{AB1} = A^{S,III}_{nn1} + A^{S,III}_{n\bar m1} + A^{S,III}_{\bar m\bar m1} + \partial_r A^{S,III}_{n\bar m2} + \partial_r A^{S,III}_{\bar m\bar m2} + \partial^2_r A^{S,III}_{\bar m\bar m3}, \quad (78)$$

**The $t$ Integral for Circular Motion**

The $t$ integral looks like the well-known identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\,\mathrm{e}^{i(\omega - m\Omega)t} = \delta(\omega - m\Omega) \quad (80)$$

which is consistent with the Fourier decomposition for circular, equatorial motion

$$Z^\pm_{\ell m\omega} = Z^\pm_{\ell m}\delta(\omega - m\Omega)\,. \quad (81)$$

Restoring the full form of the scaling coefficients, we have

$$Z^{S,\pm}_{\ell m} = \frac{2\pi}{W}\left[Z^{S\pm}_{I,\ell m} + Z^{S\pm}_{II,\ell m} + Z^{S\pm}_{III,\ell m}\right] \quad (82)$$

where $W = 2i\omega B^{\text{in}}D^{trans}$ is the usual Wronskian and

$$Z^{S\pm}_{I,\ell m} = \left[R^\mp_{\ell m}[A^{S,I}_{nn0} + A^{S,I}_{n\bar m0} + A^{S,I}_{\bar m\bar m0}] - \frac{dR^\mp_{\ell m}}{dr}[A^{S,I}_{n\bar m1} + A^{S,I}_{\bar m\bar m1}] + \frac{d^2R^\mp_{\ell m}}{dr^2}A^{S,I}_{\bar m\bar m2}\right]\Bigg|^{\theta=\pi/2}_{r=r_0} \quad (83)$$

$$Z^{S\pm}_{II,\ell m} = (im)\left[R^\mp_{\ell m}[A^{S,II}_{nn0} + A^{S,II}_{n\bar m0} + A^{S,II}_{\bar m\bar m0}] - \frac{dR^\mp_{\ell m}}{dr}[A^{S,II}_{n\bar m1} + A^{S,II}_{\bar m\bar m1}] + \frac{d^2R^\mp_{\ell m}}{dr^2}A^{S,II}_{\bar m\bar m2}\right]\Bigg|^{\theta=\pi/2}_{r=r_0} \quad (84)$$

$$Z^{S\pm}_{III,\ell m} = \left[-\partial_r\left\{R^\mp_{\ell m}(r)\tilde A^{S,III}_{AB1}(r)\right\} + \partial^2_r\left\{R^\mp_{\ell m}(r)\tilde A^{S,III}_{AB2}(r)\right\} - \partial^3_r\left\{R^\mp_{\ell m}(r)A^{S,III}_{\bar m\bar m3}(r)\right\}\right]\Bigg|^{\theta=\pi/2}_{r=r_0}, \quad (85)$$

where the exact expressions for the $A$'s and $\tilde A$'s can be extracted from the text.

One final simplification can be made when evaluating these expressions and this is in regards to the $\mathcal{L}^\dagger_2 S$ and $\mathcal{L}^\dagger_1\mathcal{L}^\dagger_2 S$ terms in Eqs. (39 - 44), (59 - 64), and (69 - 74) which can be evaluated using known identities (cf. App. A.3 of Hughes2000), which, for $s = -2$ yield

$$\mathcal{L}^\dagger_2 S = a\omega\sin\theta\,S - \sum_{k=\ell_{\min}}^{\infty} b_k\sqrt{(k-1)(k+2)}\,{}_{-1}Y_{km}(\theta), \quad (86)$$

$$\mathcal{L}^\dagger_1\mathcal{L}^\dagger_2 S = 2a\omega\sin\theta\mathcal{L}^\dagger_2 S - (a\omega\sin\theta)^2 S + \sum_{k=\ell_{\min}}^{\infty} b_k\sqrt{\frac{(k+2)!}{(k-2)!}}\,{}_0Y_{km}(\theta), \quad (87)$$

where ${}_sY_{\ell m}(\theta)$ are the spin-weighted spherical harmonics and $\ell_{\min}, b_k$ arise from using the spectral decomposition method to contstruct ${}_sS^{a\omega}_{\ell m}(\theta)$ from ${}_sY_{\ell m}(\theta)$.

**Game Plan**

My aim is to first evaluate the flux due to the spin source and compare this with Harms et al. to see whether or not I get agreement. Then we can think about the local computation.
I will use my existing Sasaki-Nakamura (SN) code to solve the homogeneous Teukolsky equation then transform to the standard radial solutions $R^\pm_{\ell m}(r)$.

# Lorenz-gauge source(s)

- Tensor harmonic basis + Fourier transform = 10 wave-like equations:

$$\Box_{\mathsf{sc}}\phi + \ldots = F_1(r)\,\delta(r - r_0) + F_2(r)\delta'(r - r_0)$$

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- Matching: new junction conditions due to $\delta'(r-r_0)$

$$c_+\phi_+ - c_-\phi_-\big|_{r_0} = \boxed{F_2(r_0)},$$

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- Additional terms in $i = 1, 2, 4, 8$ eqs.

$$\Box_{\mathsf{sc}}h^{(1)}\boxed{-\frac{4M}{r^2}\partial_r h^{(3)}} + \tilde{\mathcal{M}}^{(1)}(h) = F_1^{(1)}(r)\delta(r-r_0) + F_2^{(1)}(r)\delta'(r-r_0),$$

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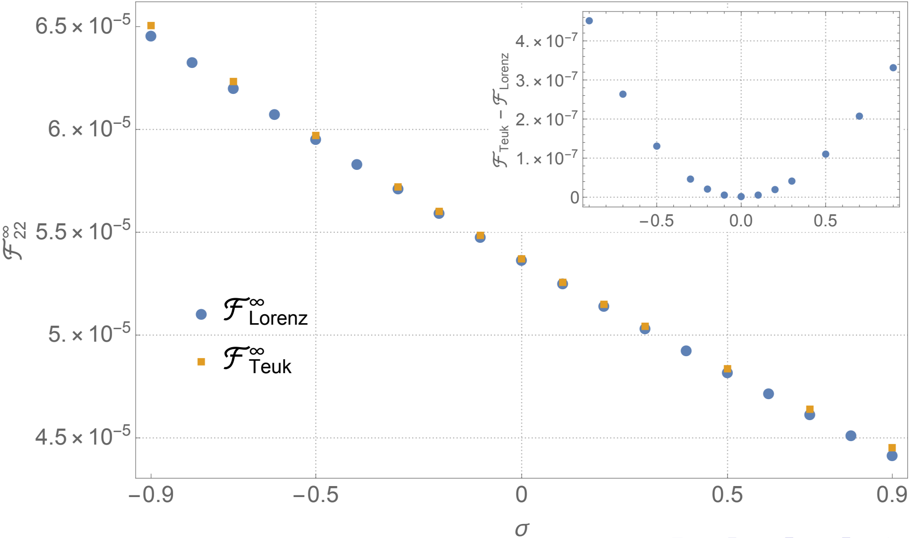
- Check: $\nabla_\alpha T^{\alpha\beta} = 0 + \mathcal{O}(\sigma^2)$.

# Status

Good agreement ($\lesssim 10^{-6}$) in fluxes between gauges



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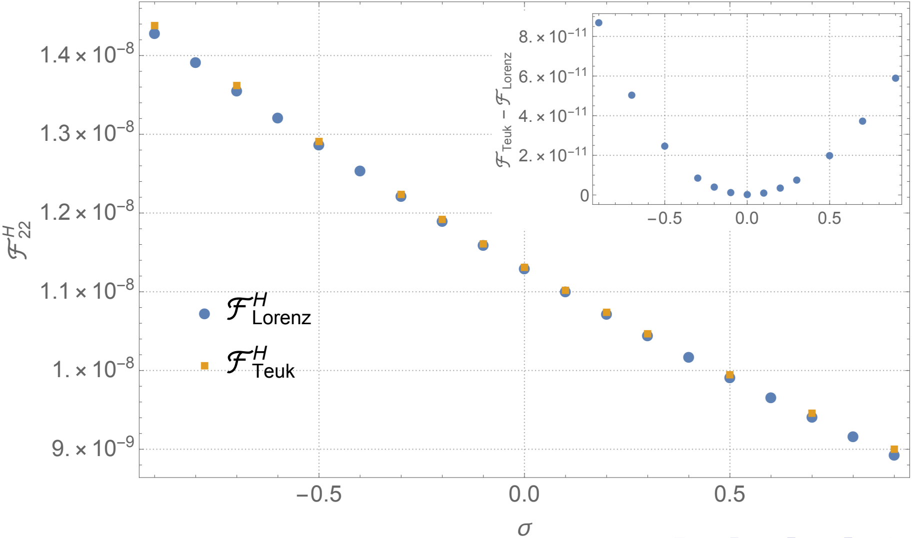
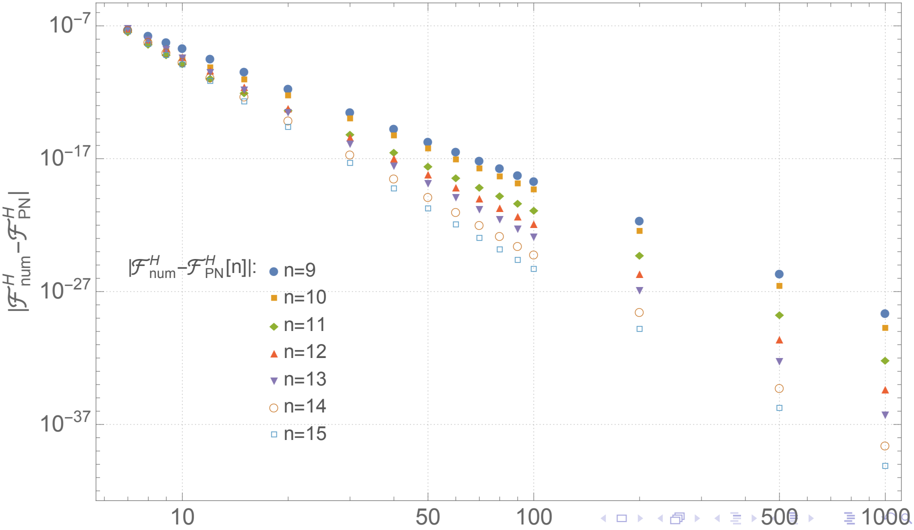


# Status

Good agreement between numerical and analytic (PN) Teukolsky

$s$=1.



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3. Augmented EMRI evolution: add $F^{\alpha,\text{L}}_{\text{diss}}$ to Warburton-Osburn-Evans?
   add $F^{\alpha,\text{ORG}}_{\text{diss}}$ to van de Meent 2017?