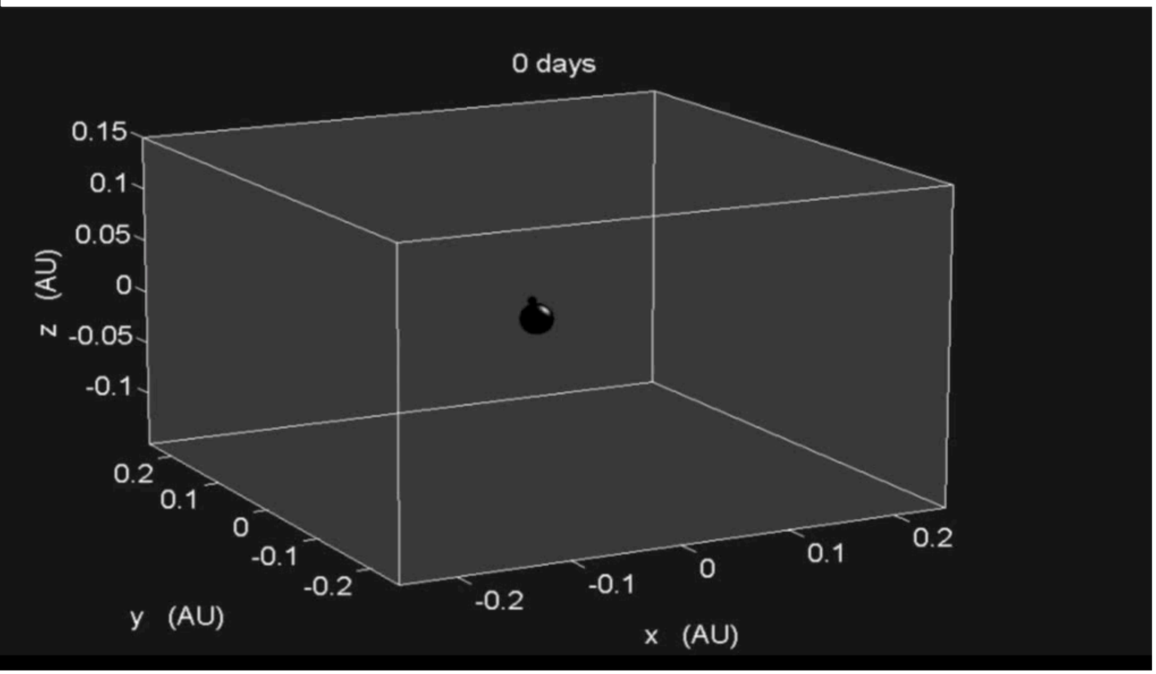
Bound orbits of slowly evolving black holes



How do orbits react when a black hole spacetime slowly changes?

Based on arXiv:1806.09022

Motivation

Very nice paper by Isoyama & Nakano (1705.03869): Down-horizon fluxes for post-Newtonian templates, foundation for examining the influence of horizon coupling on comparable mass binary black holes.

Also quantifies effect of the secular change in black hole mass and spin due to this down-horizon flux: Allow masses (m_1, m_2) and spins (s_1, s_2) to grow, with growth rate determined by flux on the horizon.

Take formulas, promote mass and spin:

$$m_{1,2} \rightarrow m_{1,2}(t)$$
$$s_{1,2} \rightarrow s_{1,2}(t)$$

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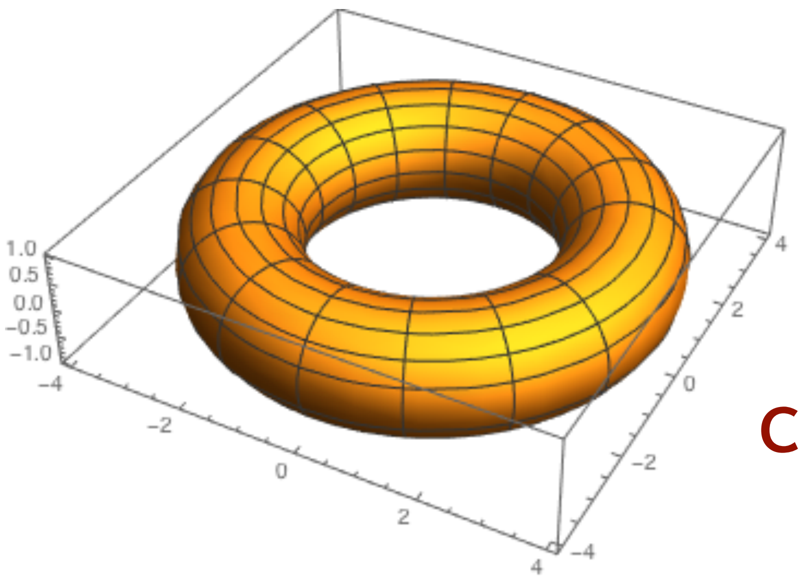
Does this calculation correctly model how inspiral is modified when masses and spins change with time?

Theorem

Consider integrable motion in a potential *V*.
Motion confined to the surface of a torus in phase space, dimensions given by motion's actions:



$$J_k = \frac{1}{2\pi} \oint p_k \, dx^k$$

Imagine potential changes: *V* continually and smoothly changes over a time interval $t_1 \leq t \leq t_2$.

Take motion to be integrable at every moment in interval; take change to be "slow" compared to orbital timescale *T*.

$$\frac{\partial V}{\partial t} \ll \frac{V}{T}$$

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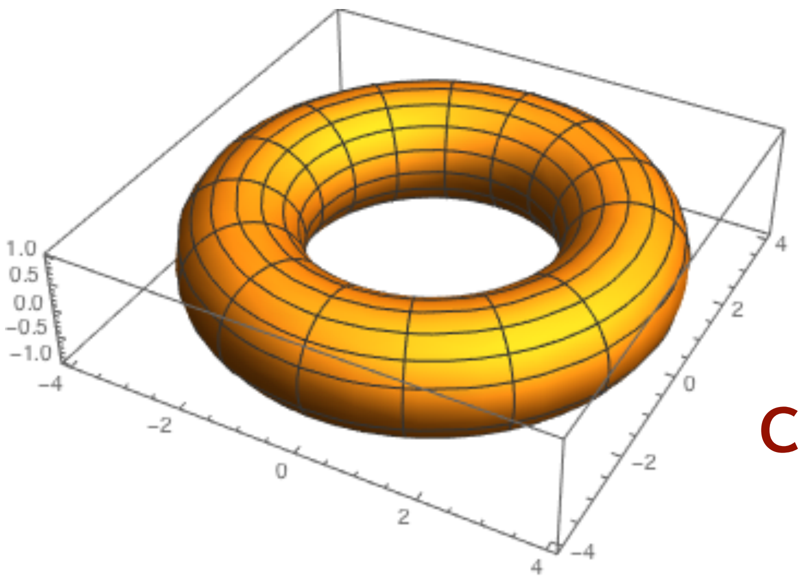

$$J_k = \frac{1}{2\pi} \oint p_k \, dx^k$$

Imagine potential changes: *V* continually and smoothly changes over a time interval $t_1 \leq t \leq t_2$.

Actions are then *adiabatically invariant*:

$$J_k \rightarrow J_k \text{ as } V \rightarrow V + \delta V$$

(Good discussion: Binney & Tremaine, "Galactic Dynamics")

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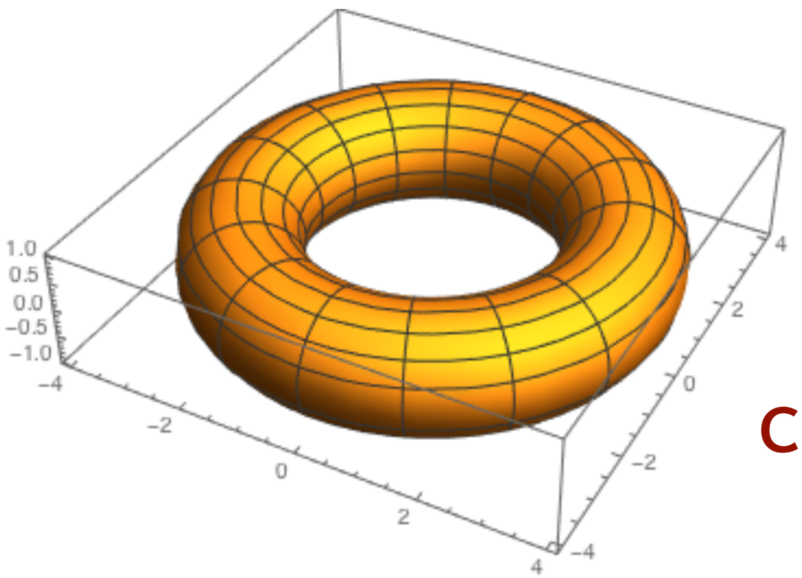

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Key point for us: This theorem makes no assumptions about "*V*," other than that it admits integrable motion and that its change is slow.

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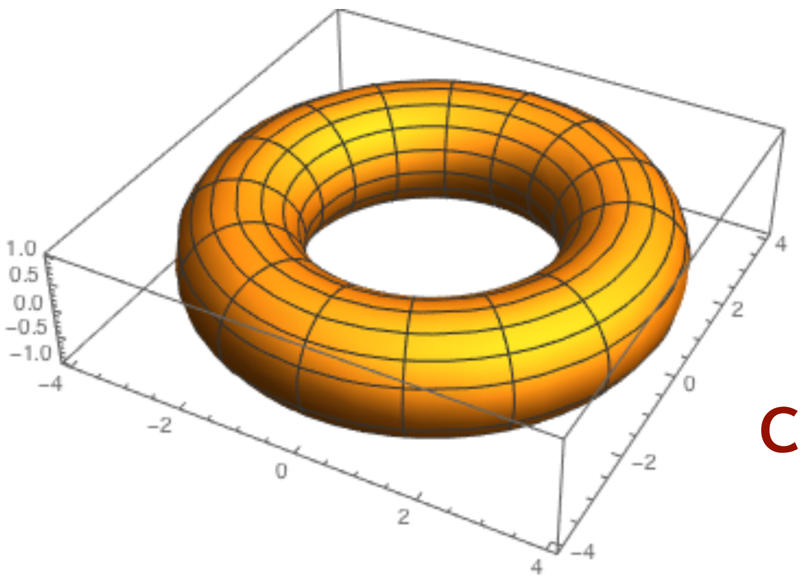

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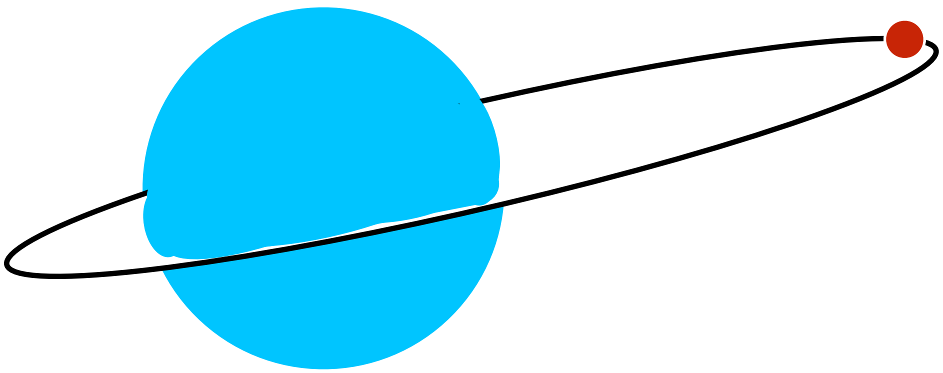
Imagine potential changes: *V* continually and smoothly changes over a time interval $t_1 \leq t \leq t_2$.

Often applied to orbits in Newtonian gravity, ***but it applies equally well to integrable orbits in general relativity.***

Newtonian example

Intuition: Examine implications of adiabatic invariance for Newtonian orbits.



Consider a Newtonian binary with total mass *M*, reduced mass μ

Orbit has semi-latus rectum *p*, eccentricity *e* [periapsis *p*/[1+*e*], apoapsis *p*/(1-*e*)], inclination *ι*

$$J_\phi = \mu\sqrt{Mp}\cos\iota \equiv L_z$$

$$J_\theta = \mu\sqrt{Mp}(1-\cos\iota)$$

$$J_r = \mu\sqrt{Mp}\left[(1-e^2)^{-1/2} - 1\right]$$

Newtonian example

Imagine that the mass increases: $M \to M + \delta M$.
Orbit must adjust for actions to be fixed:

$$0 = \delta J_k = \frac{\partial J_k}{\partial M}\delta M + \frac{\partial J_k}{\partial p}\delta p + \frac{\partial J_k}{\partial e}\delta e + \frac{\partial J_k}{\partial \iota}\delta \iota$$

Enforce this, find:

$$\frac{\delta p}{p} = -\frac{\delta M}{M} \qquad \delta e = 0 \qquad \delta \iota = 0$$

Orbit maintains shape, but shrinks in response to mass growth.

Newtonian example

This change has important implications for the orbital frequency:

Kepler's law using (p, e):

$$\Omega = \sqrt{\frac{M(1-e^2)^3}{p^3}}$$

Change mass by δM, include change δp:

$$\frac{\delta\Omega}{\Omega} = 2\frac{\delta M}{M}$$

Suppose we missed adiabatic invariance … we would miss the contribution from δp.

$$\frac{\delta\Omega_{\rm wrong}}{\Omega} = \frac{1}{2}\frac{\delta M}{M}$$

Black hole orbits: Circular and equatorial

For circular and equatorial black hole orbits,

$$J_r = 0 \qquad J_\theta = 0$$

$$J_\phi = L_z = \pm\mu\, r v \frac{1 \mp 2av^3 + a^2v^4}{\sqrt{1 - 3v^2 \pm 2av^3}}$$

[where $a = S/M^2$, $v = (M/r)^{1/2}$]

Allow $M \rightarrow M + \delta M$, $S \rightarrow S + \delta S$, enforce invariance:

$$\frac{\delta r}{r} = -\frac{1 \pm 3av^3(1-2v^2) - a^2v^4(3-10v^2) \mp 5a^3v^7}{(1 \pm av^3)(1 - 6v^2 \pm 8av^3 - 3a^2v^4)}\frac{\delta M}{M}$$
$$\pm \frac{6v^3(1 - 2v^2 \mp 4av^4(1-4v^2) - 6a^2v^7}{(1 \pm av^3)(1 - 6v^2 \pm 8av^3 - 3a^2v^4)}\frac{\delta S}{M^2}$$

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Allow $M \to M + \delta M$, $S \to S + \delta S$, enforce invariance:

$$\frac{\delta r}{r} \to -\frac{1}{1-6v^2}\frac{\delta M}{M} \pm \frac{6v^3(1-2v^2)}{1-6v^2}\frac{\delta S}{M^2}$$

($a \to 0$ limit)

Black hole orbits: Generic

When all three actions are non-zero, enforcing $\delta J_k = 0$ changes p, e, ι.

Organize the equations we need to solve:

$$0 = \frac{\partial J_k}{\partial M} + \frac{\partial J_k}{\partial S} + \frac{\partial J_k}{\partial p} + \frac{\partial J_k}{\partial e} + \frac{\partial J_k}{\partial \iota}$$

Write this $\qquad \mathsf{J} \cdot \delta\mathsf{O} = -\delta\mathsf{H} \qquad$ where

$$\mathsf{J} = \begin{pmatrix} \partial J_r/\partial p & \partial J_r/\partial e & \partial J_r/\partial \iota \\ \partial J_\theta/\partial p & \partial J_\theta/\partial e & \partial J_\theta/\partial \iota \\ \partial J_\phi/\partial p & \partial J_\phi/\partial e & \partial J_\phi/\partial \iota \end{pmatrix} \quad \delta\mathsf{H} = \begin{pmatrix} (\partial J_r/\partial M)\delta M + (\partial J_r/\partial S)\delta S \\ (\partial J_\theta/\partial M)\delta M + (\partial J_\theta/\partial S)\delta S \\ (\partial J_\phi/\partial M)\delta M + (\partial J_\phi/\partial S)\delta S \end{pmatrix}$$

$$\delta\mathsf{O} = \begin{pmatrix} \delta p \\ \delta e \\ \delta \iota \end{pmatrix}$$

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Write this $\quad \mathsf{J} \cdot \delta\mathsf{O} = -\delta\mathsf{H}$

Solve: $\quad \delta\mathsf{O} = -\mathsf{J}^{-1} \cdot \delta\mathsf{H}$

$$\delta\mathsf{O} = \begin{pmatrix} \delta p \\ \delta e \\ \delta\iota \end{pmatrix}$$

Examples showing (δp, δe, $\delta\iota$) per unit (δM, δS) given in 1806.09022; Mathematica notebook for this will be on bhptoolkit.org soon.

Application: Back reaction on spacetime of down-horizon flux

Go back to circular and equatorial case … imagine, as in Isoyama and Nakano, that the mass and spin secularly evolve due to down-horizon flux.

Consider how orbital frequency changes due to change to hole's mass and spin:

$$\delta\Omega = \frac{\partial\Omega}{\partial M}\delta M + \frac{\partial\Omega}{\partial S}\delta S + \frac{\partial\Omega}{\partial r}\delta r$$

$$\equiv \mu_\Omega \delta M + \sigma_\Omega \delta S$$

Application: Back reaction on spacetime of down-horizon flux

Go back to circular and equatorial case … imagine, as in Isoyama and Nakano, that the mass and spin secularly evolve due to down-horizon flux.

Imagine that increments of mass and spin accumulate with some rate:

$$\frac{d\delta\Omega}{dt} = \mu_\Omega \frac{dM}{dt} + \sigma_\Omega \frac{dS}{dt}$$

Balance rates of growth with down-horizon flux: $$\frac{dM}{dt} = -\dot{E}^{\mathrm{H}} , \qquad \frac{dS}{dt} = -\dot{L}_z^{\mathrm{H}}$$

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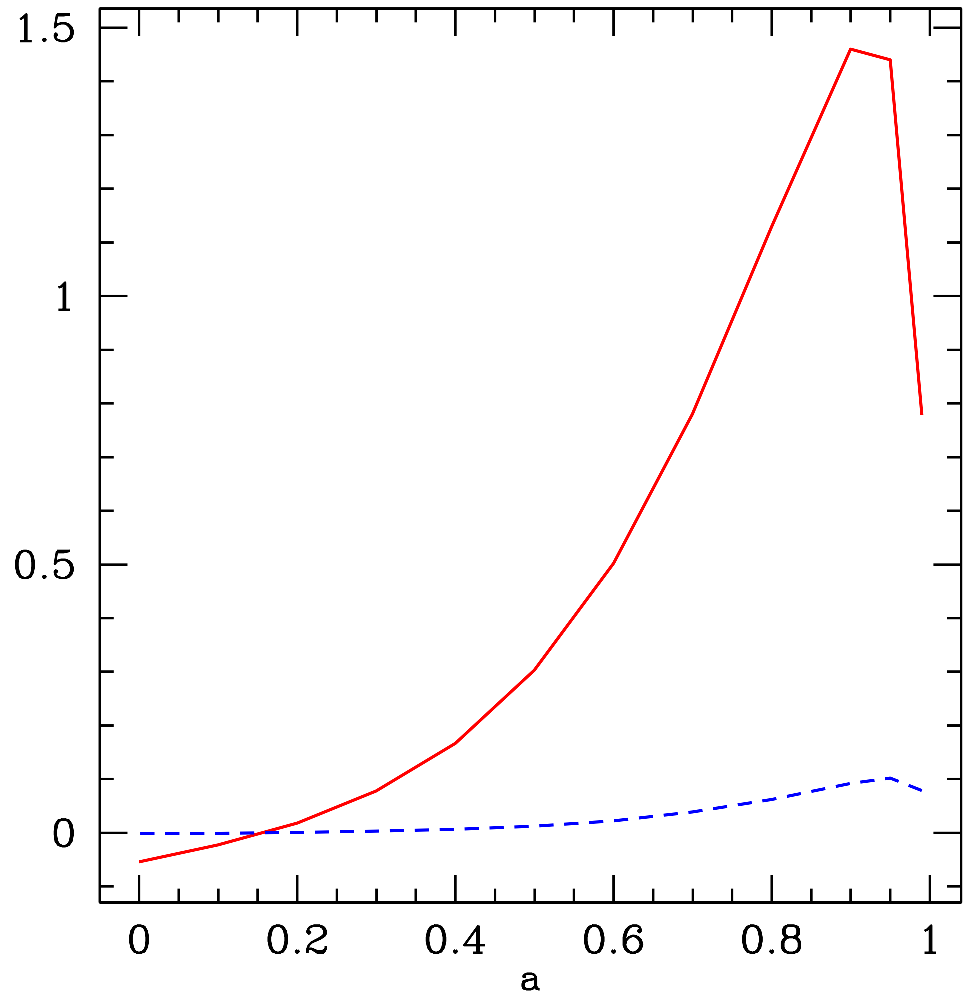
Integrate over inspiral, compute phase shift $\delta\Phi$ due to secular evolution of black hole.

Comparison: Enforcing adiabatic invariance vs not

Result: Leaving out the impact of adiabatic invariance underestimates the phase shift by a factor of roughly **20.**



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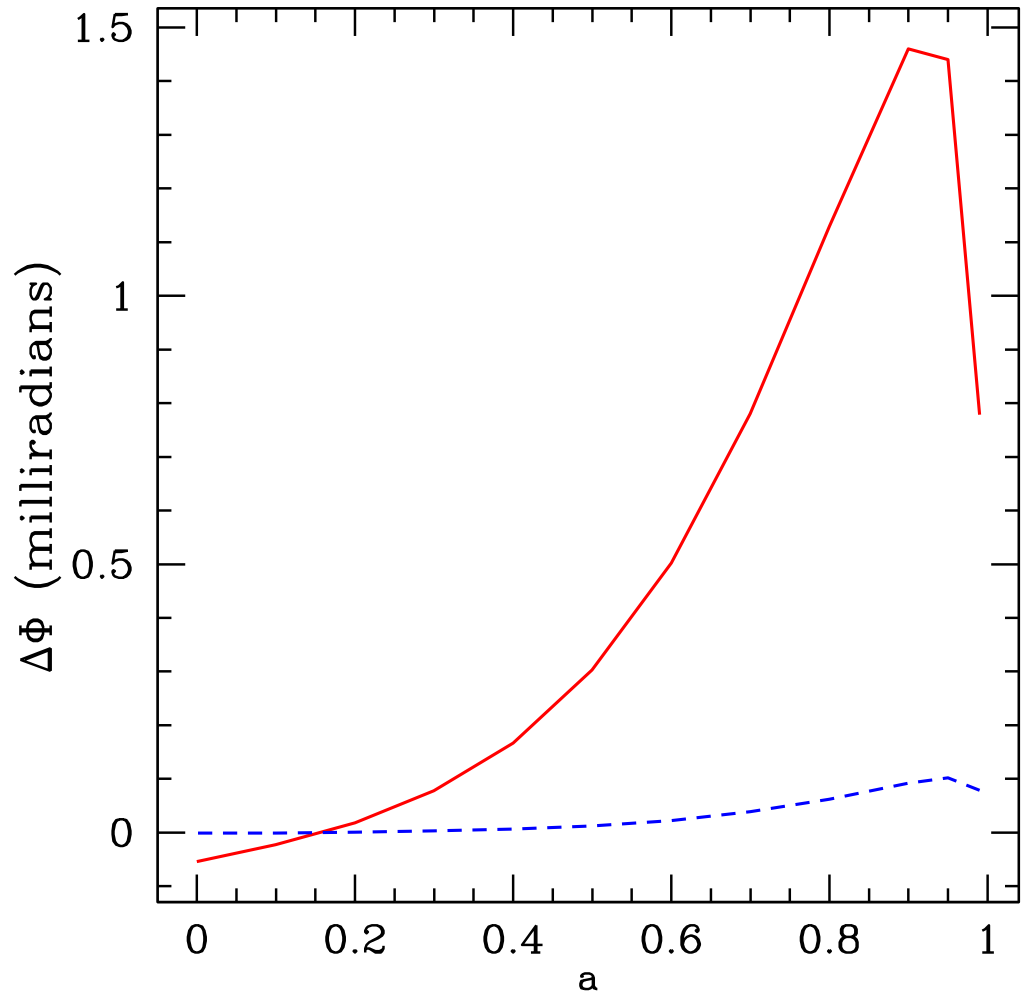

(Really small) x 20 = still really small.

Possible use: Simple check of orbit-averaged 2nd order effect

Change to orbit plus change to spacetime can be mapped to a change in orbit integrals:

$$(\delta M, \delta S; \delta p, \delta e, \delta \iota) \longrightarrow (\delta E, \delta Q)$$

These in turn can be mapped to orbit-averaged components of some piece of the second-order dissipative self force.

Possibly a useful result to bear in mind as 2nd order results reach the point where they can do calculations like this.