A naive attempt to calculate the metric perturbation in the Regge-Wheeler gauge

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OUTLINE

GRAVITATIONAL PERTURBATIONS IN SCHWARZSCHILD SPACETIME

- To first order in $\mu/M \ll 1$

$$G^{(1)}_{\mu\nu}\left[h_{\alpha\beta}\right] = 8\pi T^{(0)}_{\mu\nu} \tag{1}$$

- Harmonic decomposition of $h_{\mu\nu}$
  - Even-parity (polar):

$$Y^{\ell m}(\theta,\phi),\ Y^{\ell m}_{A}(\theta,\phi),\ Y^{\ell m}_{AB}(\theta,\phi)$$

  - Odd-parity (axial):

$$X^{\ell m}_{A}(\theta,\phi),\ X^{\ell m}_{AB}(\theta,\phi)$$

- In the Regge-Wheeler gauge

$$h^{\mathrm{RW}}_{aB} = \sum_{\ell,m} h^{\ell m}_{a} X^{\ell m}_{B}(\theta,\phi) \tag{2}$$

where $\ell \geq 2$, $h^{\ell m}_{a} = h^{\ell m}_{a}(t,r)$, $a = t, r$ and $B = \theta, \phi$.

- $(3+1)-$D RW eq. for scalar, emag and gravitational perturbations (Nakano & Sasaki, 2001)

$$\left[\nabla^\mu \nabla_\mu + s^2 \frac{2M}{r^3}\right] \Psi_s = S_s. \tag{3}$$

- Mode decomposition

$$\Psi_2 = \frac{1}{r} \sum_{\ell, m} \psi^{\ell m}(r,t) Y^{\ell m}(\theta, \phi)$$

- To reconstruct the metric perturbation from Ψ_2 we use

$$h_t^{\ell m} = -\frac{f(r)}{2} \partial_r \left(r\psi^{lm} \right) - \frac{r^2 f(r)}{(\ell-1)(\ell+2)} P^t, \tag{4}$$

$$h_r^{\ell m} = \frac{r}{2f(r)} \partial_t \psi^{\ell m} + \frac{r^2}{(\ell-1)(\ell+2) f(r)} P^r. \tag{5}$$

where $f(r) = 1 - \frac{2M}{r}$ and $P^{t,r}$ is a X_B-projection of $T^{(0)}_{\mu\nu}$

GREEN FUNCTION FOR THE $(3+1)-$D RW EQ.

- Retarded Green function of the RW equation

$$\left[\nabla^{\mu}\nabla_{\mu} + s^2\frac{2M}{r^3}\right] G_{\text{ret}}(x,x') = -\frac{4\pi}{rr'}\delta(t-t')\delta(r-r')\delta(\Omega-\Omega') \tag{6}$$

- Due to the Schwarzschild spacetime symmetries

$$G_{\text{ret}}(x,x') = -\frac{1}{rr'}\sum_{\ell=0}^{\infty}(2\ell+1)\hat{G}_{\ell}^{\text{ret}}(r,r',\Delta t)P_{\ell}(\cos\gamma)$$

where $\Delta t = t - t'$, $\gamma = \phi - \phi'$ and

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - Q_{\ell}(r)\right]\hat{G}_{\ell}^{\text{ret}} = \delta(t-t')\delta(r-r'), \tag{7}$$

which is the standard $(1+1)$-D RW eq.

$$Q_\ell(r) = f(r)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M(1-s^2)}{r^3}\right],$$
$$r_* = r + 2M\ln\left(\frac{r}{M} - 2\right).$$

- We re-write (7) using the null coordinates $v = t + r_*$ and $u = t - r_*$

$$\left[\frac{\partial^2}{\partial v \partial u} + Q_\ell(r)\right]\hat{G}^{\text{ret}} = -\frac{1}{2}\delta(v-v')\delta(u-u'). \qquad (8)$$

- Characteristic initial data

$$\hat{G}^{\text{ret}}(v,u;v,u') = G_\ell^{\text{ret}}(v,u;v',u')\theta(v-v')\theta(u-u') \qquad (9)$$

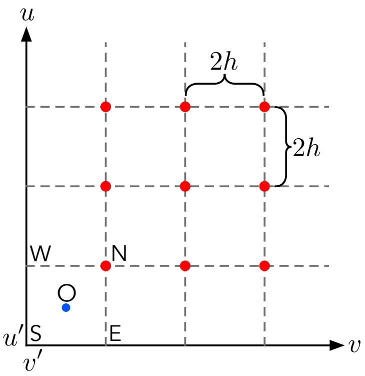
where $\theta(x)$ is the Heaviside function.

THE CHARACTERISTIC INITIAL DATA PROBLEM

- Plugging $G_\ell^{ret}(v,u;u,u')$ back into Eq. (8)

$$\left[-\frac{\partial^2}{\partial v \partial u} + Q_\ell(r)\right] G_\ell^{\text{ret}}(v,u;v',u') = 0,$$
$$G_\ell^{\text{ret}}(v=v',u;v',u') = -\frac{1}{2},$$
$$G_\ell^{\text{ret}}(v,u=u';v',u') = -\frac{1}{2}.$$

- Price and Lousto were the first to solve this kind of CID problem to $\mathcal{O}\left(h^4\right)$, being $2h$ the usual step-size of the scheme.



- We extend the scheme to $\mathcal{O}\left(h^6\right)$

$$\left(\frac{\partial g_\ell}{\partial u}\right)_{u'} = \frac{1}{2}\left(\int Q_\ell(r)\,\mathrm{d}v\right)_{u'} - \left(\frac{1}{2}\int Q_\ell(r)\,\mathrm{d}v\right)_{v',u'}$$

$$\left(\frac{\partial g_\ell}{\partial v}\right)_{v'} = \frac{1}{2}\left(\int Q_\ell(r)\,\mathrm{d}u\right)_{v'} - \left(\frac{1}{2}\int Q_\ell(r)\,\mathrm{d}u\right)_{v',u'}.$$

We calculate g_ℓ from $r = r' = 6M$ and $\Delta t = 0$ up to $r = r' = 6M$ and $\Delta t = 200M$, $\ell : 0 \to \ell_{\max} = 100$

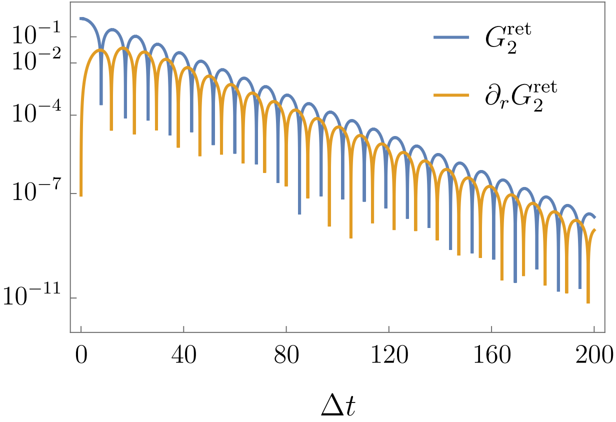
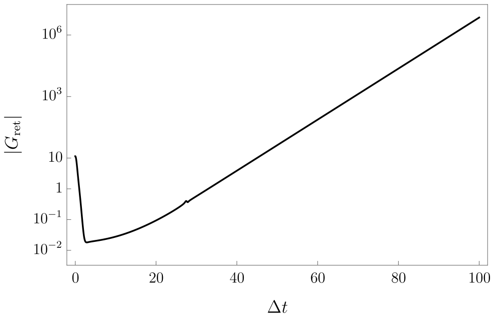
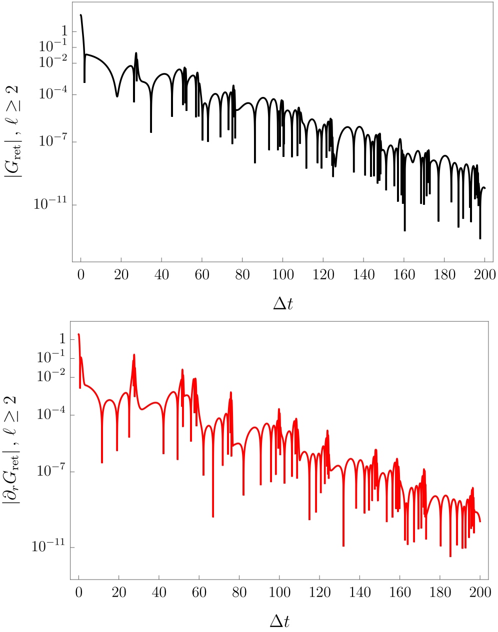


Figure: On a circular geodesic at $r = 6M$

$$G_{\text{ret}}(x,x') \approx -\frac{1}{rr'}\sum_{\ell=0}^{\ell_{\max}}(2\ell+1)G_\ell^{\text{ret}}(r,r';\Delta t)P_\ell(\cos\gamma)$$



- Growth for large Δt due to unphysical $\ell = 0$
- Problems near coincidence due to finite upper limit


1
10^{-1}
10^{-2}
10^{-4}
10^{-7}
10^{-11}
$|G_{\rm ret}|$, $\ell \geq 2$
0 20 40 60 80 100 120 140 160 180 200
Δt
1
10^{-1}
10^{-2}
10^{-4}
10^{-7}
10^{-11}
$|\partial_r G_{\rm ret}|$, $\ell \geq 2$
0 20 40 60 80 100 120 140 160 180 200
Δt

METHOD OF MATCHED EXPANSIONS

- Hadamard form (In a normal neighborhood)

$$G_{\text{ret}}(x,x') = U(x,x')\delta(\sigma) - V(x,x')\theta(-\sigma)$$

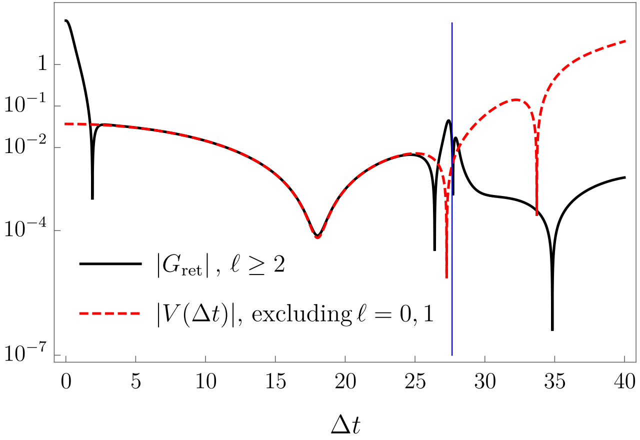
- We need the regular part, $G_{\text{ret}} - U\delta(\sigma)$
- Method of matched expansions:
  Quasilocal region

$$V(x,x') = \sum_{i,j,k} v_{ijk}(t-t')^{2i}(1-\cos\gamma)^j(r-r')^k,$$

  We computed v_{ijk} to have $V(x,x')$ to $\mathcal{O}\left((x-x')^{21}\right)$.
  Distant past

$$G_{\text{ret}}(x,x') \approx -\frac{1}{rr'}\sum_{\ell=0}^{\ell_{\max}}(2\ell+1)G_\ell^{\text{ret}}(r,r';\Delta t)P_\ell(\cos\gamma), \quad (10)$$

FRAME TITLE



- $V(x, x')$ does not do well near the end of the neighborhood
- *Gret* does not do well near coincidence

IMPROVEMENT OF ℓ-SUM (IN PROGRESS BY CASALS, NOLAN, OTTEWILL & WARDELL)

- Subtract the ℓ-mode of $U(x,x')\delta(\sigma)$ from the ℓ-mode of $G_{\text{ret}}(x,x')$,

$$U(x,x')\delta(\sigma) \approx \frac{1}{rr'} \sum_{\ell=0}^{\ell_{\max}} (2\ell+1) G_\ell^{\text{d}}(r,r';\Delta t) P_\ell(\cos\gamma),$$

$$G_\ell^{\text{d}}(x,x') = \frac{1}{2}\theta(t-t')\theta(\eta)\theta(\pi-\eta)\Delta_{2\text{d}}^{1/2} P_\ell(\cos\eta)\sqrt{\frac{\sin\eta}{\eta}}.$$

- η and $\Delta_{2\text{d}}$ are the proper time and the van-Vleck determinant in

$$\text{d}s_2^2 = \frac{f(r)}{r^2}\left(-\text{d}t^2 + \text{d}r_*^2\right)$$
$$\text{d}s_{\text{Schw}}^2 = r^2\left(\text{d}s_2^2 + \text{d}\Omega_2^2\right)$$

SOME RESULTS

- For $\Delta_{2d}^{1/2}$ and η
  - Using the transport equations introduced by A. Ottewill and B. Wardell

$$D'\Delta_{2d}^{1/2} = \frac{1}{2}(4 - \xi^{\alpha'}{}_{\alpha'})$$

  - Coordinate expansion, Casals & Nolan (work in progress)

$$\Delta_{2d}^{1/2} = \sum_{ij} q_{ij}(t-t')^i(r-r')^j.$$

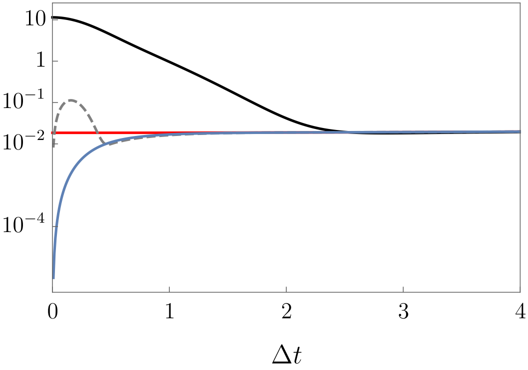


Figure: (Black) G_{ret}, (Dashed gray) $G_{\text{ret}} - G^{\text{d}}_{\text{num}}$, (Red) $V(x, x')$, (Blue) $G_{\text{ret}} - G^{\text{d}}_{\text{exp}}$ to $\mathcal{O}\left((x - x')^6\right)$

CONCLUSIONS

- We calculated the retarded Green functions of the $(1+1)$-D and $(3+1)$-D RW eqs.
- Improved ℓ-sum calculation of the $(3+1)$-D RW Green function by subtracting the modes of the direct part.
- Next: regularize any of these Green functions appropriately in order to obtain the self-force by applying a differential operator.

Thanks!