Self-force effects
in the
marginally bound zoom-whirl orbit

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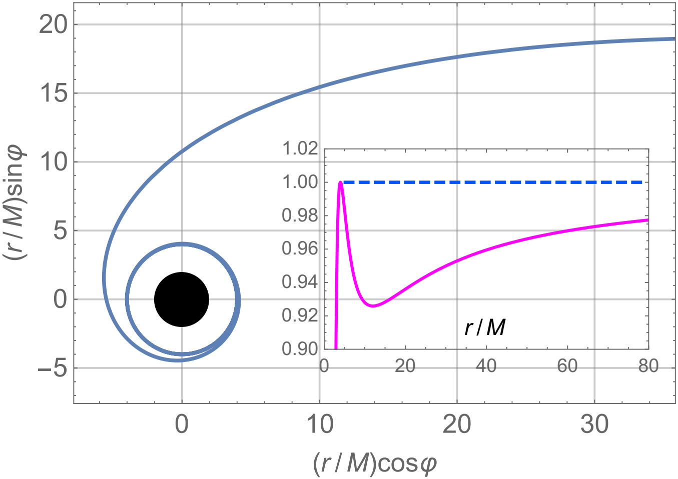
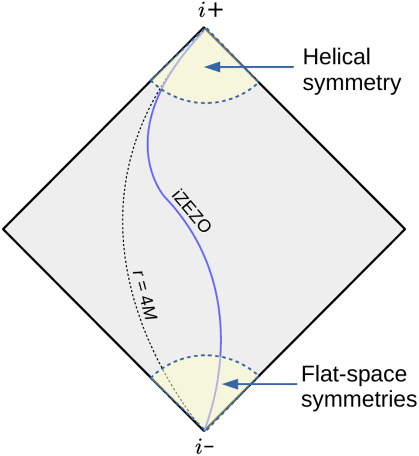
ZEZO: Zero-binding-Energy Zoom-whirl Orbit



- **iZEZO/oZEZO:** inbound/outbound ZEZO

Why is it interesting?

(neglecting radiation) Two different asymptotic symmetries near i^- and i^+ $\Rightarrow$ rigorous relationships between (Bondi-type) energy and AM on one hand and orbital frequency/redshift on other hand



Why is it interesting?

- New invariant quantities associated with post-geodesic conservative dynamics (last of the quantities suggested by Damour in 2009 paper)
- Test of 1st law and Hamiltonian formalism in v. strong field
- Method development for unbound/scatter orbits. Potential for comparisons with NR, calibration of EOB
- Handle on notion of orbital angular-momentum in self-force context

Equation of motion: geodesic case

- Represent the geodesic ZEZO with $x^\alpha = x_\mathrm{p}^\alpha(\tau)$, $u^\alpha := \dot{x}_\mathrm{p}$
- First integrals:

$$\dot{t}_\mathrm{p} = \frac{E}{1 - 2M/r_\mathrm{p}}\,, \qquad E := -u_\alpha \xi^\alpha_{(t)} = -u_t$$

$$\dot{\varphi}_\mathrm{p} = \frac{L}{r_\mathrm{p}^2}\,, \qquad L := u_\alpha \xi^\alpha_{(\varphi)} = u_\varphi$$

$$\dot{r}_\mathrm{p} = \pm\left[E^2 - V(r_\mathrm{p}; L)\right]^{1/2}$$

- Effective radial acceleration:

$$\ddot{r}_\mathrm{p} = -\frac{1}{2}\frac{\partial V(r_\mathrm{p}; L)}{\partial r_\mathrm{p}} =: F^r_\mathrm{geod}$$

- i/oZEZO defined by

$$\dot{r}_\mathrm{p} \to 0 \text{ for } r_\mathrm{p} \to \infty \quad (t \to \mp\infty)$$
$$\dot{r}_\mathrm{p} = 0 = \ddot{r}_\mathrm{p} \text{ for some } r_\mathrm{p} = R \quad (t \to \pm\infty)$$

$$\boxed{\Rightarrow E = 1, \quad L = 4M, \quad R = 4M}$$

Equation of motion with conservative self-force

- Represent the perturbed ZEZO with $x^\alpha = \hat{x}_{\rm p}^\alpha(\tau)$, $\hat{u}^\alpha := \dot{\hat{x}}_{\rm p}$,
- Define $\hat{E}(\tau) := -\hat{u}_t$ and $\hat{L}(\tau) := \hat{u}_\varphi$
- Equations of motion:

$$\dot{\hat{E}} = -F_t^{\rm self} \quad \longleftarrow \text{ Conservative piece of self-acceleration (GSF}/\mu) \tag{1}$$

$$\dot{\hat{L}} = F_\varphi^{\rm self} \tag{2}$$

$$\dot{\hat{r}}_{\rm p} = \pm\left[\hat{E}^2 - V(\hat{r}_{\rm p}; \hat{L})\right]^{1/2}$$

- Effective radial acceleration: $\ddot{\hat{r}}_{\rm p} = F^r_{\rm geod} + F^r_{\rm self}$
- i/oZEZO defined by

$$\dot{\hat{r}}_{\rm p} \to 0 \text{ for } \hat{r}_{\rm p} \to \infty \quad (t \to \mp\infty) \tag{3}$$

$$\dot{\hat{r}}_{\rm p} = 0 = \ddot{\hat{r}}_{\rm p} \text{ for some } \hat{r}_{\rm p} = \hat{R} \quad (t \to \pm\infty) \tag{4}$$

Think of (1)–(4) as 5 algebraic equations for the 5 unknowns
$\left\{\hat{R},\ \hat{E}(\hat{R}),\ \hat{E}_\infty,\ \hat{L}(\hat{R}),\ \hat{L}_\infty\right\}$

EoM with conservative self-force (cont'd)

Solutions:

$$\begin{aligned}
\hat{R} &= 4M - 8M\Delta E - 32M^2 F^r_{\rm self}(\hat{R}) \\
\hat{E}(\hat{R}) &= 1 + \Delta E \\
\hat{E}_\infty &= 1 \\
\hat{L}(\hat{R}) &= 4M + 8M\Delta E \\
\hat{L}_\infty &= 4M + 8M\Delta E - \Delta L
\end{aligned}$$

where

$$\begin{aligned}
\Delta E &:= -\int_\infty^{4M} (F_t^{\rm self}/\dot{r}_{\rm p}) dr_{\rm p} \\
\Delta L &:= \int_\infty^{4M} (F_\varphi^{\rm self}/\dot{r}_{\rm p}) dr_{\rm p}
\end{aligned}$$

IBCO frequency

$$\hat{\Omega} := \left(\frac{\hat{u}^{\varphi}}{\hat{u}^{t}}\right)\bigg|_{r_{\rm p}\to\hat{R}} = \frac{1}{\hat{R}^2}\left(1-\frac{2M}{\hat{R}}\right)\frac{\hat{L}(\hat{R})}{\hat{E}(\hat{R})} = \ldots \text{(subs. and expand in } \mu\text{)}$$

$$\boxed{\Rightarrow \hat{\Omega} = (8M)^{-1}\left(1 + 3\Delta E + 8\ F^{r}_{\rm ibco}\right).}$$

$F^{r}_{\rm ibco}$: Self-force (per μ) on the circular geodesic orbit of radius $r = 4M$ (IBCO: the Innermost Bound Circular Orbit).

Is $\hat{\Omega}$ gauge invariant?

- $x^\alpha \to x^\alpha - \xi^\alpha \;\Rightarrow\quad \delta_\xi \hat{\Omega} = \frac{1}{2}\left(\ddot{\xi}^r + \Omega\dot{\xi}^t - \dot{\xi}^\varphi\right) \quad\Rightarrow\quad$ *NO*!
- What if we restrict to transformations that retain (asymptotic) helical symmetry of the metric perturbation?
  Still not! There is a 4-parameter family of gauge transformations that retain the helical symmetry of the metric perturbation while modifying $\hat{\Omega}$:

$$\xi^\alpha = (\alpha_1 t + \alpha_2\varphi)\delta^\alpha_t + (\alpha_3 t + \alpha_4\varphi)\delta^\alpha_\varphi =: \Xi^\alpha$$

$$\Rightarrow \delta_\Xi \hat{\Omega} = \Omega(\alpha_1 + \Omega\alpha_2) - (\alpha_3 + \Omega\alpha_4)$$

- However, such transformations produce gauge pathology at infinity for any $\alpha_n \neq 0$, so are excluded with a requirement on manifest asymptotic flatness.

⇒ $\hat{\Omega}$ is invariant within the class of gauges in which the perturbed metric is both manifestly helically symmetric and manifestly asymptotically flat.

Correction due to Lorenz-gauge pathology at infinity

- Lorenz-gauge perturbation has pathology at infinity $[h_{tt}(r \to \infty) = \mathrm{const} \neq 0]$ that puts it outside the "good" class of gauges.
- However, it can be taken into the good class using a Ξ-type transformations with $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (-\eta/2, 0, 0, 0)$. $\quad (\eta := \mu/M$ mass ratio.)

$$\Rightarrow \hat{\Omega} = \hat{\Omega}^{(Lor)} - \frac{\eta}{2}\Omega$$

- So, in terms of Lorenz-gauge self-force quantities:

$$\boxed{\Rightarrow \hat{\Omega} = (8M)^{-1}\left(1 - \frac{1}{2}\eta + 3\Delta E^{(Lor)} + 8\ F^{r(Lor)}_{\mathrm{ibco}}\right)}$$

Summary of numerical method

- Two independent implementations of 1+1D evolution of Lorenz-gauge perturbation equations in *uv* coordinates (4th-order convergence).
- Pathological linear-in-t monopole and dipole modes filtered out using fits to analytical expressions for $r \to 4M$
- Self-force calculated using mode-sum regularization.
- Conservative piece extracted by combining iZEZO and oZEZO data based on symmetries under $\dot{r}_{\mathrm{p}} \to -\dot{r}_{\mathrm{p}}$.

Error dominated by

- Orbit truncation at large r. ($r_{\mathrm{max}} \sim 1000M$; run time $\propto r^3_{\mathrm{max}}$)
- Filtering residues in monopole and dipole modes

IBCO frequency: results

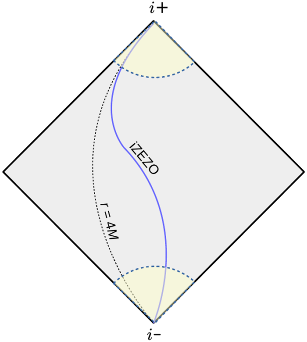
$$\begin{aligned}
\hat{\Omega} &= (8M)^{-1}\left[1+0.5536(8)\eta\right] && \text{our direct calculation}\\
&= (8M)^{-1}\left[1+0.55358516671\eta\right] && \text{IBCO redshift + 1st law}\\
&= (8M)^{-1}\left[1+0.32\eta\right] && \text{EOB (2009 model)}\\
&= (8M)^{-1}\left[1+0.553603030(1)\eta\right] && \text{EOB using 1st law}
\end{aligned}$$

Angular momentum of iZEZO system

Goal: For the time-symmetric iZEZO spacetime, define a notion $\mathcal{L}$ of angular momentum that is unambiguous and gauge-invariant through $O(\eta^2)$.

Issues:

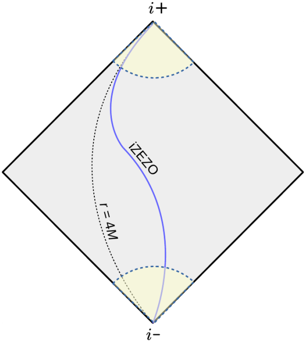
1. $\mathcal{L}_{\mathrm{ADM}}$ or $\mathcal{L}_{\mathrm{Bondi}}(u \to \infty)$ ill-defined in the time-symmetric spacetime because of helical symmetry near i^+.
2. Need to define $\mathcal{L}$ wrt the center of mass. Where is the CoM in our particular gauge?
3. At $O(\eta^2)$ must account for BH recoil
4. Lorenz-gauge pathology at infinity



Angular momentum of iZEZO system:

definition and ADM interpretation

- Define $\mathcal{L}$ as the "incoming" $\mathcal{L}_{\text{Bondi}}$ at $v \to -\infty$.
- This is the purely "mechanical" AM of the particle–BH system at infinite separation
- Can be calculated through mapping to 2-particle kinematics on flat space; no recourse to 2nd-order perturbation
- Naturally relates to PN and EOB notions



Alternative interpretation:

$\mathcal{L}$ is the total ADM angular momentum of the physical, dissipative spacetime with same initial conditions. (Note time-symmetric orbit deviates only slightly from physical orbit until well into the whirl.)

Angular momentum of iZEZO system:

$\mathcal{L}$ in a CoM-centred asymptotically-flat gauge

- Particle contribution:

$$\mathcal{L}_\mathrm{p} = \mu \left(\hat{x}_\mathrm{p} \hat{v}^y - \hat{y}_\mathrm{p} \hat{v}^x \right) = \mu \hat{v}_\varphi = \mu \hat{u}_\varphi / \dot{\hat{t}}_\mathrm{p} \to \mu \hat{L}_\infty ,$$

assuming a gauge in which $\dot{\hat{t}}_\mathrm{p} \to 1$ for $\hat{r}_\mathrm{p} \to \infty$.

- BH contribution:
  BH coordinates in CoM system are $\{X, Y\} = -\eta \{\hat{x}_\mathrm{p}, \hat{y}_\mathrm{p}\}$, with 3-velocity components $\{V^x, V^y\} = -\eta \{\hat{v}^x, \hat{v}^y\}$, so

$$\mathcal{L}_\mathrm{bh} = M \left(X \hat{V}^y - Y \hat{V}^x \right) = M \eta^2 \hat{u}_\varphi = 4\mu^2 + O(\eta^3).$$

- Total angular momentum in CoM frame:

$$\begin{array}{lcccc} \mathcal{L} = \mathcal{L}_\mathrm{p} + \mathcal{L}_\mathrm{bh} & & & & \\ \quad = 4\mu M & + & 4\mu^2 & + & \mu(8M\Delta E - \Delta L) \\ \quad O(\eta) & & O(\eta^2) & & O(\eta^2) \\ \quad \text{geodesic} & & \text{recoil} & & \text{self-force} \end{array}$$

Is the Lorenz gauge CoM-centred?

- CoM position given using Landau-Lifshitz formalism:

$$R^i = \frac{1}{16\pi M} \oint_{i^0} \left(x^i \partial_j H^{tjtk} - H^{titk} \right) dS_k$$

where $H^{\alpha\beta\gamma\delta} := \mathfrak{g}^{\alpha\gamma}\mathfrak{g}^{\beta\delta} - \mathfrak{g}^{\alpha\delta}\mathfrak{g}^{\gamma\beta}$, with "Gothic metric" $\mathfrak{g}^{\alpha\beta} = (-\hat{g})^{1/2}\hat{g}^{\alpha\beta}$.

- Simplifications:
  - Approach i^0 along hypersurface $t = \mathrm{const} \ll -M$, assume perturbation there dominated by static piece.
  - Only even-parity dipole mode $[(l, m) = (1, \pm 1)]$ contributes to R^i.
  - This mode is pure gauge away from sources

- General homogeneous solution is linear combination of 6 basis functions, which can be constructed analytically. Only one contributes to R^i. It is globally regular, and hence represents a true ambiguity within the Lorenz-gauge solutions, corresponding to displacements away from *CoM*.

- We fix $R^i = 0$ by restricting the support of above mode to $r < r_{\mathrm{p}}(\tau)$.

Correction due to Lorenz-gauge pathology at infinity

In Lorenz gauge we don't have $\dot{\hat{t}}_{\mathrm{p}} \to 1$ at i^- but rather

$$\dot{\hat{t}}_{\mathrm{p}} \to 1 - \frac{1}{2}\eta \quad \text{(Lorenz gauge)}$$

So the particle's velocity-at-infinity is $\hat{v}^i = (1-\eta/2)\hat{v}^{i(L)}$, and the particle's contribution to $\mathcal{L}$ becomes

$$\mathcal{L}_{\mathrm{p}} = \mu \hat{L}_\infty^{(\mathrm{Lor})}(1-\eta/2) \simeq \mu \hat{L}_\infty^{(\mathrm{Lor})} - 2\mu^2,$$

⇒ Our final expression for $\mathcal{L}$, in terms of Lorenz-gauge quantities, is

$$\boxed{\mathcal{L} = 4\mu M + \underset{\substack{\uparrow \\ \text{recoil}}}{4\mu^2} - \underset{\substack{\uparrow \\ \text{Lorenz gauge} \\ \text{correction}}}{2\mu^2} + \underset{\substack{\uparrow \\ \text{self-force}}}{\mu(8M\Delta E - \Delta L)^{(Lor)}}}$$

Angular momentum: results

$$\begin{aligned}
\mathcal{L} &= 4M\mu\,[1-0.3046(?)\eta] && \text{our direct calculation}\\
&= 4M\mu\,[1-0.3046742879\eta] && \text{IBCO redshift + 1st law}\\
&= 4M\mu\,[1-0.288(80)\eta] && \text{EOB (2009 model)}\\
&= 4M\mu\,[1-0.30467428782(6)\eta] && \text{EOB using 1st law}
\end{aligned}$$

Implications for 1st law of BH binaries

- First law gives "$L_{\mathrm{ADM}}(\Omega)$" and "$E_{\mathrm{ADM}}(\Omega)$" for circular orbits through $O(\eta^2)$. Relations confirmed previously for ISCO ($r = 6M$), now in stronger field ($r = 4M$).
- We have a rigorous interpretation of the 1st law's L_{ADM} for the IBCO: it is precisely the incoming Bondi AM of the iZEZO in the time-symmetric spacetime
- The 1st-law notions of energy and AM for any unstable circular geodesic orbit with radius $3M < r < 4M$ may be similarly defined by sending particles on a geodesic to infinity.
- Interpretation for $r > 4M$ follows from analytic continuation?

How to improve numerical accuracy

- Analytical treatment of large-r asymptotics
- Improved filtering of monopole+dipole modes, or...
- Alternatives to Lorenz-gauge time evolution:
  - radiation-gauge metric reconstruction
  - frequency-domain treatment
  - ...

Going forward: scatter-angle calculations

- Again high symmetry near i^+ and i^-.
- Parametrize with E (ADM) and impact parameter,

$$b := \lim_{r\to\infty} r \sin|\varphi_{\mathrm{p}}(r) - \varphi_\infty| = \frac{L}{\sqrt{E^2-1}}$$

- Does not have to switch dissipation off
- Probe geometry down to $r = 3M$.
- Initial junk radiation possibly more of a problem, but...
- Better scaling of runtime ($\propto r_{\max}^2$ instead of $r_{\max}^3$).
- Potential for comparison with NR results, now being obtained