Progress in self-consistent evolution with a time domain gravitational self-force code

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Basic Goals

- Aims to develop time domain evolution code for gravitational self-force using the effective source approach
- Builds on Peter Diener's scalar evolution code
- Calculated in the Lorenz gauge
- Uses tortoise coordinates around the source
- Transitions to hyperboloidal coordinates in inner and outer regions

Derivation of Evolution equations

- Use first order perturbation equations for the trace-reversed metric in the Lorenz gauge:

$$\Box \bar{h}_{\alpha\beta} + 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\bar{h}_{\mu\nu} = -16\pi T_{\alpha\beta}$$

- Decompose results into multipole harmonics:

$$\Box^{2d}_{sc}\bar{h}^{(i)\ell m} + M^{(i)\ell}_{(j)}\bar{h}^{(j)\ell m} = -\frac{4\pi r f}{\mu a^{(i)\ell}}\pi T^{(i)\ell m}$$

where

$$\Box^{2d}_{sc} \equiv \partial_t^2 - \partial_{r_*}^2 + \frac{f}{4}\left(\frac{f'}{r} + \frac{\ell(\ell+1)}{r^2}\right)$$

Expressions for the coupling matrix $M^{(i)\ell}_{(j)}$ were originally derived by Barack & Lousto [1].

Constraint Damping

Gundlach *et al* [2] provide a methodology to introduce constraint damping to the evolution equations. This damping requires adding term to evolution equations of the form

$$-\kappa(t_\alpha Z_\beta + t_\beta Z_\alpha),$$

where κ is a positive constant, t_α is a future-directed time-like vector field, and $Z_\alpha = \bar{h}_{\alpha\beta}{}^{;\beta}$ is the Lorenz gauge condition.

We adopt the specific choice of constraint damping given in Barack & Lousto [1]. We revisit the constraint damping problem later, as this choice is incompatible with hyperboloidal slicing as we have implemented it.

Hyperboloidal Slicing

For the hyperboloidal layer from the tortoise coordinate region to *Scri*$^+$, we construct the layer as done by Bernuzzi *et al* [3]. The following relations define the coordinate transformation $\{t, r\} \to \{\tau, \rho\}$ for the outer hyperboloidal layer.

Invariant Killing vector fields:	$\partial_t = \partial_\tau \to \tau = t - h(r_*)$
Invariant outgoing null rays:	$t - r_* = \tau - \rho \to \frac{d\rho}{dr_*} = 1 - H(\rho)$
Compactifying coordinates:	$r_* = \frac{\rho}{\Omega(\rho)}$ $\frac{dh}{dr_*} \equiv H(\rho) = 1 - \frac{\Omega^2}{\Omega - \rho\Omega'}$

In this compactification, $\Omega(\rho)$ is defined such that
$\Omega(S^+) = 0 \Rightarrow r_*(S^+) = \infty$, where S^+ is some positive real number.

Outer Limit

For the numerical implementation, the evolution equation is rearranged so that the second t or τ derivative is on the LHS, and everything else is moved to the RHS. In the outer layer, this becomes

$$\begin{aligned}\bar{h}^{(i)\ell m}_{,\tau\tau} = \frac{1-H}{1+H}\bar{h}^{(i)\ell m}_{,\rho\rho} - \frac{2H}{1+H}\bar{h}^{(i)\ell m}_{,\tau\rho} - \frac{H'}{1+H}\bar{h}^{(i)\ell m}_{,\tau} \\ - \frac{H'}{1+H}\bar{h}^{(i)\ell m}_{,\rho} - \frac{4}{1-H^2}M^{(i)\ell}_{(j)}\bar{h}^{(j)\ell m} - \frac{4}{1-H^2}V(\rho)\bar{h}^{(i)\ell m}\end{aligned}$$

In the limit $\rho \to S^+$, $H \to 1$, so the limit must be considered carefully for terms with a factor of $\frac{1}{1-H}$. Thankfully, all terms are finite in the limit.

Hyperboloidal Slicing

For the hyperboloidal layer from the tortoise coordinate region to the horizon, we perform a similar construction as before. The primary difference is that we preserve the ingoing null rays instead of the outgoing rays. This amounts to changing the equation

$$t - r_* = \tau - \rho$$

to

$$t + r_* = \tau + \rho$$

In this compactification, $\Omega(\rho)$ is defined such that $\Omega(S^-) = 0 \Rightarrow r_*(S^-) = -\infty$, where S^- is some negative real number.

Inner Limit

Looking at the RHS for the inner layer, we have

$$\bar{h}^{(i)\ell m}_{,\tau\tau} = \frac{1+H}{1-H}\bar{h}^{(i)\ell m}_{,\rho\rho} - \frac{2H}{1-H}\bar{h}^{(i)\ell m}_{,\tau\rho} - \frac{H'}{1-H}\bar{h}^{(i)\ell m}_{,\tau} + \frac{H'}{1-H}\bar{h}^{(i)\ell m}_{,\rho} - \frac{4}{1-H^2}M^{(i)\ell}_{(j)}\bar{h}^{(j)\ell m} - \frac{4}{1-H^2}V(\rho)\bar{h}^{(i)\ell m}$$

In the limit $\rho \to S^-$, $H \to -1$, so the limit must be considered carefully for terms with a factor of $\frac{1}{1+H}$. Unfortunately, in this case there are terms in the coupling matrix $M^{(i)\ell}_{(j)}$ which are infinite in the coupling equations for $i = 2, 4, 8$.

Tensor Spherical Harmonic Basis

For this work, we use the tensor spherical harmonics from Barack & Sago [4]. This basis introduces a factor of f in the $i = 3$ tensor spherical harmonic from [1]. Examining how this affects the coupling matrix in the hyperboloidal layer reveals that this change resolved infinite coefficients which would be present if we used the original basis [1].

We examined whether a similar procedure could improve the behavior of the other coefficients by introducing a factor of f^{n_i} to the other tensor spherical harmonics, where n_i are real numbers. Recalculating the coupling matrix revealed that no combination of factors of this form would resolve the issue.

Using the Constraints

We also examined the effect of adding constraints to the evolution equations in the inner hyperboloidal region to cancel the bad terms. The terms in question come in the form

$$\partial_t + \partial_{r_*}$$

In the outer layer, this combination has a factor of $(1 + H)^{-1}$ in τ, ρ coordinates, which is finite. In the inner layer, the ∂_ρ factor changes to $(1 - H)^{-1}$, which is finite in the inner layer. The ∂_τ factor stays as $(1 + H)^{-1}$, which blows up at the horizon. To fix this, we need to change the above combination to

$$\partial_t - \partial_{r_*}$$

We can do exactly what we need by subtracting some of the constraint equations from these evolution equations.

Constraint Damping

To dampen constraints and simply the equations,

we added the constraint equations to equations for $h^{(i)}, \ i = 1, 2, 4, 5, 8, 9$.

To resolve the infinities,

we subtracted constraint equations from equations for $h^{(i)}, \ i = 2, 4, 8$.

The net result:

for those three equations, we have changed the sign of the constraint damping terms, meaning that the constraint violations grow uncontrollably in the inner layer. Upon implementing these coupling coefficients, we verified that the constraint violations grow with time.

A Path Forward?

As it stands, it is unclear if hyperboloidal slicing is inherently unusable in the inner region, or if there is another available trick to resolve these difficulties.

Alternatively, another coordinate choice could be more appropriate for the inner region, though the behavior at the horizon could still be problematic, depending on how the second time derivative transforms in the new coordinate choice.

Once an effective solution is chosen, the needed coupling matrix elements can be quickly calculated in Mathematica notebooks and implemented into the code.

References

L. Barack and C.O. Lousto (2005).
Perturbations of Schwarzschild black holes in the Lorenz gauge: Formulation and numerical implementation
Phys. Rev. D 72, 104026.

C. Gundlach, J. M. Martin-García, G. Calabrese, and I. Hinder (2005)
Constraint damping in the Z4 formulation and harmonic gauge
Class. Quant. Grav. 22, 3767.

S. Bernuzzi, A. Nagar, and A. Zenginoğlu (2011)
Binary black hole coalescence in the large-mass-ratio limit: the hyperboloidal layer method and waveforms at null innity
Phys. Rev. D 84, 084026.

L. Barack and N. Sago (2007)
Gravitational self-force on a particle in circular orbit around a Schwarzschild black hole
Phys. Rev. D 75, 064021.

Questions/Discussion

- Are there better coordinates to cover the inner region?
- Is there another approach to somehow resolve the difficulties of using hyperboloidal slicing in the inner region?