Dissipative self-force with a spinning secondary: ‘Flux-balance laws’

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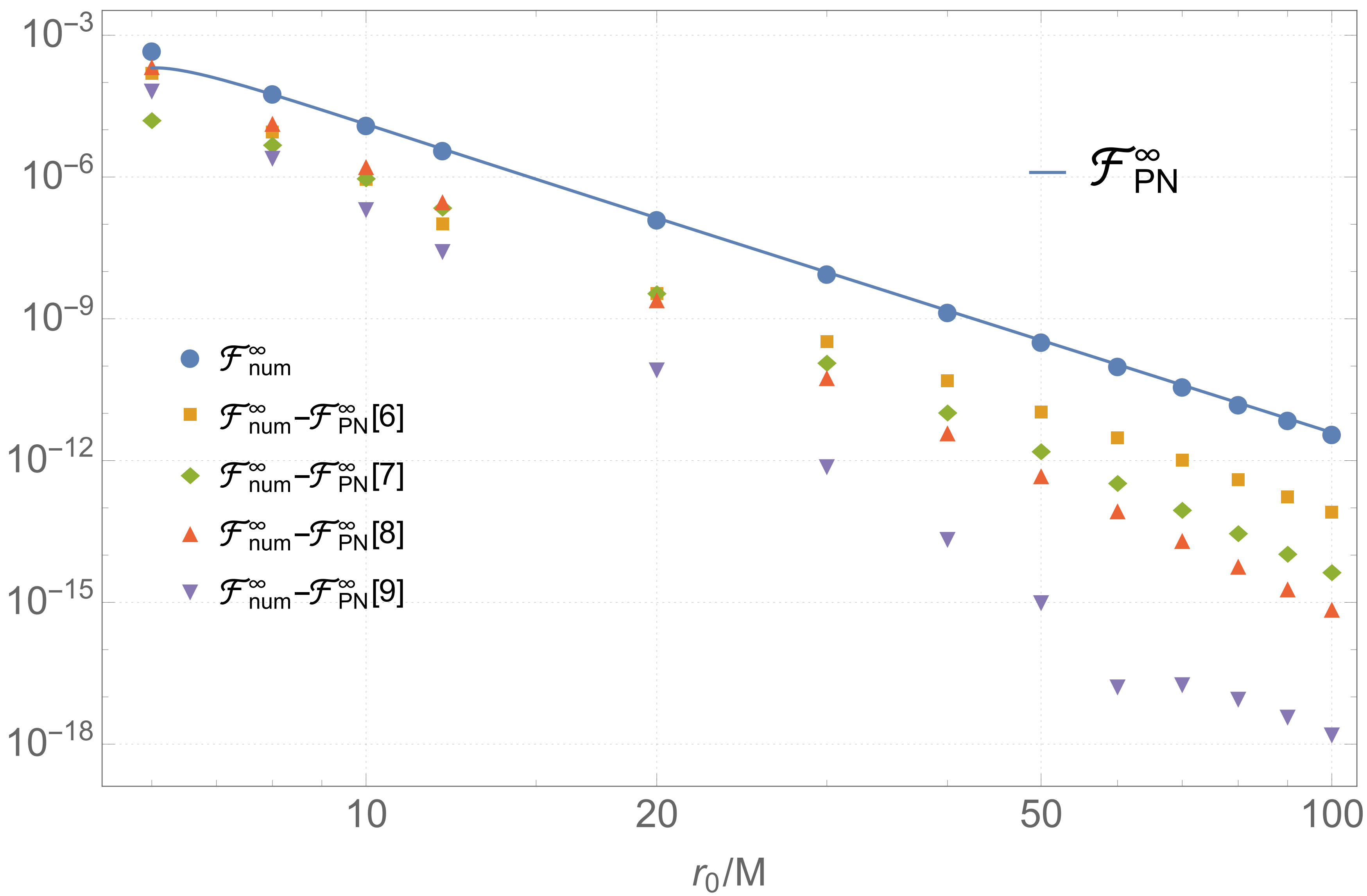
Capra 22
CBPF

Last Capra..



Working asymptotic fluxes, two gauges, analytical and numerical agreed with Tanaka et al, Harms et al

..no flux balance, apparent divergences, unclear interpretations(more later)

Generated some discussion which lead to clearing up all issues.

Adiabatic evolutions and flux balance laws

One slide review: adiabatic order: dutdt=dEdt, heuristically balance this with flux

See other talks T. Tanaka, T Osburn, S Hughes, etc

$$u^\alpha \nabla_\alpha u_\beta = \epsilon \langle f_\beta \rangle + \mathcal{O}(\epsilon^2)$$

Killing vectors->conservation laws. Specifically, both Schw and Kerr possess two killing vectors

$$\xi^\alpha = \partial_t, \quad \eta^\alpha = \partial_\varphi$$

$$u^\alpha \nabla_\alpha (\xi^\beta u_\beta) = 0$$

$$u_t = -E$$

$$u_\varphi = L$$

then, e.g.

$$\frac{dE}{d\tau} = -\langle f_t \rangle$$

Heuristically we can make the identification

$$-\frac{1}{u^t}\langle f_t \rangle \equiv \frac{dE^\infty}{dt} + \frac{dE^H}{dt}$$

Relating local orbit-averaged SF to Teukolsky amplitudes: Key points

Following some algebraic manipulations, the MiSaTaQuWa SF expression can be rewritten as

$$\langle f_\alpha \xi^\alpha \rangle = \frac{1}{2}\langle u^\alpha u^\beta \mathcal{L}_\xi h^{\rm ret}_{\alpha\beta}\rangle$$

i.e. the familiar

$$\langle f_t \rangle = \frac{1}{2}\langle u^\alpha u^\beta \partial_t h_{\alpha\beta}\rangle$$

$$\begin{aligned}
\frac{1}{2}\langle u^\alpha u^\beta \mathcal{L}_\xi h^{\rm ret}_{\alpha\beta}\rangle &= \int_T d\tau \int_T d^4x' u^\alpha u^\beta \mathcal{L}_\xi G^{\mathcal{R}}_{\alpha\beta\gamma'\delta'}(x,x') T^{\gamma'\delta'} + \mathcal{O}(\epsilon) \\
&= \int d^4x \int d^4x' \frac{\delta^4(x-z(t))}{\sqrt{-g}} u^\alpha u^\beta \mathcal{L}_\xi G^{\mathcal{R}}_{\alpha\beta\gamma'\delta'}(x,x') T^{\gamma'\delta'} + \mathcal{O}(\epsilon) \\
&= \int d^4x \int d^4x' T^{\alpha\beta} \mathcal{L}_\xi G^{\mathcal{R}}_{\alpha\beta\gamma'\delta'}(x,x') T^{\gamma'\delta'} + \mathcal{O}(\epsilon)
\end{aligned}$$

Using the symmetry $\mathcal{L}_\xi G(x,x') = -\mathcal{L}_{\xi'} G(x,x')$

$$\begin{aligned}
\frac{1}{2}\langle u^\alpha u^\beta \mathcal{L}_\xi h^{\rm ret}_{\alpha\beta}\rangle &= \int d^4x \int d^4x' T^{\alpha\beta} \frac{1}{2}\left(\mathcal{L}_\xi G^{\mathcal{R}}_{\alpha\beta\gamma'\delta'}(x,x') - \mathcal{L}_{\xi'} G^{\mathcal{R}}_{\alpha\beta\gamma'\delta'}(x,x')\right) T^{\gamma'\delta'} + \mathcal{O}(\epsilon) \\
&= \int d^4x \int d^4x' T^{\alpha\beta} \mathcal{L}_\xi G^{\rm Rad}_{\alpha\beta\gamma'\delta'}(x,x') T^{\gamma'\delta'} + \mathcal{O}(\epsilon) = \frac{1}{2}\langle u^\alpha u^\beta \mathcal{L}_\xi h^{\rm rad}_{\alpha\beta}\rangle
\end{aligned}$$

Relating local orbit-averaged SF to Teukolsky amplitudes: Key points

Ultimately we have that

$$\langle f_\alpha \xi^\alpha \rangle = \frac{1}{2} \langle u^\alpha u^\beta \mathcal{L}_\xi h^{\mathrm{rad}}_{\alpha\beta} \rangle$$

The radiative part of the MP can be related to the Teukolsky amplitudes using CCK reconstruction (see T Tanaka's talk on monday)

$$h^{\mathrm{rad}}_{\alpha\beta} = \mathcal{D}\Phi_{\mathrm{H}}$$

$$\ddots$$

(lots of work)

$$\left\langle \frac{D\mathcal{E}}{d\tau} \right\rangle = \sum \frac{1}{4\pi\omega} |Z^{(m)}_{lm}|^2$$

Situation with a spinning secondary: ‘background’

Equations of motion: Mathisson-Pappepetrou-Dixon (to linear in the spin) (SPELL)

$$\frac{Dp^\alpha}{d\tau} = -\frac{1}{2}R^\alpha{}_{\beta\gamma\delta}u^\beta S^{\gamma\delta}$$

$$\frac{DS^{\alpha\beta}}{d\tau} = 2p^{[\alpha}u^{\beta]} = \mathcal{O}(\sigma^2)$$

At linear in sigma $\quad p^\alpha = \mu u^\alpha + \mathcal{O}(\sigma^2)$

For this talk we will assume

$$\epsilon \ll \sigma \ll 1$$

i.e.

$$\epsilon^2 \ll \sigma\epsilon \ll \epsilon$$

Situation with a spinning secondary: Perturbed spacetime

MPD-Harte: Everything is ‘upgraded’ to perturbed spacetime

$$\frac{\hat{D}\hat{p}^{\alpha}}{d\tau} = -\frac{1}{2}\hat{R}^{\alpha}{}_{\beta\gamma\delta}u^{\beta}\hat{S}^{\gamma\delta} + \mathcal{O}(\epsilon^2) + \mathcal{O}(\sigma^2)$$

$$\frac{\hat{D}\hat{S}^{\alpha\beta}}{d\tau} = \mathcal{O}(\epsilon^2) + \mathcal{O}(\sigma^2)$$

with

$$\hat{g}_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \epsilon h^{(m)R}_{\alpha\beta} + \epsilon\sigma h^{(\sigma)R}_{\alpha\beta} + \mathcal{O}(\epsilon^2) + \mathcal{O}(\epsilon^2\sigma^2)$$

Where the ‘m’ and ’σ’ metric perturbations are sourced by

$$T^{(m)}_{\alpha\beta}(x) = \int d\tau \frac{\delta^4(x^\mu - z^\mu(\tau))}{\sqrt{-g}} \hat{m} u^\alpha(\tau) u^\beta(\tau),$$

$$T^{(\sigma)}_{\alpha\beta}(x) = \int d\tau \nabla_\delta \left(\frac{\delta^4(x^\mu - z^\mu(\tau))}{\sqrt{-g}} \right) u^{(\alpha}(\tau) S^{\beta)\delta}(\tau).$$

Situation with a spinning secondary

Equations of motion MPDH & metric perturbation with dipolar source

MPD-Harte: Everything is 'upgraded' to perturbed spacetime

As usual, write as derivatives wrt background metric, and everything else as 'forcing'

$$\frac{D\hat{p}^\alpha}{d\tau} = -\frac{1}{2} R^{(0)\alpha}{}_{\beta\gamma\delta} u^\beta S^{\gamma\delta} + a^\alpha + \mathcal{O}(m_2^2) + \mathcal{O}(\sigma^2)$$

$$\frac{D\hat{S}^{\alpha\beta}}{d\tau} = -u^\alpha S^{\beta[\delta} g^{\gamma]\lambda} \left(h^{(m)}_{\lambda\beta;\alpha} + h^{(m)}_{\alpha\lambda;\beta} - h^{(m)}_{\alpha\beta;\lambda} \right) + \mathcal{O}(m_2^2) + \mathcal{O}(\sigma^2)$$

<u>Question</u>: Under an adiabatic type assumption (i.e. orbit averaging the RHS) how much can be filled in by asymptotic flux data?

Situation with a spinning secondary: situation last year was which flux balance?

Under the MPD equations, the conserved charges are

$$\mathcal{E} = u^\alpha \xi_\alpha + \frac{1}{2} S^{\alpha\beta} \nabla_\alpha \xi_\beta .$$

Last year, we didn't know what to identify with the asymptotic fluxes, i.e.

$$\langle \frac{du_t}{dt} \rangle \equiv \langle \frac{dE^\infty}{dt} \rangle + \langle \frac{dE^H}{dt} \rangle$$

or

$$\langle \frac{dE}{dt} \rangle \equiv \langle \frac{dE^\infty}{dt} \rangle + \langle \frac{dE^H}{dt} \rangle$$

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Approach: follow formal procedure as in non-spinning case

Procedure:

- CD the conserved quantity
- Write the result in terms of the radiative MP
- Follow metric reconstruction in reverse to write expression in terms of Teukolsky amplitudes

$$\left\langle \frac{D\mathcal{E}}{d\tau} \right\rangle = \left\langle \xi^{\beta} a_{\beta} + \frac{1}{2} u^{\alpha} \nabla_{\alpha} \left(S^{\gamma\beta} \right) \nabla_{\gamma} \xi_{\beta} + \frac{1}{2} u^{\alpha} S^{\gamma\beta} \nabla_{\alpha} \nabla_{\gamma} \xi_{\beta} \right\rangle$$

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$$\left\langle \frac{D\mathcal{E}}{d\tau} \right\rangle = \left\langle \xi^\beta a_\beta + \frac{1}{2} u^\alpha \nabla_\alpha \left(S^{\gamma\beta} \right) \nabla_\gamma \xi_\beta + \frac{1}{2} u^\alpha S^{\gamma\beta} \nabla_\alpha \nabla_\gamma \xi_\beta \right\rangle$$
$$= \frac{1}{2} \left\langle u^\alpha u^\beta \mathcal{L}_\xi h_{\alpha\beta} - S^{\gamma\delta} u^\beta \nabla^{(0)}_\delta \mathcal{L}_\xi h^{(m)}_{\gamma\beta} \right\rangle$$

The RHS here is the 'local part' of the flux balance law we derive

Notably, each of these terms are *NOT* radiative. You only achieve l-mode convergence in combination.

Approach: follow formal procedure as in non-spinning case

To get something radiative, look at the GF representation

$$\langle u^\alpha u^\beta \mathcal{L}_\xi h^{(\sigma)}_{\alpha\beta}\rangle = \int d^4x \int d^4x' T^{(m)\alpha\beta} \mathcal{L}_\xi G_{\alpha\beta\alpha'\beta'} T^{(\sigma)\alpha'\beta'}$$

Approach: follow formal procedure as in non-spinning case

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$$\langle S^{\gamma\delta} u^\beta \nabla^{(0)}_\delta \mathcal{L}_\xi h^{(m)}_{\gamma\beta} \rangle = \int d\tau \int d^4x \frac{\delta^4(x - z(\tau))}{\sqrt{g}} u^\beta S^{\gamma\delta} \nabla^{(0)}_\delta \mathcal{L}_\xi h^{(m)}_{\gamma\beta}$$

Approach: follow formal procedure as in non-spinning case

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$$\langle u^\alpha u^\beta \mathcal{L}_\xi h^{(\sigma)}_{\alpha\beta}\rangle = \int d^4x \int d^4x' T^{(m)\alpha\beta} \mathcal{L}_\xi G_{\alpha\beta\alpha'\beta'} T^{(\sigma)\alpha'\beta'}$$

$$\langle S^{\gamma\delta} u^\beta \nabla^{(0)}_\delta \mathcal{L}_\xi h^{(m)}_{\gamma\beta}\rangle = \int d\tau \int d^4x \frac{\delta^4(x - z(\tau))}{\sqrt{g}} u^\beta S^{\gamma\delta} \nabla^{(0)}_\delta \mathcal{L}_\xi h^{(m)}_{\gamma\beta}$$

$$= -\int d^4x \left(\int d\tau \nabla^{(0)}_\delta \left(\frac{\delta^4(x - z(\tau))}{\sqrt{g}} u^\beta S^{\gamma\delta}\right)\right) \mathcal{L}_\xi h^{(m)}_{\gamma\beta}$$

From which follows the expected flux formulae:

Approach: follow formal procedure as in non-spinning case

To get something radiative, look at the GF representation

$$\langle u^\alpha u^\beta \mathcal{L}_\xi h^{(\sigma)}_{\alpha\beta}\rangle = \int d^4x \int d^4x' T^{(m)\alpha\beta}\mathcal{L}_\xi G_{\alpha\beta\alpha'\beta'}T^{(\sigma)\alpha'\beta'}$$

$$\begin{aligned}\langle S^{\gamma\delta}u^\beta \nabla^{(0)}_\delta \mathcal{L}_\xi h^{(m)}_{\gamma\beta}\rangle &= \int d\tau \int d^4x \frac{\delta^4(x-z(\tau))}{\sqrt{g}} u^\beta S^{\gamma\delta}\nabla^{(0)}_\delta \mathcal{L}_\xi h^{(m)}_{\gamma\beta} \\ &= -\int d^4x \left(\int d\tau \nabla^{(0)}_\delta \left(\frac{\delta^4(x-z(\tau))}{\sqrt{g}} u^\beta S^{\gamma\delta}\right)\right)\mathcal{L}_\xi h^{(m)}_{\gamma\beta} \\ &= -\int d^4x \int d^4x' T^{(\sigma)\alpha\beta}\mathcal{L}_\xi G_{\alpha\beta\alpha'\beta'}T^{(m)\alpha'\beta'}\end{aligned}$$

Using the symmetry $\mathcal{L}_\xi G(x,x') = -\mathcal{L}_{\xi'} G(x,x')$ **the two terms are identical modulo x and x'**

$$\frac{1}{2}\left\langle u^\alpha u^\beta \mathcal{L}_\xi h^{(s)}_{\alpha\beta} - S^{\gamma\delta}u^\beta \nabla^{(0)}_\delta \mathcal{L}_\xi h^{(m)}_{\gamma\beta}\right\rangle = \langle u^\alpha u^\beta \mathcal{L}_\xi h^{(s)\mathrm{rad}}_{\alpha\beta}\rangle$$

Summary of flux balance law

Ultimately the loss of energy is the expected extension of the usual expression

$$\left\langle \frac{D\mathcal{E}}{d\tau} \right\rangle = \sum \frac{1}{4\pi\omega} |Z_{lm}^{(m)} + Z_{lm}^{(\sigma)}|^2$$

This is equivalent to the local calculation with the retarded metric perturbation

$$\left\langle \frac{D\mathcal{E}}{d\tau} \right\rangle = \frac{1}{2} \left\langle u^\alpha u^\beta \mathcal{L}_\xi h_{\alpha\beta} - S^{\gamma\delta} u^\beta \nabla_\delta^{(0)} \mathcal{L}_\xi h_{\gamma\beta}^{(m)} \right\rangle$$

Explicit verification

We specialise to circular obits, Schwarzschild, spin aligned with orbital AM

The introduction of spin introduces two main changes to existing codes for the metric *(see Sarp's talk from last year)*

1. Spin altered background trajectory: e.g. allowed frequencies

$$\omega = m\Omega_\varphi = \omega_0 + \omega_\sigma \sigma + \mathcal{O}(\sigma^2)$$

Likewise, 4-velocity appearing in T has a spin dependent piece!

2. Dipolar term in stress energy

More singular source, need careful analysis of Lorenz gauge matching conditions

Derive new Teukolsky source (this appears in the literature for flux calculations)

—> CCK procedure remains unchanged

Explicit verification

Numerical approaches

1. **Lorenz gauge**
2. **Radiation gauge**

Analytical approach

2. **Radiation gauge**

Technically similar to previous calculations (see C. Murra's talk), expand at each PN order to linear in spin

Results

r_0	$\mathcal{F}^{\mathrm{H}}_{\sigma}$	$\mathcal{F}^{\infty}_{\sigma}$	$F^{\mathrm{local}}_{\sigma}$	$\lvert 1 - \langle \mathcal{F} u^t \rangle_\sigma / F^{\mathrm{local}}_{\sigma} \rvert$
6	$-2.44110277062 \times 10^{-6}$	$-5.05052142614 \times 10^{-4}$	$-7.62945928003 \times 10^{-4}$	1.96×10^{-11}
8	$-5.85126152707 \times 10^{-8}$	$-6.2795524478 \times 10^{-5}$	$-8.27935402002 \times 10^{-5}$	5.08×10^{-12}
10	$-4.02409747537 \times 10^{-9}$	$-1.35283840476 \times 10^{-5}$	$-1.66725567025 \times 10^{-5}$	1.79×10^{-11}
12	$-4.91730395266 \times 10^{-10}$	$-3.96761534542 \times 10^{-6}$	$-4.69443695538 \times 10^{-6}$	3.09×10^{-11}
20	$-1.70447749342 \times 10^{-12}$	$-1.36368164644 \times 10^{-7}$	$-1.49916383527 \times 10^{-7}$	1.61×10^{-10}
30	$-2.14463437625 \times 10^{-14}$	$-9.69553949111 \times 10^{-9}$	$-1.03086338554 \times 10^{-8}$	4.7×10^{-10}
40	$-9.92781195010 \times 10^{-16}$	$-1.49558022979 \times 10^{-9}$	$-1.56494549394 \times 10^{-9}$	1.45×10^{-9}
50	$-9.25922620715 \times 10^{-17}$	$-3.51467899595 \times 10^{-10}$	$-3.64338708148 \times 10^{-10}$	2.97×10^{-9}
60	$-1.33975153331 \times 10^{-17}$	$-1.07706168184 \times 10^{-10}$	$-1.10964686283 \times 10^{-10}$	4.3×10^{-9}
70	$-2.62071409834 \times 10^{-18}$	$-3.96302737321 \times 10^{-11}$	$-4.06516696731 \times 10^{-11}$	7.28×10^{-9}
80	$-6.38761880534 \times 10^{-19}$	$-1.66688751664 \times 10^{-11}$	$-1.7043065291 \times 10^{-11}$	1.04×10^{-8}
90	$-1.84096376783 \times 10^{-19}$	$-7.76490004651 \times 10^{-12}$	$-7.91929323499 \times 10^{-12}$	1.37×10^{-8}
100	$-6.05434134454 \times 10^{-20}$	$-3.92050069646 \times 10^{-12}$	$-3.9904602329 \times 10^{-12}$	1.94×10^{-8}

TABLE I. Contribution at $\mathcal{O}(\sigma)$ to the radiated flux for a spinning body moving on a circular orbit of radius r_0 about a Schwarzschild black hole. The data in the second, third and forth columns can be found digitally in the Black Hole Perturbation Toolkit [29].

Results-radiation gauge divergences (which cancel)

The PN expansion of the terms making up the flux balance law for large l:

$$\left(S^{\gamma\delta}u^{\beta}\nabla^{(0)}_{\delta}\mathcal{L}_{\xi}h_{\gamma\beta}\right)^{+}=\frac{6(\ell+1)}{\ell-1}\sigma y^{11/2}+\frac{3\left(27\ell^{3}+40\ell^{2}-8\ell-9\right)}{(\ell-1)(2\ell-1)(2\ell+3)}\sigma y^{13/2}+\mathcal{O}(y^{7})$$

$$\left(S^{\gamma\delta}u^{\beta}\nabla^{(0)}_{\delta}\mathcal{L}_{\xi}h_{\gamma\beta}\right)^{-}=\frac{6\ell}{2+\ell}\sigma y^{11/2}+\frac{3\left(27\ell^{3}+41\ell^{2}-7\ell-12\right)}{(\ell+2)(2\ell-1)(2\ell+3)}\sigma y^{13/2}+\mathcal{O}(y^{7})$$

$$\left(u^{\alpha}u^{\beta}\mathcal{L}_{\xi}h_{\alpha\beta}\right)^{+}=\frac{6(\ell+1)}{\ell-1}\sigma y^{11/2}+\frac{3\left(27\ell^{3}+40\ell^{2}-8\ell-9\right)}{(\ell-1)(2\ell-1)(2\ell+3)}\sigma y^{13/2}+\mathcal{O}(y^{7})$$

$$\left(u^{\alpha}u^{\beta}\mathcal{L}_{\xi}h_{\alpha\beta}\right)^{-}=\frac{6\ell}{2+\ell}\sigma y^{11/2}+\frac{3\left(27\ell^{3}+41\ell^{2}-7\ell-12\right)}{(\ell+2)(2\ell-1)(2\ell+3)}\sigma y^{13/2}+\mathcal{O}(y^{7})$$

— individual terms are explicitly not radiative, but in combination all divergences disappear

Might need to be careful in tests of convergence!

Conclusions/implications

The flux balance law we have derived, tells you how to evolve the specific combination of the 4-velocity and spin tensor

$$\mathcal{E} = u^\alpha \xi_\alpha + \frac{1}{2} S^{\alpha\beta} \nabla_\alpha \xi_\beta .$$

So by simultaneously integrating

$$\frac{D\hat{S}^{\alpha\beta}}{d\tau} = -u^\alpha S^{\beta[\delta} g^{\gamma]\lambda} \left(h^{(m)}_{\lambda\beta;\alpha} + h^{(m)}_{\alpha\lambda;\beta} - h^{(m)}_{\alpha\beta;\lambda} \right) + \mathcal{O}(m_2^2) + \mathcal{O}(\sigma^2)$$

which requires local information, the system can be evolved.

Open question: can the RHS here be related to asymptotic information somehow?

Conclusions/implications— To do list

Formalism

- Relax aligned spin assumption
- Action angle formulation- validate for resonances
- Carter constant calculation

Practical calculation

Verify things for Kerr/different orbital configurations

We now have SF codes for a dipolar source, can start investigating how to generate data and include in evolution.