Spectroscopy of extremal and near-extremal Kerr black holes

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Motivations

- Extremal Kerr (EK) black holes are the "last frontier " between rotating black holes and naked singularities
- There are observational evidences that support the existence of extremal black holes in nature : **Lijun Gou et al. '14**

Teukolsky Equation

- The radial factor of the perturbation obeys the ODE :

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d\,{}_sR_{lm\omega}(r)}{dr}\right)+\left(\frac{K^2-2is(r-M)K}{\Delta}+4is\omega r-{}_s\lambda_{lmc}\right){}_sR_{lm\omega}(r)=0 \quad (1)$$

where

$$K\equiv(r^2+a^2)\omega-am \quad (2)$$

- For $a<M$ this equation has two regular singular points at $r=r_+$ and $r=r_-$ and an irregular one at $r=\infty$
- For $a=M$ this equation has two irregular singular points at $r=M$ and $r=\infty$
- This different structure of the ODE requires a different approach when dealing with the extremal case
- For now on $M=1/2$

Sub-extremal Case $a < M$

- The asymptotic behavior

$$R^{\text{in}}_{lm\omega} \sim \begin{cases} B^{\text{trans}}_{lm\omega} \Delta^{-s} e^{-i\tilde{\omega} r_*}, & \text{for} \quad r \to r_+ \\ B^{\text{ref}}_{lm\omega} \dfrac{e^{i\omega r_*}}{r^{2s+1}} + B^{\text{inc}}_{lm\omega} \dfrac{e^{-i\omega r_*}}{r}, & \text{for} \quad r \to \infty \end{cases} \tag{3}$$

$$R^{\text{up}}_{lm\omega} \sim \begin{cases} C^{\text{ref}}_{lm\omega} \Delta^{-s} e^{-i\tilde{\omega} r_*} + C^{\text{up}}_{lm\omega} e^{i\tilde{\omega} r_*}, & \text{for} \quad r \to r_+ \\ C^{\text{trans}}_{lm\omega} \dfrac{e^{i\omega r_*}}{r^{2s+1}}, & \text{for} \quad r \to \infty \end{cases} \tag{4}$$

where $\tilde{\omega} \equiv \omega - m\Omega_H$ and $\Omega_H = a/(r_+^2 + a^2)$

- Define $\bar{R}^{in}_{lm\omega}$ and $\bar{R}^{up}_{lm\omega}$ as

$$\bar{R}^{\text{in}}_{lm\omega} \equiv \frac{R^{\text{in}}_{lm\omega}}{B^{\text{trans}}_{lm\omega}} \quad \text{and} \quad \bar{R}^{\text{up}}_{lm\omega} \equiv \frac{R^{\text{up}}_{lm\omega}}{C^{\text{trans}}_{lm\omega}} \tag{5}$$

Extremal Case $a = M$

- The asymptotic behavior

$$ {}^{\mathrm{x}}R^{\mathrm{in}}_{lm\omega} \sim \begin{cases} {}^{\mathrm{x}}B^{\mathrm{trans}}_{lm\omega}\, e^{i\frac{k}{2x}} \dfrac{e^{-i\omega \ln(x)}}{x^{2s}}, & \text{for} \quad x \to 0^+ \\ {}^{\mathrm{x}}B^{\mathrm{ref}}_{lm\omega} \dfrac{e^{i\omega(x+\ln(x))}}{x^{2s+1}} + {}^{\mathrm{x}}B^{\mathrm{inc}}_{lm\omega} \dfrac{e^{-i\omega(x+\ln(x))}}{x}, & \text{for} \quad x \to \infty \end{cases} \tag{6} $$

$$ {}^{\mathrm{x}}R^{up}_{lm\omega} \sim \begin{cases} {}^{\mathrm{x}}C^{\mathrm{ref}}_{lm\omega}\, e^{i\frac{k}{2x}} \dfrac{e^{-i\omega \ln(x)}}{x^{2s}} + {}^{\mathrm{x}}C^{\mathrm{inc}}_{lm\omega}\, e^{-i\frac{k}{2x}+i\omega\ln(x)}, & \text{for} \quad x \to 0^+ \\ {}^{\mathrm{x}}C^{\mathrm{trans}}_{lm\omega} \dfrac{e^{i\omega(x+\ln(x))}}{x^{1+2s}}, & \text{for} \quad x \to \infty \end{cases} \tag{7} $$

where $k \equiv \omega - m(\Omega_H = 1)$ and $x \equiv r - r_H = r - 1/2$

- Define ${}^{\mathrm{x}}\bar{R}^{\mathrm{in}}_{lm\omega}$ and ${}^{\mathrm{x}}\bar{R}^{\mathrm{up}}_{lm\omega}$ as

$$ {}^{\mathrm{x}}\bar{R}^{\mathrm{in}}_{lm\omega} \equiv \frac{{}^{\mathrm{x}}R^{in}_{lm\omega}}{{}^{\mathrm{x}}B^{\mathrm{trans}}_{lm\omega}} \quad \text{and} \quad {}^{\mathrm{x}}\bar{R}^{\mathrm{up}}_{lm\omega} \equiv \frac{{}^{\mathrm{x}}R^{\mathrm{up}}_{lm\omega}}{{}^{\mathrm{x}}C^{\mathrm{trans}}_{lm\omega}} \tag{8} $$

GF of Teukolsky Equation

- Construct the Green Function (GF) of the radial Teukolsky equation as

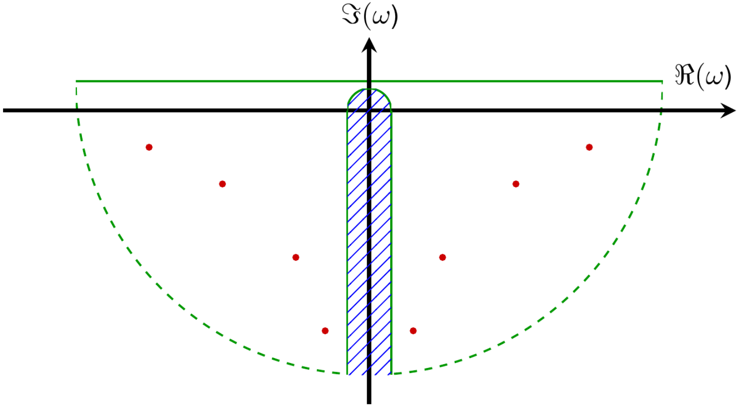
$$G_{lm\omega}(r|r') = \frac{\bar{R}^{up}_{lm\omega}(r)\bar{R}^{in}_{lm\omega}(r')}{W_{lm\omega}}\Theta(r-r') + \frac{\bar{R}^{up}_{lm\omega}(r')\bar{R}^{in}_{lm\omega}(r)}{W_{lm\omega}}\Theta(r'-r) \tag{9}$$

- For the sub-extremal case

$$W_{lm\omega} \equiv \Delta^{s+1}\left(\bar{R}^{\text{in}}_{lm\omega}\frac{d\bar{R}^{\text{up}}_{lm\omega}}{dr} - \bar{R}^{\text{up}}_{lm\omega}\frac{d\bar{R}^{\text{in}}_{lm\omega}}{dr}\right) = 2i\omega\frac{B^{\text{inc}}_{lm\omega}}{B^{\text{trans}}_{lm\omega}} \tag{10}$$

- For the extremal case

$$^{\text{x}}W_{lm\omega} \equiv \Delta^{s+1}\left(^{\text{x}}\bar{R}^{\text{in}}_{lm\omega}\frac{d^{\text{x}}\bar{R}^{\text{up}}_{lm\omega}}{dx} - {}^{\text{x}}\bar{R}^{\text{up}}_{lm\omega}\frac{d^{\text{x}}\bar{R}^{\text{in}}_{lm\omega}}{dx}\right) = 2i\omega\frac{^{\text{x}}B^{\text{inc}}_{lm\omega}}{^{\text{x}}B^{\text{trans}}_{lm\omega}} \tag{11}$$



- Poles : quasi-normal modes (QNMs) contribution
- Branch cuts (BCs) : power-law decays contribution
  - At $\omega = 0$ for $0 \geqslant a \geqslant M$, **Leaver '86**
  - At $\omega = m$ for $a = M$, **M.Casals and P.Zimmerman '18**

Superradiance

- An initially ingoing scattered wave at infinity will experience a fractional variation of energy

$$Z_{slm\omega} \equiv \frac{dE_{out}}{dt}\left(\frac{dE_{in}}{dt}\right)^{-1} - 1 = \begin{cases} -\dfrac{8\omega M k r_+}{|W_{lm\omega}|^2}, & \text{for} \quad s = 0 \\ -\dfrac{2\omega^3}{M k r_+ |W_{lm\omega}|^2}, & \text{for} \quad s = 1 \\ -\dfrac{4\omega^5}{k(2Mr_+)^3(k^2+4\pi^2 T_H^2)}\dfrac{1}{|W_{lm\omega}|^2}, & \text{for} \quad s = 2 \end{cases} \tag{12}$$

with $T_H = (M^2 - a^2)^{1/2}/(4\pi M r_+)$

- Energy extraction when $Z_{lm\omega} > 0 \longleftrightarrow (\omega - m\Omega_H)\omega < 0$
- superradiant bound frequency : $\omega_{SR} \equiv m\Omega_H \to m(a \to M)$

MST Method : Extremal case

- Developed in **M. Casals and P. Zimmerman '18**
- Expressions for ${}^{\text{x}}R^{\text{in}}_{lm\omega}$ and ${}^{\text{x}}R^{\text{up}}_{lm\omega}$ in terms of confluent hypergeometric functions with coefficients a_n^ν

$$
{}^{\text{x}}W_{lm\omega} = \frac{\sin(\pi(\nu+i\omega))}{\sin(2\pi\nu)} \frac{\left(\bar{K}_\nu(-ik)^{-2\nu-1} + \bar{\mathcal{C}}\bar{K}_{-\nu-1}\right)}{(-ik)^{s+i\omega-\nu-1} e^{-i\pi\chi_{-s}/2} e^{-i\pi(\nu+\frac{1}{2})}}
\frac{2^{1+s-i\omega}\omega^{\nu+s}(i\omega)^{1-\nu-i\omega} e^{-3\pi\omega/2} e^{-\pi i}\sum_{n=-\infty}^{\infty}(-1)^n a_n^\nu}{\sum_{n=-\infty}^{\infty}\frac{\Gamma(q_n^\nu+\chi_s)}{\Gamma(q_n^\nu-\chi_s)}a_n^\nu}, \quad \text{Re}(\omega) > 0, \tag{13}
$$

where

$$
\bar{\mathcal{C}} \equiv ie^{-2\pi i\nu}(i\omega)^{2\nu}\omega^{-2\nu}, \qquad \bar{K}_\nu \equiv (2\omega)^{-\nu-1}e^{i\pi s}\frac{\sum_{n=p}^{\infty} C_{n,n-p}}{\sum_{n=-\infty}^{p} D_{n,p-n}}, \tag{14}
$$

MST Method : Extremal case

- Developed in **M. Casals and P. Zimmerman ‘18**
- Expressions for ${}^{\mathrm{x}}R^{\mathrm{in}}_{lm\omega}$ and ${}^{\mathrm{x}}R^{\mathrm{up}}_{lm\omega}$ in terms of confluent hypergeometric functions with coefficients a_n^ν

$$D_{n,j} \equiv \frac{\Gamma(q_n^\nu + \chi_s)\Gamma(1-2q_n^\nu)(q_n^\nu+\chi_s)_j}{\Gamma(q_n^\nu - \chi_s)\Gamma(1-q_n^\nu+\chi_s)(2q_n^\nu)_j\, j!} a_n^\nu (-2i\omega)^{n+j}, \tag{15}$$

$$C_{n,j} \equiv \frac{\Gamma(q_n^\nu + \chi_s)\Gamma(2q_n^\nu - 1)(1-q_n^\nu+\chi_{-s})_j}{\Gamma(q_n^\nu - \chi_s)\Gamma(q_n^\nu+\chi_{-s})(2-2q_n^\nu)_j\, j!} a_n^\nu (-ik)^{j-n}, \tag{16}$$

and

$$\chi_s \equiv s - i\omega, \quad q_n^\nu \equiv n + \nu + 1 \tag{17}$$

$$\alpha_n a_{n+1}^\nu + \beta_n a_n^\nu + \gamma_n a_{n-1}^\nu = 0, \quad n \in \mathbb{Z}, \tag{18}$$

MST Method : Sub-extremal case

- Developed in **Mano, Suzuki and Takasugi '96**
- Expressions for $R^{\text{in}}_{lm\omega}$ and $R^{\text{up}}_{lm\omega}$ in terms of hypergeometric functions
- Similar expressions
- Define the Wronskian factor

$$W^f_{lm\omega} \equiv \frac{2^{i\omega}\kappa^{2s}e^{i\kappa\epsilon_+(1+2\log\kappa/(1+\kappa))}e^{i\omega(\ln\omega-\frac{1-\kappa}{2})}}{e^{-\frac{\pi}{2}\omega}e^{i\omega\kappa}e^{\frac{\pi}{2}i(\nu+2-s)}\Gamma(1-s-2i\epsilon_+)}W_{lm\omega}, \tag{19}$$

where

$$\epsilon_+ \equiv \frac{1}{2}\left(1+\frac{1}{\kappa}\right)\tilde{\omega}, \quad \kappa \equiv \sqrt{1-(2a)^2}. \tag{20}$$

Physics behind ν

Near extremal Kerr (NEK): damped modes (DMs) and zero-damped modes (ZDMs)

- In the eikonal limit
  - DMs only exist if $\mu \equiv |m|/(l+1/2) \lessapprox 0.74$ (as in **H. Yang et al '13**)
  - Same behaviour at EK (as in **M. Richartz '16**)
- For general l and m,
  - Define $\nu_c \equiv \nu(\omega = m)$, $\delta_{SR}^2 = -(\nu_c + 1/2)^2$ and $\mathcal{F}_s^2 \equiv \delta_{SR}^2 + 1/4$, then :

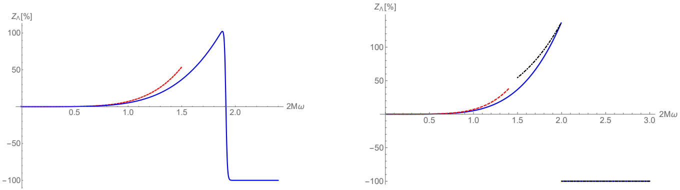
δ_{SR}^2	$\mathcal{F}_s^2$	$\exists$ DMs in NEK	Z_Λ for $a = M$ near $\omega = m$
< 0	< 0	Yes	continuous and monotonous
> 0	> 0	No	discontinuous and oscillatory

Numerical Results

Superradiance

Superradiance

- $Z_{222\omega}$, $\delta^2_{SR} > 0$,$\mathcal{F}^2_s > 0$



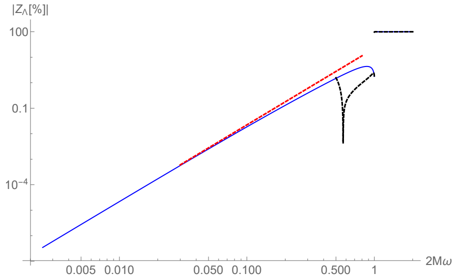
(a) s=2,l=2,m=2,a=0.99M, $2M\omega_{SR} \approx 1.91$

(b) s=2,l=2,m=2,a=M, $2M\omega_{SR} = 2$

Figure: Full numerical value of $Z_{222\omega}$, its small ω and $\omega \to m(a = M)$ asymptotic are represented as blue, red and black lines respectively.

Superradiance

- $Z_{111\omega}$, $\delta^2_{SR} > 0$,$\mathcal{F}^2_s > 0$

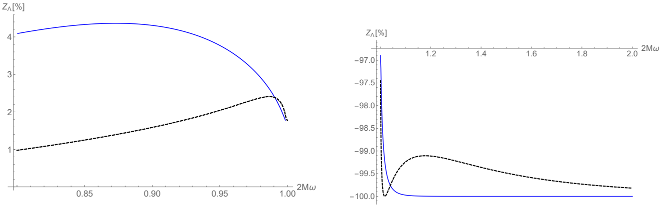


(a) s=1,l=1,m=1,a=M, $2M\omega_{SR} = 1$

Figure: Full numerical value of $Z_{111\omega}$, its small ω and $\omega \to m(a = M)$ asymptotic are represented as blue, red and black lines respectively.

Superradiance

- $Z_{111\omega}$, $\delta^2_{SR} > 0$,$\mathcal{F}^2_s > 0$



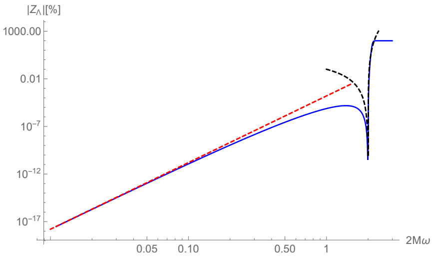
(a) Zoom-in for $\omega \to m^-$

(b) Zoom-in for $\omega \to m^+$

Figure: Full numerical value of $Z_{111\omega}$ and its $\omega \to m(a = M)$ asymptotic are represented as blue and black lines respectively.

Superradiance

- $Z_{032\omega}$, $\delta^2_{SR} < 0$,$\mathcal{F}^2_s < 0$

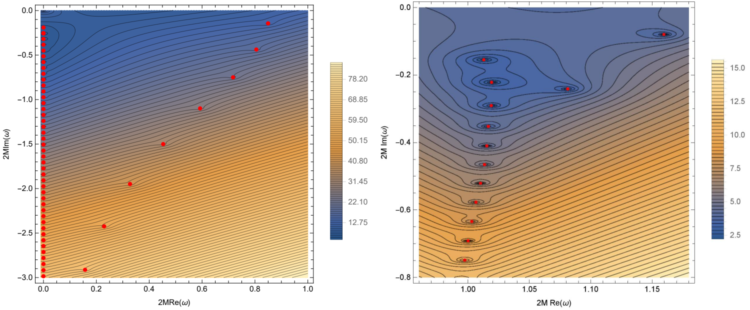


(a) s=0,l=3,m=2,a=M, $2M\omega_{SR} = 2$

Figure: Full numerical value of $Z_{032\omega}$, its small ω and $\omega \to m(a = M)$ asymptotic are represented as blue, red and black lines respectively.

NEK Quasi-Normal Modes

Near extremal Kerr QNMs

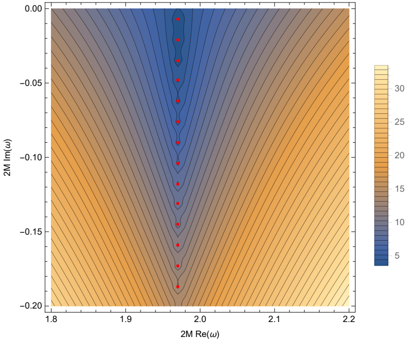


(a) $s = -2, l = 2, m = 0, a = 0.998M,$ $2M\omega_{SR} = 0, \mu = 0 < 0.74$

(b) $s = -2, l = 2, m = 1, a = 0.998M,$ $2M\omega_{SR} = 0.938, \mu = 0.4 < 0.74$

Figure: Contour plots of $\log_{10} |W^{f}_{lm\omega}|$ in the near-extremal Kerr (NEK) case. Both damped modes (DM’s) and zero damped modes (ZDM’s) are found. QNM are marked in red.

Near extremal Kerr QNMs



(a) $s = -2, l = 2, m = 2, a = 0.9999M$, $2M\omega_{SR} \approx 1.972, \mu = 0.8 > 0.74$

Figure: Contour plots of $\log_{10}|W^f_{lm\omega}|$ in the near-extremal Kerr (NEK) case. Only zero damped modes (ZDM's) are found. QNM are marked in red.

Accumulation of QNM's

Formation of the Superradiant BC

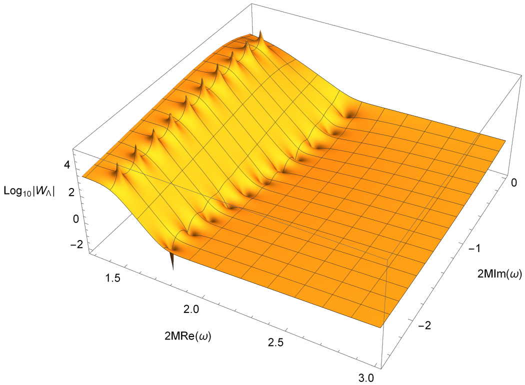


Figure: 3D plot for the absolute value of the Wronskian for the mode $s = 0, l = 3, m = 2, a = 0.95M$ and $2M\omega_{SR} \approx$ 1.447.

Formation of the Superradiant BC

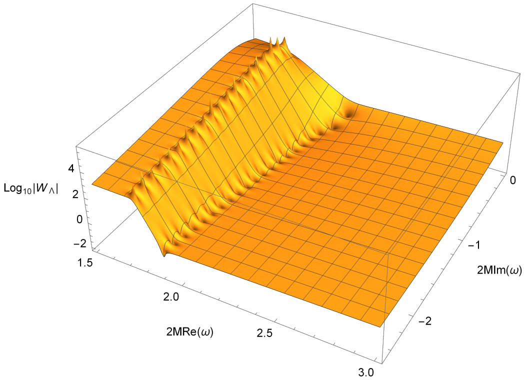


Figure: 3D plot for the absolute value of the Wronskian for the mode $s = 0, l = 3, m = 2$, $a = 0.99M$ and $2M\omega_{SR} \approx$ 1.735.

Formation of the Superradiant BC

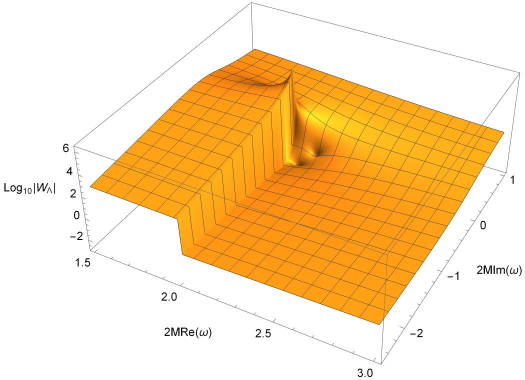


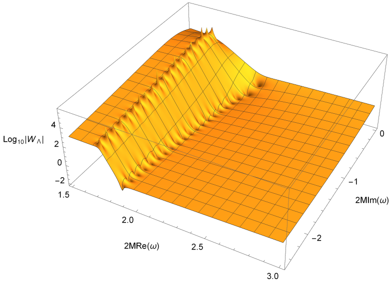
Figure: 3D plot for the absolute value of the Wronskian for the mode $s = 0, l = 3, m = 2$, $a = M$ and $2M\omega_{SR} = 2$.

Formation of the Superradiant BC

- Accumulation of zeros of $W_{lm\omega}$, QNM's
- Accumulation of poles of $W_{lm\omega}$, what are those?
- Poles come from $\Gamma(1 - s - 2i\epsilon_+)$
  - $1 - s - 2i\epsilon_+ = -n$
- Equivalent to the condition

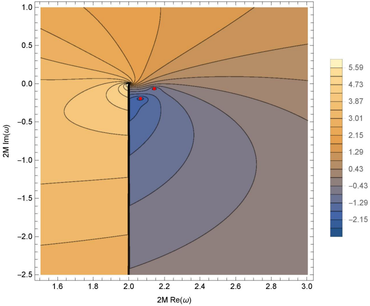
$$\omega = m\Omega_H - 2\pi i T_H(n - s + 1),$$

found in **U. Keshet and A. Neitzske '08** as the condition for totally reflected modes (TRMs).

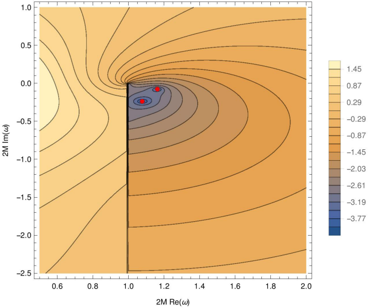


EK Quasi-Normal Modes

Extreme Kerr QNMs



(a) $s = 0, l = 3, m = 2, \mathcal{F}_s^2 < 0$



(b) $s = -2, l = 2, m = 1, \mathcal{F}_s^2 < 0$

Figure: Contour plots of $\log_{10} |^{\mathrm{x}}W_{lm\omega}(2M)^{2s}|$ in the complex-frequency plane. Both damped modes (DM's) and the branch cut (BC) can be seen.

Extreme Kerr QNMs

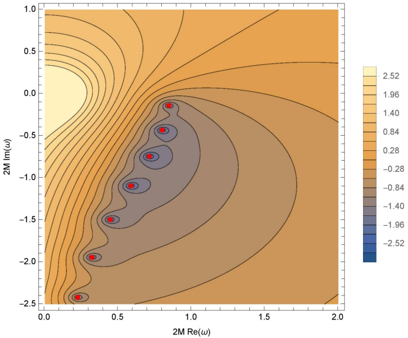


Figure: Contour plots of $\log_{10}|{}^{\mathrm{X}}W_{lm\omega}(2M)^{2s}|$ in the complex-frequency plane, for the mode $s = -2, l = 2, m = 0$ for which $\mathcal{F}_s^2 < 0$. DM's and BC can be seen.

Extreme Kerr QNMs

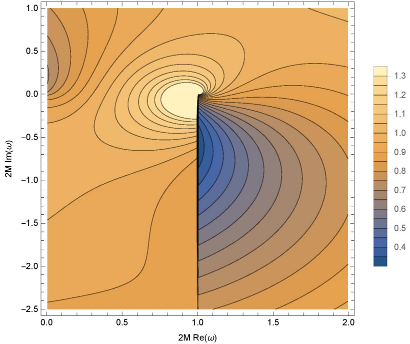
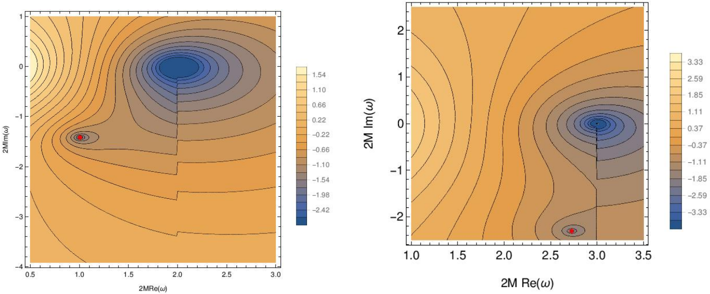


Figure: Contour plots of $\log_{10}|{}^{\mathsf{X}}W_{lm\omega}(2M)^{2s}|$ in the complex-frequency plane, for the mode $s=1, l=1, m=1$ for which $\mathcal{F}_s^2>0$. Only the BC can be seen in this case.

Extreme Kerr QNMs



(a) $s = -2, l = 2, m = 2, \mathcal{F}_s^2 > 0$ (b) $s = -2, l = 3, m = 3, \mathcal{F}_s^2 > 0$

Figure: Contour plots of $\log_{10} |^{\mathsf{X}}W_{lm\omega}(2M)^{2s}|$ in the extremal Kerr (EK) case. The superradiant BC and also a NSDM (non-standard DM) can bee seen.

Conclusions

Conclusions

- Calculated Z_Λ in the extremal case agreeing with **Starobinskii'73**
- Formation of the extremal BC by an accumulation of QNM's (**Glampedakis and Andersson '00**) and TRM's
- ZDM and DM behavior as described in **H. Yang et al '13** and **Richartz'16**
- NSDMs were found for s=-2, l=m=2 (as anticipated in **G. B. Cook and M. Zalutskiy'14**) and l=m=3
- No evidence of BC nor QNM on the upper complex-frequency plane - mode stability
- Open questions:
  - Exact condition for the existence of DM's (and NSDM)
  - Waveforms for modes with no QNMs

Thanks for listening