Eccentric orbit EMRIs: Informing the post-Newtonian expansion through black hole perturbation theory and multipole moment analysis

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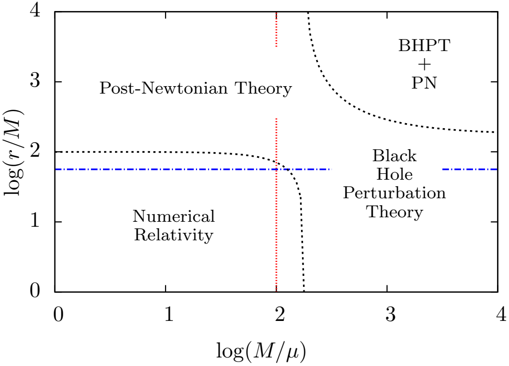
- Computing finite-order PN series with perturbation theory – analytic approach (dissipative sector)

- Obtaining logarithmic PN contributions with multipole moment analysis (fluxes at infinity)

- Next steps:
  - The rest of the dissipative sector
  - The conservative sector
  - Eventually, Teukolsky formalism on Kerr

Goal: Develop the overlap region between BHPT and PN theory

- Mass ratio μ/M and orbital speed v/c both small
- Use analytic techniques from both formalisms, extend various expansions in PN order and e



Goal: Develop the overlap region between BHPT and PN theory

- BHPT side:
  - Take RWZ equations in PN limit. Use small PN parameters: $1/c, 1/p, y$, etc.
  - Analytic expansion procedure: Follow Kavanagh, Ottewill, Wardell (2015) & Hopper, Kavanagh, Ottewill (2016)

  - Homogeneous problem: MST method, Detweiler-Chandrasekar transformation
  - Source problem: Schwarzschild geodesics, Extended Homogeneous Solutions
  - Obtain desired quantities to finite PN order, finite order in e

  - Also, reuse results from old numeric fitting methods

- PN side:
  - Work with the multipole moment formalism $I_{ij}, J_{ij}, I_{ijk}, \cdots$
  - Combine derivations with some BHPT results to determine logarithmic PN terms to all orders in e_t
  - Will include some expansions at higher order in the mass ratio

Goal: Develop the overlap region between BHPT and PN theory

- End result: High order series in $y = (M\Omega_\varphi)^{2/3}$ or $(1/p)$ and e for various quantities of interest

- Many terms computed to *all* orders in e or e_t

- Sample of references/related work:

  Arun, Blanchet, Iyer, Qusailah (2008a,b); Arun, Blanchet, Iyer, Sinha (2009); Damour, Iyer, Nagar (2009); Goldberger and Ross (2010); Blanchet, Detweiler, Le Tiec, and Whiting (2010,2011); Fujita (2012); Bini and Damour (2013, 2014, etc); Shah, Friedman, and Whiting (2014); Shah (2014); Fujita (2014); Johnson-McDaniel (2014); Johnson-McDaniel, Shah, and Whiting (2015); Sago and Fujita (2015); Forseth, Evans, Hopper (2016); Bini, Damour, Geralico (2016, etc.); Loutrel and Yunes (2017)

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The RWZ formalism for eccentric orbits on Schwarzschild

- Fourier-space form:

$$\left[f\frac{\partial}{\partial r}\left(f\frac{\partial}{\partial r}\right)+\omega^2+V_l(r)\right]X_{lmn}(r)=\frac{1}{T_r}\int_0^{T_r}S_{lm}e^{i\omega t}dt, \qquad f=(1-2M/r)$$

$S_{lm}=\delta[r-r_p(t)]G_{lm}+\delta'[r-r_p(t)]F_{lm}$. V_l and S_{lm} parity dependent

- Homogeneous solutions: $X^{\rm up}_{lmn}=X^{+}_{lmn}$ and $X^{\rm in}_{lmn}=X^{-}_{lmn}$
  - Use MST method

- Inhomogeneous problem: Apply Extended Homogeneous Solutions:

$$X_{lmn}=C^{+}_{lmn}X^{+}_{lmn}(r)\theta[r-r_p(t)]+C^{-}_{lmn}X^{-}_{lmn}(r)\theta[r_p(t)-r]$$

$$C^{\pm}_{lmn}=\frac{1}{W_{lmn}T_r}\int_0^{2\pi}\left[\frac{1}{f_p}G_{lm}(\chi)\hat{X}^{\mp}_{lmn}+\left(\frac{2M}{r_p^2f_p^2}\hat{X}^{\mp}_{lmn}-\frac{1}{f}\frac{d\hat{X}^{\mp}_{lmn}}{dr}\right)F_{lm}(\chi)\right]\left(\frac{dt}{d\chi}\right)e^{i\omega t(\chi)}d\chi$$

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The normalized MST homogeneous solutions

- Odd-parity homogeneous solutions found through MST methods
- Work in the dissipative sector – immediately divide off the amplitude at $r = \infty$ or $r = 2M$.
- Example: Normalized odd-parity $\hat{X}^{+}_{lmn}$ given by

$$\hat{X}_{\text{up}} = \frac{e^{iz}(-2iz)^{\nu+1}}{A_{\text{up}}^{\text{sum}}}(-2i\epsilon)^{-i\epsilon}\left(1-\frac{\epsilon}{z}\right)^{-i\epsilon}\sum_{j=-\infty}^{\infty} a_j(-2iz)^j \frac{\Gamma(b-2)\Gamma(b)}{\Gamma(b^*+2)\Gamma(b^*)}U(b,d,-2iz)$$

$$A_{\text{up}}^{\text{sum}} = \sum_{j=-\infty}^{\infty} a_j \frac{\Gamma(j+\nu-1-i\epsilon)\Gamma(j+\nu+1-i\epsilon)}{\Gamma(j+\nu+3+i\epsilon)\Gamma(j+\nu+1+i\epsilon)}$$

  - $b = j+\nu+1-i\epsilon, \quad d = 2j+2\nu+2$. U the irregular confluent hypergeometric function
  - Expansion parameters: $z = (r\omega)\eta, \quad \epsilon = (2M\omega)\eta^3, \quad \eta = 1/c$

- $\hat{X}^{-}_{lmn}$ follows similarly

Expanding the homogeneous solutions

- $\hat{X}^{\pm}_{lmn}$ most easily expanded in 4 steps:
  - Expand the continued fraction equation $\alpha_0 R_1 + \beta_0 + \gamma_0 L_{-1} = 0$ for ν
  - Use ν to develop the series coefficients a_j.
  - Transform the relevant hypergeometric function into a form more suitable for the PN regime (See Kavanagh, Ottewill, Wardell (2015)). Include needed j in summation. Expand
  - Expand the prefactor and put everything together

- Result $(l = 2)$:

$$\hat{X}_{\rm up} = -\frac{3}{z^2\eta^2} - \left(\frac{1}{2} + \frac{5\epsilon}{2z^3}\right) + \left(\frac{3i\epsilon\gamma_E}{z^2} - \frac{5i\epsilon}{z^2} + \frac{3\epsilon\pi}{2z^2} + \frac{3i\epsilon\log(2\epsilon\eta^3)}{z^2}\right)\eta - \left(\frac{15\epsilon^2}{7z^4} + \frac{7\epsilon}{4z} + \frac{z^2}{8}\right)\eta^2 + \left(\frac{i\epsilon\gamma_E}{2} - \frac{5i\epsilon}{6} + \frac{\epsilon\pi}{4} - \frac{25i\epsilon^2}{6z^3} + \frac{5i\epsilon^2\gamma_E}{2z^3} + \frac{5\epsilon^2\pi}{4z^3} - \frac{iz^3}{15} + \frac{i\epsilon}{2}\log(2\epsilon\eta^3) + \frac{5i\epsilon^2\log(2\epsilon\eta^3)}{2z^3}\right)\eta^3 + \mathcal{O}(\eta^4)$$

$$\hat{X}_{\rm in} = \frac{z^3}{\epsilon^3\eta^6} - \frac{z^5}{14\epsilon^3\eta^4} + \frac{13iz^3}{12\epsilon^2\eta^3} + \left(-\frac{13z^4}{42\epsilon^2} + \frac{z^7}{504\epsilon^3}\right)\frac{1}{\eta^2} - \frac{13iz^5}{168\epsilon^2\eta} + \left[-\frac{95z^3}{48\epsilon} - \frac{\pi^2 z^3}{6\epsilon} + \frac{z^6}{54\epsilon^2} - \frac{z^9}{33264\epsilon^3} + \frac{107z^3}{210\epsilon}\log\left(\frac{\epsilon}{z}\eta^2\right)\right] + \left(-\frac{169iz^4}{504\epsilon} + \frac{13iz^7}{6048\epsilon^2}\right)\eta + \mathcal{O}(\eta^2)$$

Even-parity homogeneous solutions

- Normalized even-parity solutions found using Detweiler-Chandrasekar transformation:

$$\hat{X}_{\pm}^{\text{even}} = \left(\frac{4}{\lambda_l \pm 6i\epsilon}\right)\left[\frac{3\epsilon}{2}\left(1-\frac{\epsilon}{z}\right)\frac{d\hat{X}_{\pm}^{\text{odd}}}{dz} + \left(\frac{1}{4}\lambda_l + \frac{9\epsilon^2\left(1-\frac{\epsilon}{z}\right)}{2(l-1)(l+2)z^2+6z\epsilon}\right)\hat{X}_{\pm}^{\text{odd}}\right],$$

where $\lambda_l = (l-1)l(l+1)(l+2)$

Expanding the source motion

- To move to the inhomogeneous solutions, must handle source motion

  $\rightarrow$ Geodesics on Schwarzschild

- Parameterize in terms of semi-latus rectum p, eccentricity e, relativistic anomaly χ (Note: $1/p = \mathcal{O}(\eta^2)$)
- $r_p(\chi)$ and $\varphi_p(\chi)$ have simple forms. $t(\chi)$ and T_r more complicated – must integrate ODE

$$\frac{dt_p}{d\chi} = \frac{p^2 M}{(p-2-2e\cos(\chi))(1+e\cos(\chi))^2}\left(\frac{(p-2)^2-4e^2}{p-6-2e\cos(\chi)}\right)^{1/2}$$

  - Can rapidly integrate by expanding in $1/p$ and e

- Fundamental frequencies given by

$$\Omega_r = \frac{2\pi}{T_r} = \left[1-\frac{3e^2}{2}+\mathcal{O}\left(e^4\right)\right]\left(\frac{1}{p}\right)^{3/2} + \left[-3+\frac{15e^2}{2}+\mathcal{O}\left(e^4\right)\right]\left(\frac{1}{p}\right)^{5/2} + \mathcal{O}\left(\frac{1}{p^{7/2}}\right)$$
$$\Omega_\varphi = \frac{\varphi(2\pi)}{T_r} = \left[1-\frac{3e^2}{2}+\mathcal{O}\left(e^4\right)\right]\left(\frac{1}{p}\right)^{3/2} + \left(3e^2+\mathcal{O}\left(e^4\right)\right)\left(\frac{1}{p}\right)^{5/2} + \mathcal{O}\left(\frac{1}{p^{7/2}}\right)$$

Expanding the inhomogeneous solution coefficients $C^{\pm}_{lmn}$

- Parameterization of the source motion allows for evaluation of the source integral

- First need 3 additional expansions:
  - $X^{\pm}_{lmn}$ at the location of the particle
  - The Wronskian W_{lmn}
  - The source terms G_{lm} and F_{lm}

- For the first two, make substitutions in z/ϵ expansions (now, $c = 1$):

$$z = r\omega = \frac{pM\omega}{1 + e\cos(\chi)} = \frac{M\omega'}{\sqrt{p}(1 + e\cos(\chi))} \qquad \epsilon = 2M\omega = \frac{2M\omega'}{p^{3/2}} \tag{1}$$

  - Introduced PN-adjusted frequency $\omega' = \omega p^{3/2} = \mathcal{O}(1)$ to avoid evaluating $\omega = m\Omega_\varphi + n\Omega_r$
  - In this way, W_{lmn} (χ, e, parity)-independent

- G_{lm} and F_{lm}: Decompose the stress-energy tensor over tensor spherical harmonics
  - See Martel and Poisson (2005), Hopper and Evans (2010)

The full $C^{\pm}_{lmn}$

- When everything plugged in and expanded in $1/p$ and e, source integral becomes straightforward (though large) sum of complex exponentials

- Bottleneck in the code, but still manageable. ~ 30 minutes to get to 7PN, e^{10} for $l = 2$.
  - Higher powers of e more expensive than higher powers of $1/p$

- Additional simplifications still available:
  - Expression for $C^{\pm}_{lmn}$ involves the integral of $X^{\mp}_{lmn}$ divided by W_{lmn} – any z-independent factors in $X^{\mp}$ will cancel. Thus, can omit from the start
  - Can also apply factorizations. See: Johnson-McDaniel (2014)

- Larger l start at higher PN order – only need finite l for dissipative quantities of interest

The full $C^{\pm}_{lmn}$

- Example low-order expansions (odd parity):

$$C^{+}_{2m1} = \left[\left(\frac{8(\omega')^2}{15} - \frac{16(\omega')^3}{45}\right) e + \mathcal{O}\left(e^2\right)\right]\left(\frac{1}{p}\right)^{3/2} + \left[\left(-\frac{2(\omega')^2}{5} - \frac{2m(\omega')^2}{15} - \frac{16(\omega')^3}{45}\right.\right.$$
$$\left.- \frac{2(\omega')^4}{21} + \frac{4(\omega')^5}{105}\right) e + \mathcal{O}\left(e^2\right)\Bigg]\left(\frac{1}{p}\right)^{5/2} + \left[\left(-\frac{16i(\omega')^3}{9} + \frac{16}{15} i\gamma(\omega')^3 + \frac{32i(\omega')^4}{27}\right.\right.$$
$$\left.- \frac{32}{45} i\gamma(\omega')^4 + \frac{8(\omega')^3\pi}{15} - \frac{16(\omega')^4\pi}{45} + \frac{16}{15} i(\omega')^3 \log\left(\frac{4(\omega')}{p^{3/2}}\right) - \frac{32}{45} i(\omega')^4 \log\left(\frac{4(\omega')}{p^{3/2}}\right)\right) e$$
$$\left. + \mathcal{O}\left(e^2\right)\right]\frac{1}{p^3} + \mathcal{O}\left(\frac{1}{p^{7/2}}\right)$$

$$C^{-}_{2m1} = \left[\left(-\frac{32}{105} - \frac{64(\omega')}{315}\right) e + \mathcal{O}\left(e^2\right)\right]\left(\frac{1}{p}\right)^{9/2} + \left[\left(-\frac{64}{35} - \frac{16m}{105} - \frac{352(\omega')}{315}\right) e + \mathcal{O}\left(e^2\right)\right]\left(\frac{1}{p}\right)^{11/2}$$
$$+ \left[\left(\frac{2192i(\omega')}{1575} + \frac{4384i(\omega')^2}{4725}\right) e + \mathcal{O}\left(e^2\right)\right]\frac{1}{p^6} + \mathcal{O}\left(\frac{1}{p^{13/2}}\right)$$

The radiated energy and angular momentum

- Now move to observables. First, focus on the fluxes at infinity
  - Future: Horizon fluxes, waveforms
- Energy flux:

$$\left\langle \frac{dE}{dt} \right\rangle^{\infty} = \frac{1}{64\pi} \sum_{lmn} (l+2)(l+1)(l)(l-1)\,\omega^2\,|C^{+}_{lmn}|^2$$

$$= \frac{32}{5}\left(\frac{\mu}{M}\right)^2 y^5 \left[\mathcal{L}_0 + y\mathcal{L}_1 + y^{3/2}\mathcal{L}_{3/2} + y^2\mathcal{L}_2 + y^{5/2}\mathcal{L}_{5/2} + \cdots\right]$$

- Angular momentum flux:

$$\left\langle \frac{dL}{dt} \right\rangle^{\infty} = \frac{1}{64\pi} \sum_{lmn} (l+2)(l+1)(l)(l-1)\,m\omega\,|C^{+}_{lmn}|^2$$

$$= \frac{32}{5}\left(\frac{\mu}{M}\right)^2 M y^{7/2} \left[\mathcal{J}_0 + y\mathcal{J}_1 + y^{3/2}\mathcal{J}_{3/2} + y^2\mathcal{J}_2 + y^{5/2}\mathcal{J}_{5/2} + \cdots\right]$$

- Converted PN variable from $1/p$ to $y = (M\Omega_\varphi)^{2/3}$ to be consistent with prior work
- $\mathcal{L}_i = \mathcal{L}_i(e)$ and $\mathcal{J}_i = \mathcal{J}_i(e)$ expansions in e

The fluxes at infinity: Finding closed forms

- Crucially, many eccentricity functions $\mathcal{L}(e)$ and $\mathcal{J}(e)$ found to yield closed or compact forms. Examples:

$$\mathcal{L}_0(e) = \frac{1}{(1-e^2)^{7/2}}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right) \qquad \text{Peters-Mathews (1963)}$$

$$\mathcal{L}_1(e) = -\frac{1}{(1-e^2)^{9/2}}\left(\frac{1247}{336} + \frac{15901}{672}e^2 + \frac{9253}{384}e^4 + \frac{4037}{1792}e^6\right)$$

$$\mathcal{J}_0(e) = \frac{1}{(1-e^2)^2}\left(1 + \frac{7}{8}e^2\right)$$

- In these cases, can combine a finite-in-e expansion with suspected form to generate exact result, valid to all orders in e
- Singular factor known from asymptotic analysis (Forseth, Evans, Hopper (2016), Loutrel and Yunes (2017))

Side note: Expanding the fluxes through numeric fitting

- Another BHPT-PN hybrid method can be used to expand the fluxes: numeric fitting
  - BHPT fluxes calculated numerically for 50+ choices of p, 30+ choices of e
  - Perform double numeric fit for PN expansion
  - Numeric quantities converted to analytic form when possible via integer relation algorithm

- Procedure used for the energy flux at infinity by Forseth, Evans, Hopper (2016). Extended and used for the angular momentum flux by Munna, Evans, Hopper, Forseth (2019) (in preparation)

- Method more expensive and less versatile than analytic expansions
- However, most current results found in this way

- In what follows, narrow focus to energy regime – we have angular momentum analog for all results as well

The fluxes: Past work and current results

$$\langle \dot{E} \rangle = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 y^5 \Big[\mathcal{L}_0 + y\mathcal{L}_1 + y^{3/2}\mathcal{L}_{3/2} + y^2\mathcal{L}_2 + y^{5/2}\mathcal{L}_{5/2} + y^3\Big(\mathcal{L}_3 + \log(y)\mathcal{L}_{3L}\Big)$$
$$+ y^{7/2}\mathcal{L}_{7/2} + y^4\Big(\mathcal{L}_4 + \log(y)\mathcal{L}_{4L}\Big) + y^{9/2}\Big(\mathcal{L}_{9/2} + \log(y)\mathcal{L}_{9/2L}\Big)$$
$$+ y^5\Big(\mathcal{L}_5 + \log(y)\mathcal{L}_{5L}\Big) + y^{11/2}\Big(\mathcal{L}_{11/2} + \log(y)\mathcal{L}_{11/2L}\Big)$$
$$+ y^6\Big(\mathcal{L}_6 + \log(y)\mathcal{L}_{6L} + \log^2(y)\mathcal{L}_{6L^2}\Big) + y^{13/2}\Big(\mathcal{L}_{13/2} + \log(y)\mathcal{L}_{13/2L}\Big)$$
$$+ y^7\Big(\mathcal{L}_7 + \log(y)\mathcal{L}_{7L} + \log^2(y)\mathcal{L}_{7L^2}\Big) + y^{15/2}\Big(\mathcal{L}_{15/2} + \log(y)\mathcal{L}_{15/2L}$$
$$+ \log^2(y)\mathcal{L}_{15/2L^2}\Big) + y^8\Big(\mathcal{L}_8 + \log(y)\mathcal{L}_{8L} + \log^2(y)\mathcal{L}_{8L^2}\Big)$$
$$+ y^{17/2}\Big(\mathcal{L}_{17/2} + \log(y)\mathcal{L}_{17/2L} + \log^2(y)\mathcal{L}_{17/2L^2}\Big) + y^9\log(y)^3\mathcal{L}_{9L^3} + \cdots\Big]$$

Prior PN advancements (Peters-Mathews, etc.)

The fluxes: Past work and current results

$$
\begin{aligned}
\langle \dot{E} \rangle = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 y^5 \Big[& \mathcal{L}_0 + y\mathcal{L}_1 + y^{3/2}\mathcal{L}_{3/2} + y^2\mathcal{L}_2 + y^{5/2}\mathcal{L}_{5/2} + y^3\left(\mathcal{L}_3 + \log(y)\mathcal{L}_{3L}\right) \\
& + y^{7/2}\mathcal{L}_{7/2} + y^4\left(\mathcal{L}_4 + \log(y)\mathcal{L}_{4L}\right) + y^{9/2}\left(\mathcal{L}_{9/2} + \log(y)\mathcal{L}_{9/2L}\right) \\
& + y^5\left(\mathcal{L}_5 + \log(y)\mathcal{L}_{5L}\right) + y^{11/2}\left(\mathcal{L}_{11/2} + \log(y)\mathcal{L}_{11/2L}\right) \\
& + y^6\left(\mathcal{L}_6 + \log(y)\mathcal{L}_{6L} + \log^2(y)\mathcal{L}_{6L^2}\right) + y^{13/2}\left(\mathcal{L}_{13/2} + \log(y)\mathcal{L}_{13/2L}\right) \\
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& + \log^2(y)\mathcal{L}_{15/2L^2}\Big) + y^8\left(\mathcal{L}_8 + \log(y)\mathcal{L}_{8L} + \log^2(y)\mathcal{L}_{8L^2}\right) \\
& + y^{17/2}\left(\mathcal{L}_{17/2} + \log(y)\mathcal{L}_{17/2L} + \log^2(y)\mathcal{L}_{17/2L^2}\right) + y^9\log(y)^3\mathcal{L}_{9L^3} + \cdots \Big]
\end{aligned}
$$

Forseth, Evans, Hopper (2016) (lowest order in the mass ratio)

The fluxes: Past work and current results

$$\langle \dot{E} \rangle = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 y^5 \Big[\mathcal{L}_0 + y\mathcal{L}_1 + y^{3/2}\mathcal{L}_{3/2} + y^2\mathcal{L}_2 + y^{5/2}\mathcal{L}_{5/2} + y^3\left(\mathcal{L}_3 + \log(y)\mathcal{L}_{3L}\right)$$
$$+ y^{7/2}\mathcal{L}_{7/2} + y^4\left(\mathcal{L}_4 + \log(y)\mathcal{L}_{4L}\right) + y^{9/2}\left(\mathcal{L}_{9/2} + \log(y)\mathcal{L}_{9/2L}\right)$$
$$+ y^5\left(\mathcal{L}_5 + \log(y)\mathcal{L}_{5L}\right) + y^{11/2}\left(\mathcal{L}_{11/2} + \log(y)\mathcal{L}_{11/2L}\right)$$
$$+ y^6\left(\mathcal{L}_6 + \log(y)\mathcal{L}_{6L} + \log^2(y)\mathcal{L}_{6L^2}\right) + y^{13/2}\left(\mathcal{L}_{13/2} + \log(y)\mathcal{L}_{13/2L}\right)$$
$$+ y^7\left(\mathcal{L}_7 + \log(y)\mathcal{L}_{7L} + \log^2(y)\mathcal{L}_{7L^2}\right) + y^{15/2}\Big(\mathcal{L}_{15/2} + \log(y)\mathcal{L}_{15/2L}$$
$$+ \log^2(y)\mathcal{L}_{15/2L^2}\Big) + y^8\left(\mathcal{L}_8 + \log(y)\mathcal{L}_{8L} + \log^2(y)\mathcal{L}_{8L^2}\right)$$
$$+ y^{17/2}\left(\mathcal{L}_{17/2} + \log(y)\mathcal{L}_{17/2L} + \log^2(y)\mathcal{L}_{17/2L^2}\right) + y^9\log(y)^3\mathcal{L}_{9L^3} + \cdots\Big]$$

Extended fitting and analytic MST expansions (lowest order in the mass ratio)

Summary of finite-order BHPT results

$$
\begin{aligned}
\langle \dot{E} \rangle = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 y^5 \Big[& \mathcal{L}_0 + y\mathcal{L}_1 + y^{3/2}\mathcal{L}_{3/2} + y^2\mathcal{L}_2 + y^{5/2}\mathcal{L}_{5/2} + y^3\left(\mathcal{L}_3 + \log(y)\mathcal{L}_{3L}\right) \\
& + y^{7/2}\mathcal{L}_{7/2} + y^4\left(\mathcal{L}_4 + \underline{\log(y)\mathcal{L}_{4L}}\right) + y^{9/2}\left(\mathcal{L}_{9/2} + \log(y)\mathcal{L}_{9/2L}\right) \\
& + y^5\left(\mathcal{L}_5 + \underline{\log(y)\mathcal{L}_{5L}}\right) + y^{11/2}\left(\mathcal{L}_{11/2} + \log(y)\mathcal{L}_{11/2L}\right) \\
& + y^6\left(\mathcal{L}_6 + \log(y)\mathcal{L}_{6L} + \underline{\log^2(y)\mathcal{L}_{6L^2}}\right) + y^{13/2}\left(\mathcal{L}_{13/2} + \log(y)\mathcal{L}_{13/2L}\right) \\
& + y^7\left(\mathcal{L}_7 + \log(y)\mathcal{L}_{7L} + \underline{\log^2(y)\mathcal{L}_{7L^2}}\right) + y^{15/2}\Big(\mathcal{L}_{15/2} + \log(y)\mathcal{L}_{15/2L} \\
& + \log^2(y)\mathcal{L}_{15/2L^2}\Big) + y^8\left(\mathcal{L}_8 + \log(y)\mathcal{L}_{8L} + \underline{\log^2(y)\mathcal{L}_{8L^2}}\right) \\
& + y^{17/2}\left(\mathcal{L}_{17/2} + \log(y)\mathcal{L}_{17/2L} + \log^2(y)\mathcal{L}_{17/2L^2}\right) + \underline{y^9\log(y)^3\mathcal{L}_{9L^3}} + \cdots \Big]
\end{aligned}
$$

<u>Green</u>: Closed expression

Blue: $e^{26} - e^{30}$

Summary of finite-order BHPT results

$$
\begin{aligned}
\langle \dot{E} \rangle = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 y^5 \Big[& \mathcal{L}_0 + y\mathcal{L}_1 + y^{3/2}\mathcal{L}_{3/2} + y^2\mathcal{L}_2 + y^{5/2}\mathcal{L}_{5/2} + y^3\left(\mathcal{L}_3 + \log(y)\mathcal{L}_{3L}\right) \\
& + y^{7/2}\mathcal{L}_{7/2} + y^4\left(\mathcal{L}_4 + \underline{\log(y)\mathcal{L}_{4L}}\right) + y^{9/2}\left(\mathcal{L}_{9/2} + \log(y)\mathcal{L}_{9/2L}\right) \\
& + y^5\left(\mathcal{L}_5 + \underline{\log(y)\mathcal{L}_{5L}}\right) + y^{11/2}\left(\mathcal{L}_{11/2} + \log(y)\mathcal{L}_{11/2L}\right) \\
& + y^6\left(\mathcal{L}_6 + \log(y)\mathcal{L}_{6L} + \underline{\log^2(y)\mathcal{L}_{6L^2}}\right) + y^{13/2}\left(\mathcal{L}_{13/2} + \log(y)\mathcal{L}_{13/2L}\right) \\
& + y^7\left(\mathcal{L}_7 + \log(y)\mathcal{L}_{7L} + \underline{\log^2(y)\mathcal{L}_{7L^2}}\right) + y^{15/2}\Big(\mathcal{L}_{15/2} + \log(y)\mathcal{L}_{15/2L} \\
& + \log^2(y)\mathcal{L}_{15/2L^2}\Big) + y^8\left(\mathcal{L}_8 + \log(y)\mathcal{L}_{8L} + \underline{\log^2(y)\mathcal{L}_{8L^2}}\right) \\
& + y^{17/2}\left(\mathcal{L}_{17/2} + \log(y)\mathcal{L}_{17/2L} + \log^2(y)\mathcal{L}_{17/2L^2}\right) + \underline{y^9\log(y)^3\mathcal{L}_{9L^3}} + \cdots \Big]
\end{aligned}
$$

Green: Closed expression

Blue: $e^{26} - e^{30}$

Red: $e^{12} - e^{20}$

Black: e^0 or e^2

Table of contents

The energy flux at infinity: "Pure" PN form

- PN expansion via PN derivations:

$$\left\langle \frac{dE}{dt} \right\rangle_{\infty}^{\mathrm{PN}} = \frac{32}{5}\nu^2 x^5 \Big[\mathcal{R}_0 + x\mathcal{R}_1 + x^{3/2}\mathcal{R}_{3/2} + x^2\mathcal{R}_2 + x^{5/2}\mathcal{R}_{5/2} + x^3\Big(\mathcal{R}_3 + \mathcal{R}_{3L}\log(x)\Big) + x^{7/2}\mathcal{R}_{7/2} + x^4\Big(\mathcal{R}_4 + \mathcal{R}_{4L}\log(x)\Big) + \cdots\Big]$$

- $x = ((m_1 + m_2)\Omega_\varphi)^{2/3}$
- $\mathcal{R}_i = \mathcal{R}_i(e_t, \nu)$ functions of e_t (time eccentricity) and $\nu = m_1 m_2/(m_1 + m_2)^2$

$$\mathcal{R}_1 = \frac{1}{(1-e_t^2)^{9/2}}\Big[-\frac{1247}{336} + \frac{10475 e_t^2}{672} + \frac{10043 e_t^4}{384} + \frac{2179 e_t^6}{1792} + \nu\Big(\frac{35}{12} + \frac{1081 e_t^2}{36} + \frac{311 e_t^4}{12} + \frac{851 e_t^6}{576}\Big)\Big]$$

- PN derivations much more laborious than BHPT expansions, but come with 2 benefits:
  1) Obtain all orders in ν
  2) Easier to reach arbitrary order in eccentricity (e_t)

Relating the PN expansion to the BHPT-PN expansion

- Key differences:
  - $\mathcal{L}_i$ remain lowest order in ν
  - Recall: $y = (m_1 \Omega_\varphi)^{2/3} \quad (m_1 \gg m_2) \qquad \Longrightarrow y = x + \mathcal{O}(\nu)$
  - $\mathcal{R}_i$ now functions of e_t, not Darwin e
- Can connect the two expansions at lowest order in ν by relating e and e_t
- Can calculate this relationship to arbitrary PN order (0th order in ν):

$$\frac{e^2}{e_t^2} = 1 + 6y + \frac{17 - 21e_t^2 + 15\sqrt{1 - e_t^2}}{1 - e_t^2} y^2 + \frac{26 - 107e_t^2 + 54e_t^4 + \left(150 - 90e_t^2\right)\sqrt{1 - e_t^2}}{\left(1 - e_t^2\right)^2} y^3 + \mathcal{O}(y^4)$$

- Thus, can convert back and forth between the two expansions as needed

Newtonian-order terms

- Quick side note: because $e/e_t = 1 + \mathcal{O}(y)$, flux terms from Newtonian orbital quantities will have $\mathcal{R}_i = \mathcal{L}_i$

$$\mathcal{L}_0 = \frac{1}{(1-e^2)^{7/2}}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right)$$

$$\mathcal{R}_0 = \frac{1}{(1-e_t^2)^{7/2}}\left(1 + \frac{73}{24}e_t^2 + \frac{37}{96}e_t^4\right)$$

$$\mathcal{L}_{3/2} = \frac{4\pi}{(1-e^2)^5}\left(1 + \frac{1375}{192}e^2 + \frac{3935}{768}e^4 + \frac{10007}{36864}e^6 + \frac{2321}{884736}e^8 + \cdots\right)$$

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PN derivations: The multipole moment approach

- Deriving the fluxes entails computation of PN-corrected multipole moments
  - Lowest-order multipole in the fluxes is the Newtonian mass quadrupole

- These moments then plugged into instantaneous flux expressions and tail integrals to yield the $\mathcal{R}_i(e_t, \nu)$

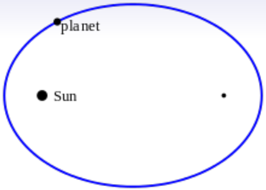
$$\mathcal{R}_0 = \frac{1}{32} \dddot{I}_{ij} \dddot{I}_{ij}, \qquad \mathcal{R}_{3/2} = \frac{1}{32} \dddot{I}_{ij}(t) \int_0^\infty I_{ij}^{(5)}(t-\tau) \ln\left(\frac{\tau}{2r_0}\right) d\tau$$

  - I_{ij} the simple (dimensionless) Newtonian mass quadrupole

Fourier decomposition in the Kepler problem

- Certain low order enhancement functions found to have particularly simple forms in Fourier space

$$I_{ij} = \sum_{n=-\infty}^{\infty} I_{ij}^{(n)} e^{in\Omega_r t} = \sum_{n=-\infty}^{\infty} I_{ij}^{(n)} e^{inl}$$

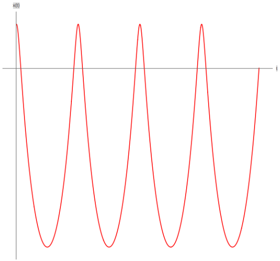


- Let $g(n, e_t) = 1/16\, n^6 |I_{ij}^{(n)}|^2$

- In the course of PN derivations, discovered that

$$\mathcal{R}_0 = \sum_{n=1}^{\infty} g(n, e_t), \qquad \mathcal{R}_{3/2} = 4\pi \sum_{n=1}^{\infty} \frac{n}{2}\, g(n, e_t),$$

$$\mathcal{R}_{3L} = -\frac{856}{105} \sum_{n=1}^{\infty} \frac{n^2}{4}\, g(n, e_t)$$



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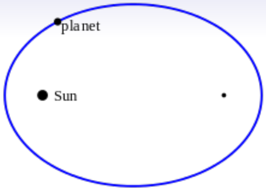

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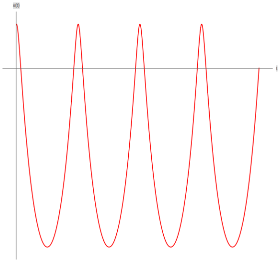

The leading logarithm series

- First two appearances (one integer, one half-integer) of each new power of logarithm

$$\left\langle \frac{dE}{dt} \right\rangle_\infty^{\mathrm{LL}} = \frac{32}{5}\nu^2 x^5 \Big[\mathcal{R}_0 + x^{3/2}\mathcal{R}_{3/2} + x^3 \log(x)\mathcal{R}_{3L} + x^{9/2}\log(x)\mathcal{R}_{9/2L} + x^6 \log^2(x)\mathcal{R}_{6L^2} + x^{15/2}\log^2(x)\mathcal{R}_{15/2L^2} + x^9 \log^3(x)\mathcal{R}_{9L^3} + \cdots \Big]$$

- Circular orbit limit of each of those terms known from Johnson-McDaniel (using BHPT)

- Discovery: $g(n, e_t)$ can be used to generate fully eccentric expressions

$$\mathcal{R}_{(3k)L(k)} = \frac{1}{k!}\left(-\frac{856}{105}\right)^k \sum_{n=1}^{\infty}\left(\frac{n}{2}\right)^{2k} g(n, e_t) = \mathcal{L}_{(3k)L(k)}(e \to e_t)$$

$$\mathcal{R}_{(3k+3/2)L(k)} = \frac{4\pi}{k!}\left(-\frac{856}{105}\right)^k \sum_{n=1}^{\infty}\left(\frac{n}{2}\right)^{2k+1} g(n, e_t) = \mathcal{L}_{(3k+3/2)L(k)}(e \to e_t)$$

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The 3PN logarithm series

- Leading log series naturally connected to its 3PN correction:

$$\left\langle \frac{dE}{dt} \right\rangle_{\infty}^{\text{3PN L}} = \frac{32}{5}\nu^2 x^5 \left[x^3 \mathcal{R}_3 + x^{9/2}\mathcal{R}_{9/2} + x^6 \log(x)\mathcal{R}_{6L} + x^{15/2}\log(x)\mathcal{R}_{15/2L} + \cdots \right]$$

- Example: full 3PN flux given by

$$\begin{aligned} \mathcal{R}_3 + \mathcal{R}_{3L}\log x = \frac{1}{(1-e_t^2)^{13/2}} \Bigg[& \frac{2193295679}{9979200} + \frac{20506331429}{19958400} e_t^2 + \cdots - \frac{8977637}{11354112} e_t^{10} \\ & + \sqrt{1-e_t^2}\left(-\frac{14047483}{151200} + \frac{36863231}{100800} e_t^2 + \cdots + \frac{185}{48} e_t^8 \right) + \mathcal{O}(\nu) \Bigg] \\ + \left(2\,\gamma_E - \frac{70}{107}\pi^2 + \frac{116761}{29960} \right) \mathcal{R}_{3L} & + \log\left(\left[\frac{8\left(1-e_t^2\right)}{1+\sqrt{1-e_t^2}} \right]^2 x \right) \mathcal{R}_{3L} - \frac{1712}{105}\chi(e_t) \end{aligned}$$

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$$\chi(e_t) = \sum_{n=1}^{\infty} \frac{n^2}{4} \log\left(\frac{n}{2}\right) g(n, e_t)$$

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- Strong suspicion: All 3PN logarithms (both $\mathcal{L}$ and $\mathcal{R}$ varieties) have forms similar to $\mathcal{R}_3$
  - Evidence in PN/BHPT eulerlog functions (see Sec. IV C of Munna and Evans (2019))
  - Caveat: half-integer terms will not be finite in the first section

- Consequences two-fold

1) Generalized set of functions like $\chi(e_t)$

$$\chi_k(e_t) = \sum_{n=1}^{\infty} \left(\frac{n}{2}\right)^k \log\left(\frac{n}{2}\right) g(n, e_t)$$

2) Can use BHPT to extract compact forms for integral 3PN logs at lowest order in ν
  - Express χ_k and corresponding LL in terms of e
  - Take finite-in-e BHPT expansion, determine last line of suspected form, solve overdetermined system of equations for remainder
  - Use e/e_t relation to convert back to $\mathcal{R}(e_t)$ form – result valid to all orders in e and e_t
  - Successfully found $\mathcal{L}_{6L}/\mathcal{R}_{6L}$. Reduced $\mathcal{L}_{9L2}$ and $\mathcal{R}_{9L2}$ to rational series

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The 1PN and 4PN logarithm series

- Now, can look for the 1PN corrections to the those results. First, the 1PN logarithm series

$$\left\langle \frac{dE}{dt} \right\rangle_{\infty}^{\mathrm{1L}} = \frac{32}{5}\nu^2 x^5 \left[\mathcal{R}_1 + x^{5/2}\mathcal{R}_{5/2} + x^4 \log(x)\mathcal{R}_{4L} + x^{11/2}\log(x)\mathcal{R}_{11/2L} + \cdots \right]$$

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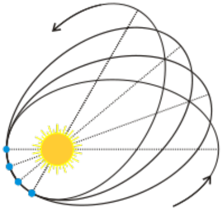

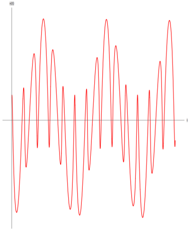

The 1PN and 4PN logarithm series

- Also must calculate the Newtonian mass octupole and current quadrupole (easy)

- The Fourier expansions become

$$I_{ijk} = \sum_{n=-\infty}^{\infty} I_{ijk}^{(n)} e^{inl}, \qquad J_{ij} = \sum_{n=-\infty}^{\infty} J_{ij}^{(n)} e^{inl}, \qquad I_{ij} = \sum_{n=-\infty}^{\infty} \sum_{p=-2}^{2} I_{ij}^{(n,p)} e^{i(n+pk)l}$$

- $k = \Omega_\varphi / \Omega_r - 1 = \mathcal{O}(x)$. $\qquad I_{ij}^{(n,\pm 2)}$ not even closed in form

The 1PN and 4PN logarithm series

- Result: Find that our single sequence of Fourier sums is replaced by 7

$$\mathcal{R}_i = \mathcal{R}_i^{\mathrm{MQ01}} + \mathcal{R}_i^{\mathrm{MQ02}} + \mathcal{R}_i^{\mathrm{MQ03}} + \nu(\mathcal{R}_i^{\mathrm{MQ11}} + \mathcal{R}_i^{\mathrm{MQ12}}) + \mathcal{R}_i^{\mathrm{MO}} + \mathcal{R}_i^{\mathrm{CQ}}$$

- Each component a complicated Fourier sum involving the aforementioned moments

- Derivation follows from that of LLs, involves identification of the various 1PN corrections
  - Of particular note: Includes corrections at next order in ν

- Expansion much slower in Mathematica (primarily due to $I_{ij}^{(n,\pm 2)}$), but can extract each term to at least e_t^{120}

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Finding $\mathcal{L}_4$ completely

- Finally, can extract $\mathcal{L}_4$ to all orders in e, just as was done with $\mathcal{L}_{6L}$. Need the following:
  - $\mathcal{L}_{4L}$
  - 1PN correction to $\chi(e_t)$, converted to Darwin e (calculated to e_t^{120}/e^{120})
  - Fitting results for $\mathcal{L}_4$ to at least e^{26}
  - Suspected form (like $\mathcal{L}_3/\mathcal{R}_3$)

- Can transform back to $\mathcal{R}_4$ with e/e_t relation to 4PN

Finding $\mathcal{L}_4$ completely

$$
\begin{aligned}
\mathcal{L}_4(e) = {} & \frac{1}{(1-e^2)^{15/2}}\Bigg[\frac{18510752431}{44144100} - \frac{40934075709731e^2}{6356750400} - \frac{131458534402891e^4}{2542700160} - \frac{3215698875850801e^6}{50854003200} \\
& - \frac{586522182193681e^8}{31294771200} - \frac{3028139270269e^{10}}{45203558400} + \frac{670101511e^{12}}{14057472} + \sqrt{1-e^2}\Bigg(-\frac{1654225499}{3175200} \\
& + \frac{2426725501e^2}{3175200} + \frac{186636561079e^4}{12700800} + \frac{72788261801e^6}{8467200} - \frac{16274063783e^8}{7526400} - \frac{4982855e^{10}}{18432}\Bigg)\Bigg] \\
& - \frac{1369\pi^2}{126(1-e^2)^{15/2}}\left(1 + \frac{104549e^2}{2738} + \frac{1113487e^4}{5476} + \frac{2644503e^6}{10952} + \frac{10829823e^8}{175232} + \frac{573939e^{10}}{350464}\right) \\
& + 2\left[\gamma_{\mathrm{E}} + \log\left(\frac{8(1-e^2)}{1+\sqrt{1-e^2}}\right)\right]\mathcal{L}_{4L}(e) + \mathcal{L}_4^{\chi}(e)
\end{aligned}
$$

- Timely result, as the PN community is currently researching the full orbital motion at 4PN

Table of contents

- Comparing post-Newtonian (PN) theory and black hole perturbation theory (BHPT): Eccentric orbits on Schwarzschild
- Computing finite-order PN series with perturbation theory – analytic approach (dissipative sector)
- Obtaining logarithmic PN contributions with multipole moment analysis (fluxes at infinity)
- Next steps:
  - The rest of the dissipative sector
  - The conservative sector
  - Eventually, Teukolsky formalism on Kerr

Other dissipative sector information

- Look to apply same simplifications/procedures at the horizon

- Currently have the lowest order (e.g. 0, 1, 2, 3, $4L$) horizon enhancement functions exactly
- 4PN may have form similar to $\mathcal{L}_3$

- Waveforms (at both infinity and the horizon) slightly more complicated, but should be amenable to similar techniques

Analytic RWZ and the conservative sector

- Eventually would like to move into the conservative sector
- Presents new set of difficulties
  - Example: PN order no longer increases with l. Require lengthy general-l expansions
- Common/significant example: generalized redshift invariant $\langle u^t \rangle$
  - Average ratio of $dt/d\tau$ over an orbit
  - 1st order correction given by $T_r/\mathcal{T}_r^2 \langle H^R \rangle, \quad H^R = (1/2) g^{(1),R}_{\mu\nu} u^\mu u^\nu$
- Found to y^4 and e^{10} in Hopper, Kavanagh, Ottewill (2016); y^4, e^{20} in Bini, Damour, Geralico (2016)
- Seek to extend those results. Goal: Combine finite results with expected forms and PN techniques to yield terms exactly (as with $\mathcal{L}_{6L}/\mathcal{L}_4$)

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Analytic results on Kerr

- Finally, move to the more difficult case of a central Kerr black hole
- No RWZ functions on Kerr, but similar Teukolsky formalism applies
- Develop expansions in both the dissipative and conservative sectors

Conclusions/Summary

Successful implementation of BHPT and PN theory for eccentric orbits

BHPT side: Analytic expansion of the RWZ equation

- Expand homogeneous solutions in $1/c$
- Convert to $1/p$ (or y) and e for the source problem
- Alternatively, use numeric to analytic fitting
- Regardless, great yield in the fluxes at infinity

PN side: Multipole moment derivations lead to logarithm series

- Combine BHPT results with multipole moment derivations, obtain leading logarithm and 1PN logarithm series all orders in eccentricity
- Use expected forms and finite-order BHPT computations to extract 3PN and 4PN logarithms to all orders in eccentricity at lowest order in ν

Next steps:

- Apply dissipative techniques to fluxes at horizon and waveforms
- Eventually: extend methods to conservative effects and Kerr spacetime

Acknowledgements

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