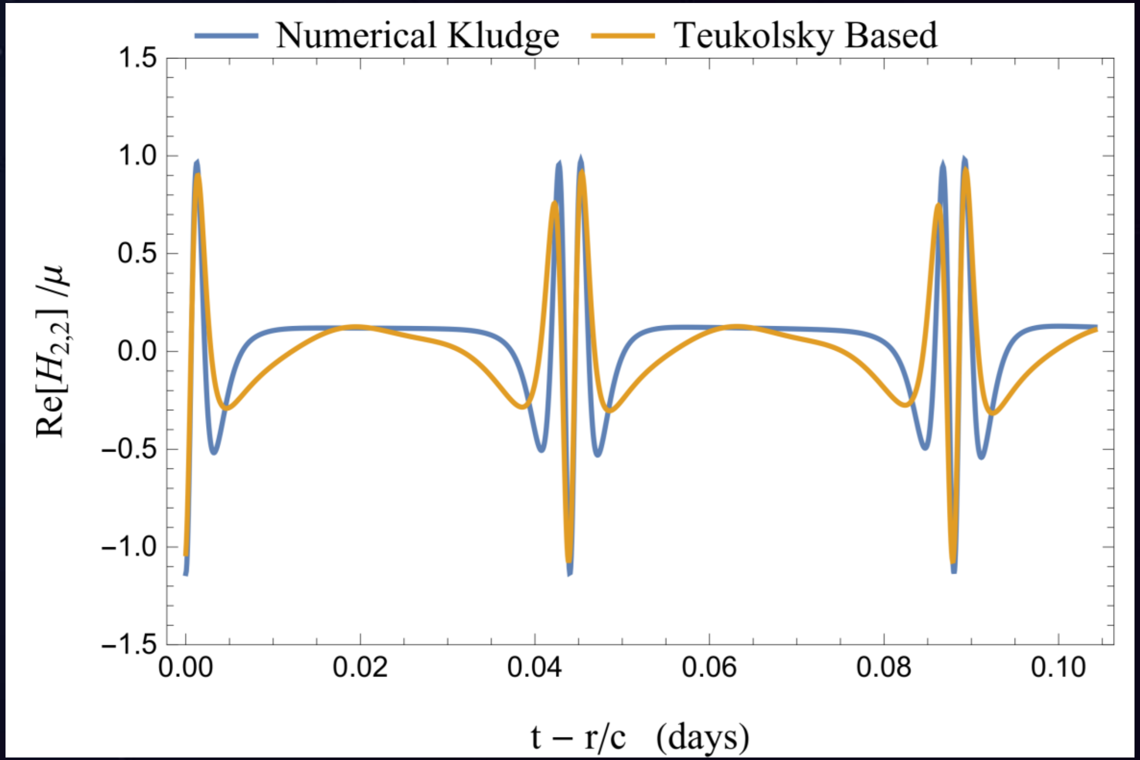
Using self-force results to test kludge vs. Teukolsky waveforms for eccentric EMRIs



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Overview

- *Background:*

  Extreme mass-ratio inspirals (EMRIs) and LISA

  What role will orbital eccentricity play?

- *Theory:*

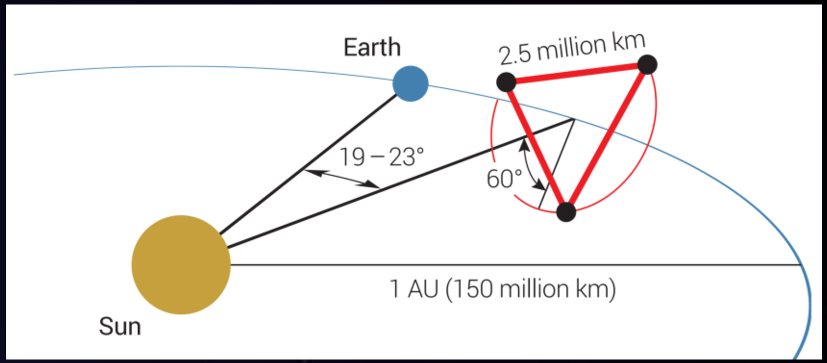
  Black hole perturbation theory and the self-force

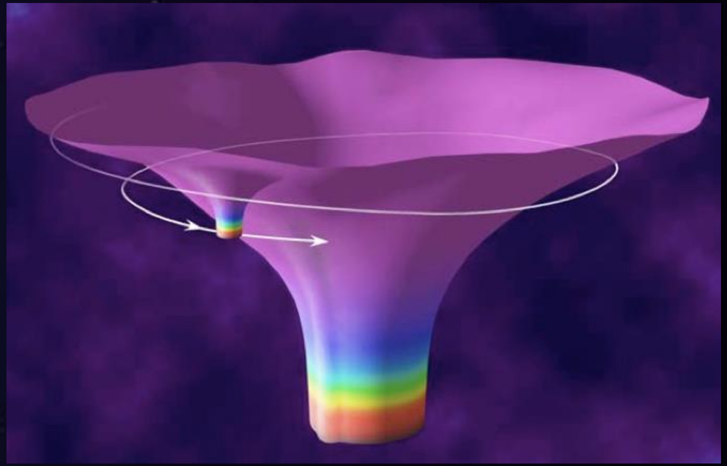
  Inspiral trajectories: kludge vs. self-force

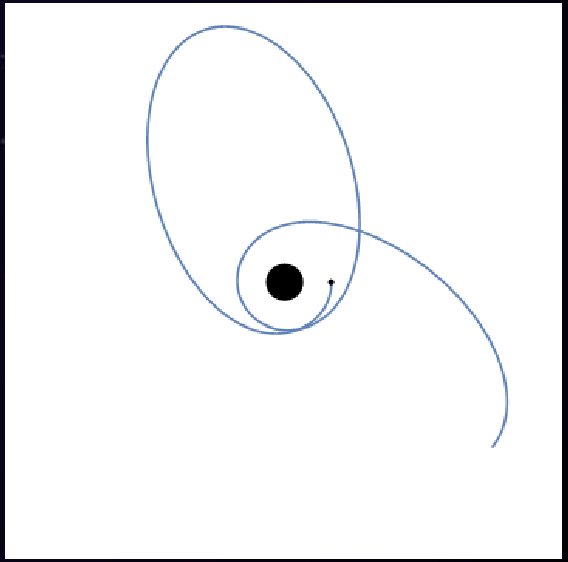
- *Application:*

  Rapidly computing self-forced inspirals

  Waveform generation: kludge vs. Teukolsky

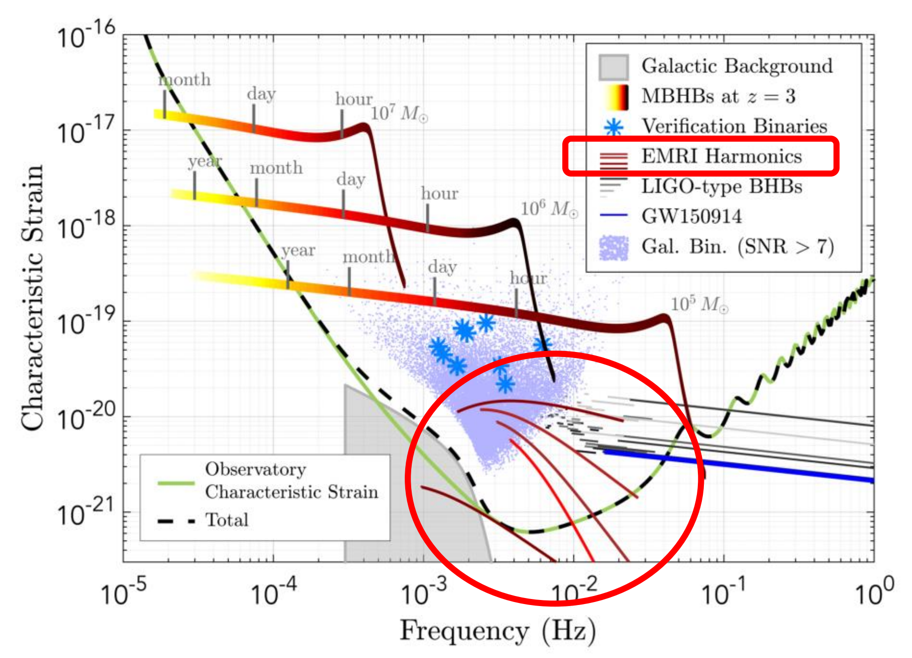


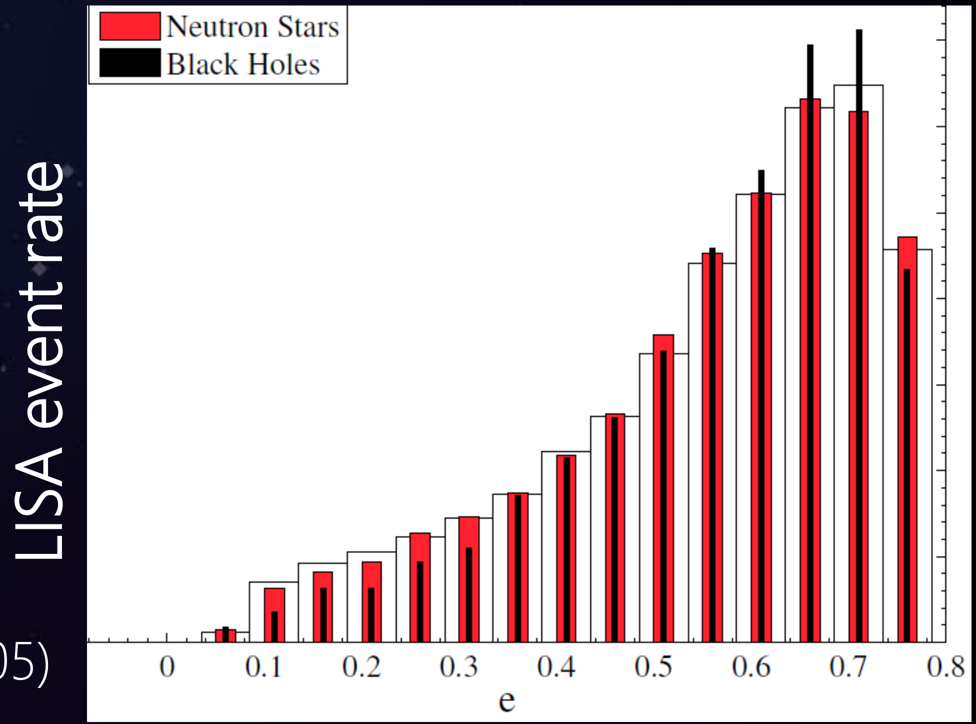




LISA and EMRIs

- EMRIs radiate gravitational waves at frequencies where LISA is most sensitive
- EMRIs aren't the loudest signals, but their long durations allow high SNRs with matched filtering
- These features will facilitate precision tests of general relativity through LISA observations
- Unlike LIGO-Virgo sources, EMRIs are expected to exhibit orbital eccentricity, so handling eccentricity is a priority during modeling
- The eccentricity can be as high as: *e ≈ 0.75* !

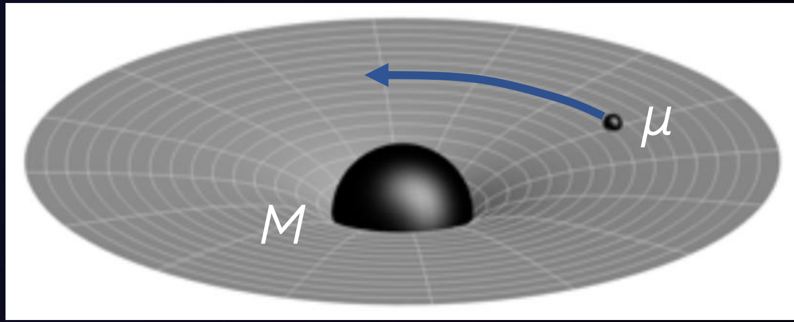




Hopman & Alexander (2005)

EMRI Modeling: Theory

- *Assumptions:* general relativity is valid throughout, environmental effects are negligible
- *Further simplifications for this prototype:* consider only non-spinning binary components
- Because the mass-ratio ($\eta = \mu/M$) is extremely small, can expand the field equations in powers of η



$$g_{\mu\nu} = \boxed{g^{(0)}_{\mu\nu}} + \eta \boxed{g^{(1)}_{\mu\nu}} + \mathcal{O}(\eta^2)$$

Schwarzschild metric

metric perturbation

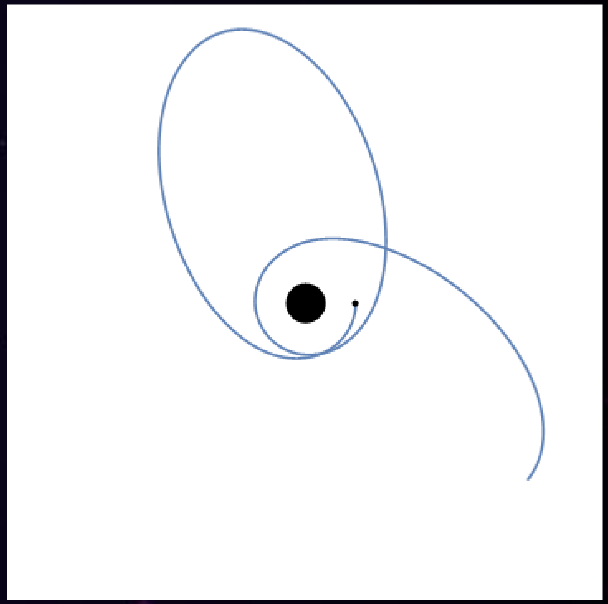
- Additional terms (2nd order) are also important, will incorporate in future work
- Einstein's equations govern the metric perturbation (Lorenz gauge):

$$\boxed{\Box \bar{g}^{(1)}_{\mu\nu} + R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}\, \bar{g}^{(1)}_{\alpha\beta} = -16\pi\, T_{\mu\nu}}$$

Calculating the metric perturbation

$$\Box \bar{g}^{(1)}_{\mu\nu} + R^{\alpha\ \beta}_{\ \mu\ \nu}\, \bar{g}^{(1)}_{\mu\nu} = -16\pi\, T_{\mu\nu}$$

- Separation of variables: (assume the time dependence of the orbital motion involves a discrete frequency spectrum)



$$g^{(1)}_{\mu\nu}(t,r,\theta,\varphi) = \sum_{l,m,\omega} \sum_{j=0}^{9} h^{(j)}_{lm\omega}(r)\, e^{-i\omega t}\, S^{lm(j)}_{\mu\nu}(\theta,\varphi)$$

- Solve a system of ODEs for each mode, use the method of "extended homogeneous solutions" for non-circular orbits

Examples:

Detweiler & Poisson (2004)

Barack & Sago (2007)

Barack, Ori & Sago (2008)

Akcay (2011)

Akcay, Warburton & Barack (2013)

Osburn, Forseth, Evans & Hopper (2014)

Radiation reaction: kludges vs. fluxes vs. self-force

- Kludges involve qualitative mechanisms for modeling inspirals and/or waveforms

- Because the waveform is computed independently, inspiral models are examined first

- To meet LISA requirements of <1 radian accuracy for the accumulated orbital phase: must avoid weak field, slow motion, and/or adiabatic approximations (*need the self-force*)

- To calculate the self-force, consider how the local metric perturbation interacts with the inspiraling small body

Inspiral models / **Approximations**	analytic kludge	numerical kludge	post-Newtonian	flux balance	self-force
weak field	x	x			✓
slow motion	x		x		
adiabatic	x	x		x	

Calculating the self-force

- The (retarded) metric perturbation diverges at the position of the small body

- Various regularization schemes exist to access the finite self-force

- The "mode-sum" regularization scheme is implemented in this work

- The self-force is pre-computed for a dense array of orbital configurations

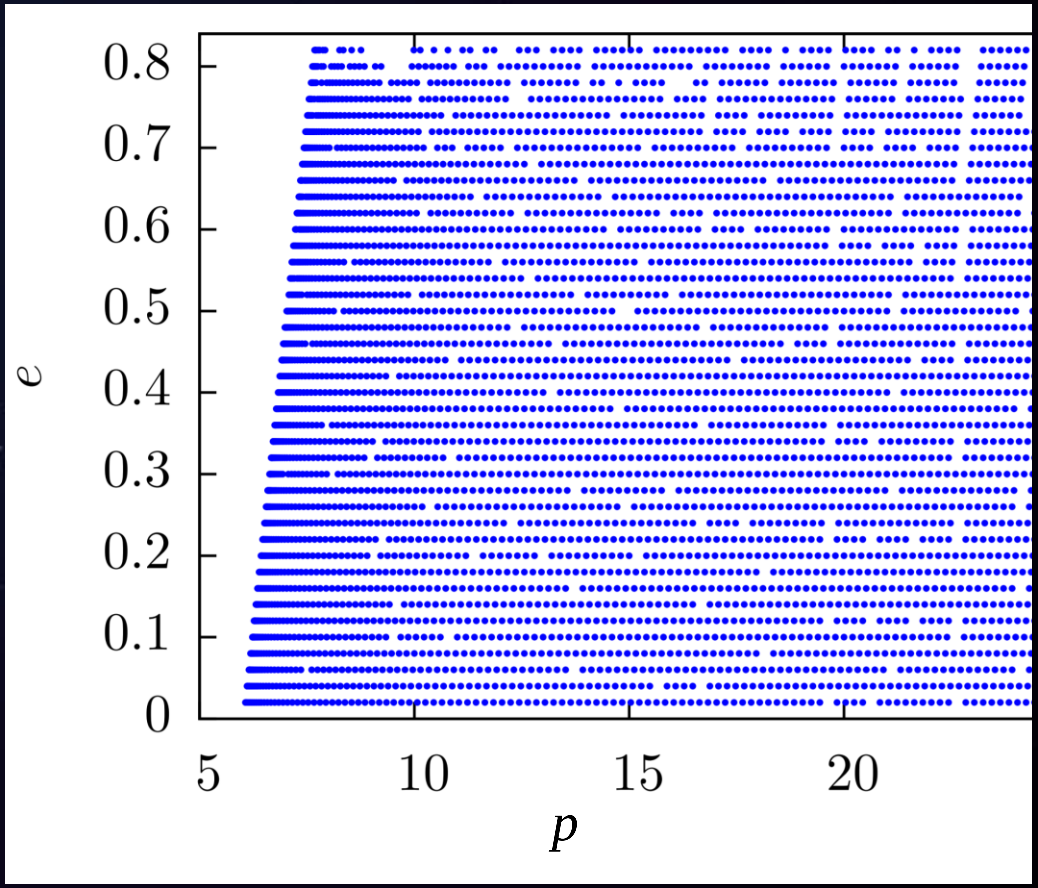
- Through interpolation, arbitrary orbital configurations are accessible

calculated from retarded metric perturbation

$$F^{\alpha} = \sum_{l=0}^{\infty} \underbrace{\left(\boxed{F^{\alpha}_{l\,(ret)}} - \boxed{F^{\alpha}_{l\,(S)}} \right)}_{\text{each spherical harmonic mode is finite}}$$

accessible from local expansion

self-force



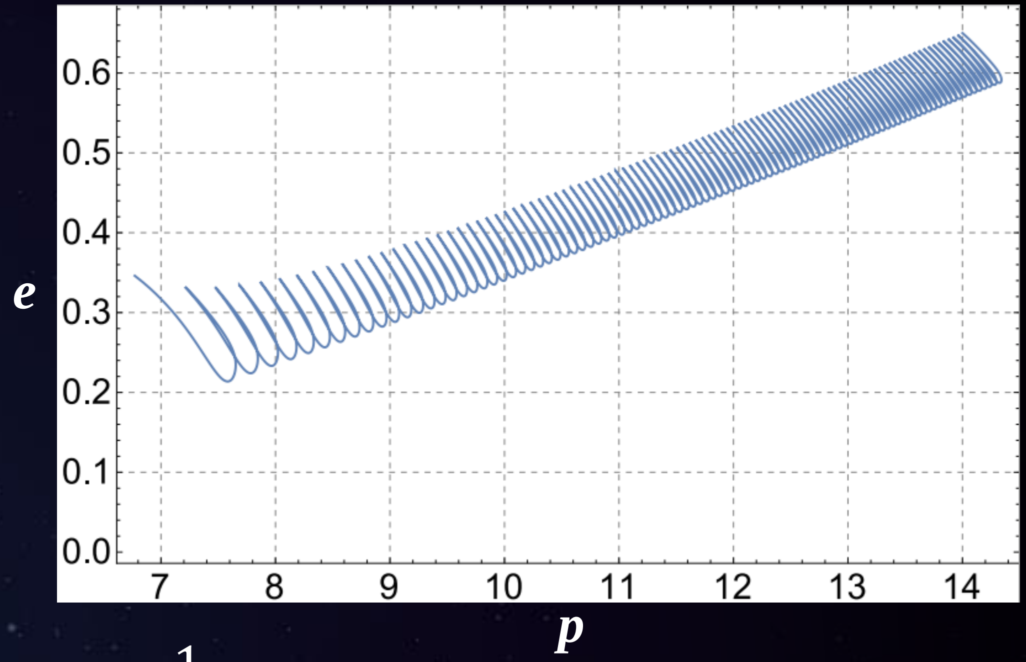
Self-forced inspirals

- The inspiral is parameterized by an evolving set of elements describing tangent geodesics

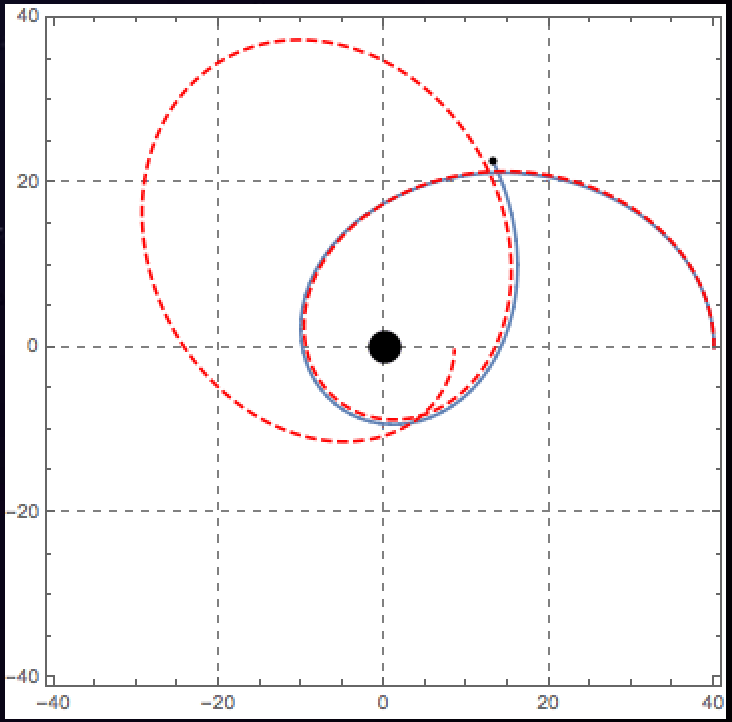
$$\frac{de}{dt} = B_{\alpha}^{(e)} F^{\alpha} \qquad \frac{dp}{dt} = B_{\alpha}^{(p)} F^{\alpha} \qquad \frac{d\chi_0}{dt} = B_{\alpha}^{(\chi_0)} F^{\alpha}$$

Pound & Poisson (2008)

Gair, Flanagan, Drasco, Hinderer & Babak (2011)



$$\eta = \frac{1}{64}$$



- It is straightforward to couple our self-force interpolation model to the ODEs governing the orbital elements

Warburton, Akcay, Barack, Gair & Sago (2012)

Osburn, Warburton & Evans (2016)

- The trajectory is reconstructed from the orbital elements

Rapid inspiral computation: near-identity transformation

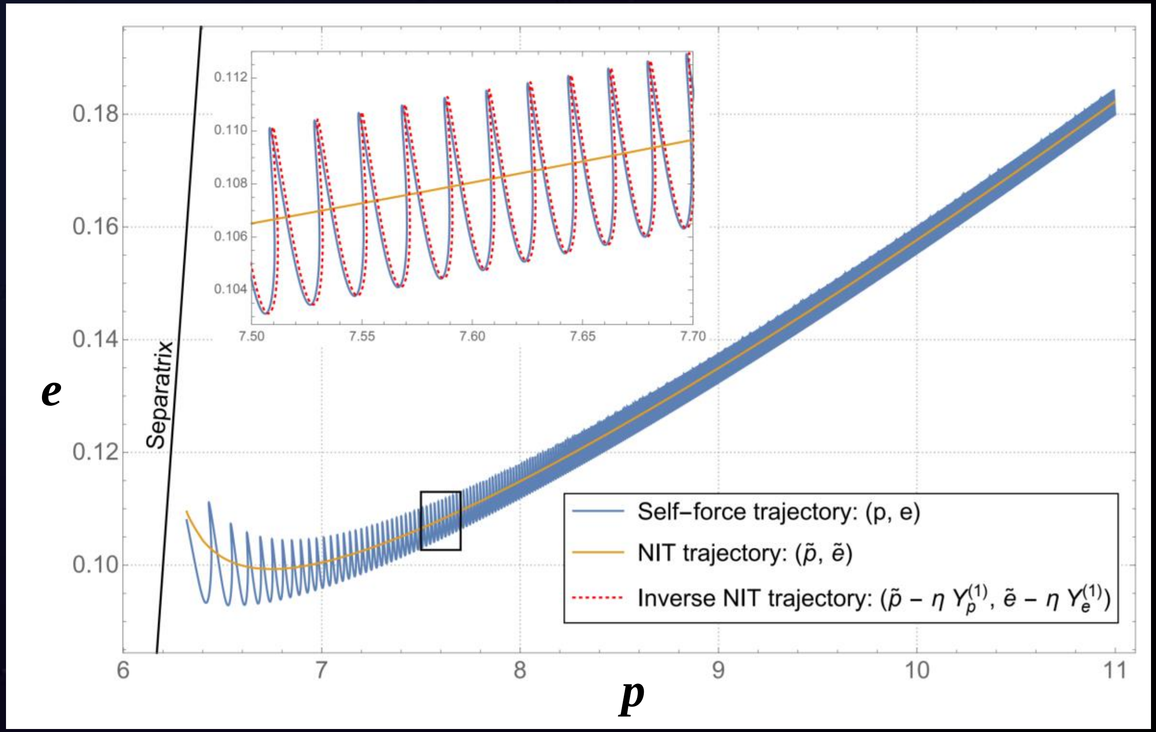
- Although the accuracy of self-forced inspirals is vital for LISA data analysis, additional enhancements are necessary to compete with the speed of kludge models

- The near-identity transformation (NIT) accomplishes this by eliminating fast oscillations in the equations of motion with modified orbital elements

$$\underbrace{\{e, p, t, \varphi_p\}}_{\text{The physical parameters oscillate rapidly}} \underset{\text{inverse NIT (approximate)}}{\overset{\text{NIT}}{\rightleftarrows}} \underbrace{\{\tilde{e}, \tilde{p}, \tilde{t}, \tilde{\varphi}_p\}}_{\text{The NIT parameters are smooth}}$$

~10 ms NIT inspiral computation time

Van De Meent & Warburton 2018



Interfacing with the Black Hole Perturbation Toolkit

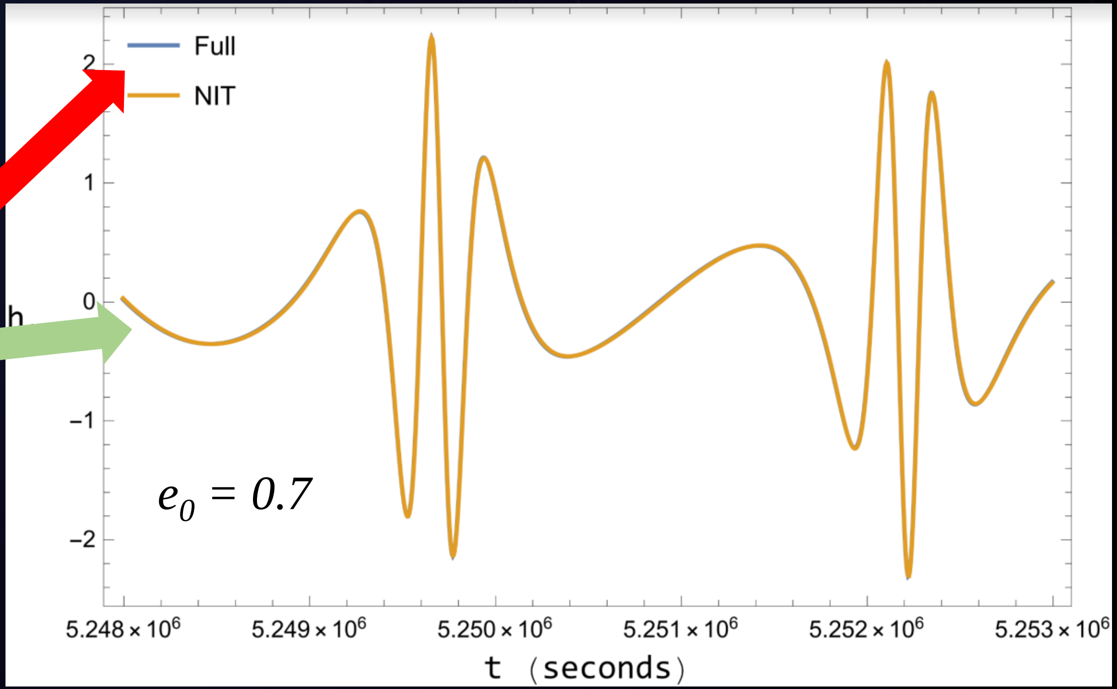
- The "Fast Self-Forced Inspirals" module of the Black Hole Perturbation Toolkit provides tools for computing inspirals with either the full self-force or NIT equations of motion

- This module's self-force model involved a compact range of orbital configurations ($e < 0.2$) to streamline distribution

- An interface between a broader library of self-force data and the Toolkit module was developed

different inspiral methods

same waveform output, but the NIT inspiral is much faster



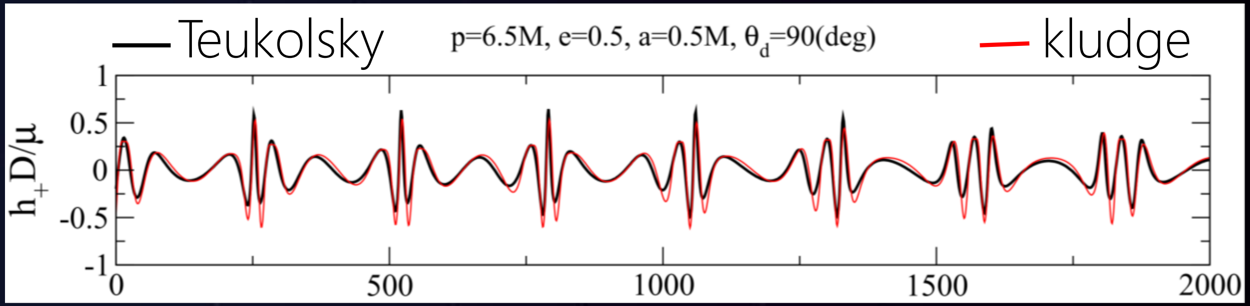
- Future versions will include default data with the full range of orbital parameters ($e < 0.75$)

https://bhptoolkit.org/Fast_Self-Forced_Inspirals/

Waveform calculation methods

- Trajectories are post-processed to generate their associated waveforms
- Different waveform methods are compared by inputting identical self-forced trajectories
- Eccentric waveform comparisons exist for fixed snapshots (no radiation reaction):

Babak, Fang, Gair, Glampedakis & Hughes (2008)



- Advancements of this work: compare waveforms during entire self-forced inspiral, higher *e*
- Possible waveform generation methods to compare:

compare these two to assess kludge accuracy

Accuracy	Method	Extra Approximations
↑	time-domain Teukolsky	none
	evolving Teukolsky snapshots	geodesic for past worldline
	kludge	weak field (maybe slow motion)

Warburton, Osburn & Evans (2017)

Kludge waveforms vs. Teukolsky waveforms

General waveform description:

$$h_+ - i\,h_\times = \frac{1}{r}\sum_{l,m} H_{lm}(t)\ {}_{-2}Y_{lm}(\theta,\varphi)$$

Kludge waveforms:

Artificially map inspiral to Minkowski spacetime and solve wave equation

$$H_{2,2}^{kludge}(t) =$$

$$4\mu\sqrt{\frac{\pi}{5}}\,\mathrm{e}^{-2i\varphi_p}\left(\dot{r}_p^2 - 4i\,r_p\,\dot{r}_p\,\dot{\varphi}_p + r_p\left(\ddot{r}_p - r_p\left(2\dot{\varphi}_p^2 + i\,\ddot{\varphi}_p\right)\right)\right)$$

The kludge waveform depends only on the instantaneous position (2nd derivative of quadrupole moment)

Evolving Teukolsky snapshots:

Solve Teukolsky equation for a dense set of geodeiscs and interpolate Fourier coefficients

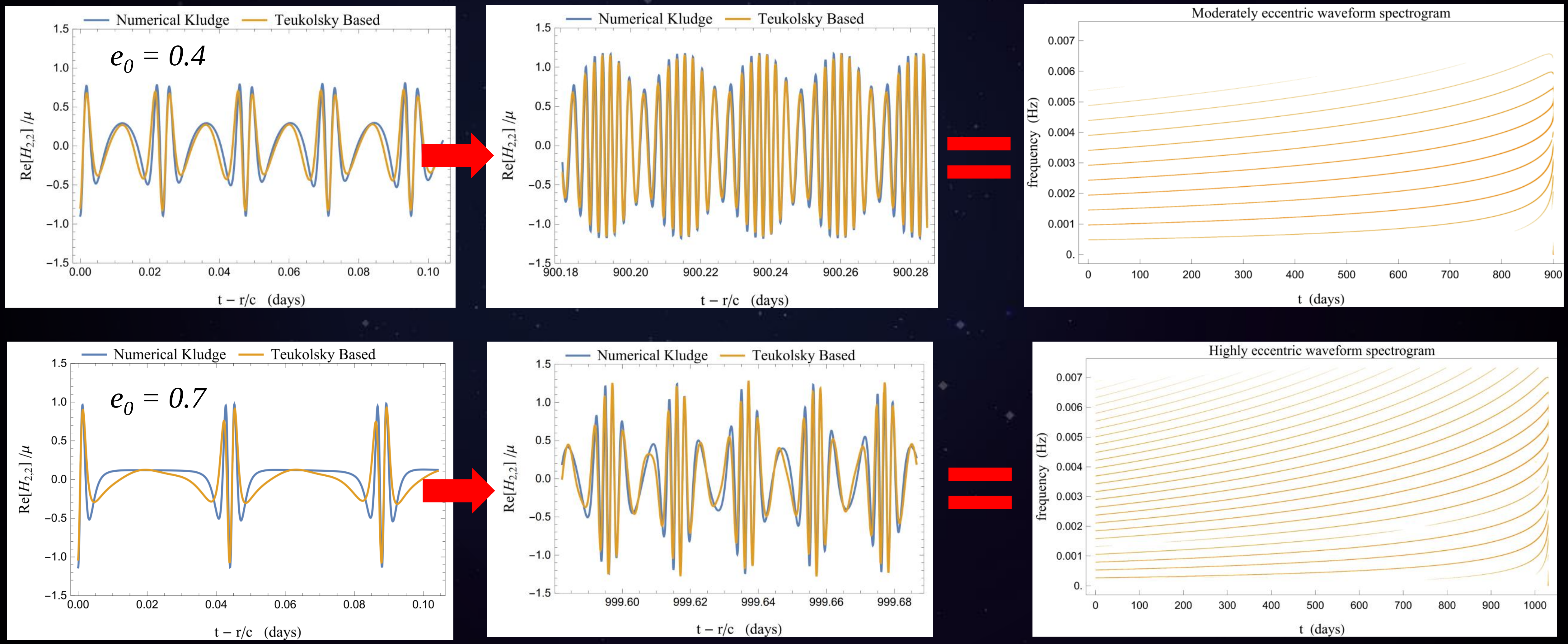
$$H_{2,2}^{Teuk}(t) = \sum_n C_{2,2,n}\, e^{-i(n\Omega_r + 2\Omega_\varphi)t}$$

The Teukolsky coefficients and fundamental frequencies evolve during the inspiral, use interpolant to update their values

Eccentric waveform comparisons

$$h_+ - i\,h_\times = \frac{1}{r}\sum_{l,m} H_{lm}(t)\;{}_{-2}Y_{lm}(\theta,\varphi)$$

$M = 10^6\,M_\odot$ $\eta = 10^{-5}$



Comparing waveforms: overlaps

$$h_+ - i\,h_\times = \frac{1}{r}\sum_{l,m} H_{lm}(t)\ {}_{-2}Y_{lm}(\theta,\varphi)$$

- LISA data analysis is performed by representing the waveform in the frequency domain:

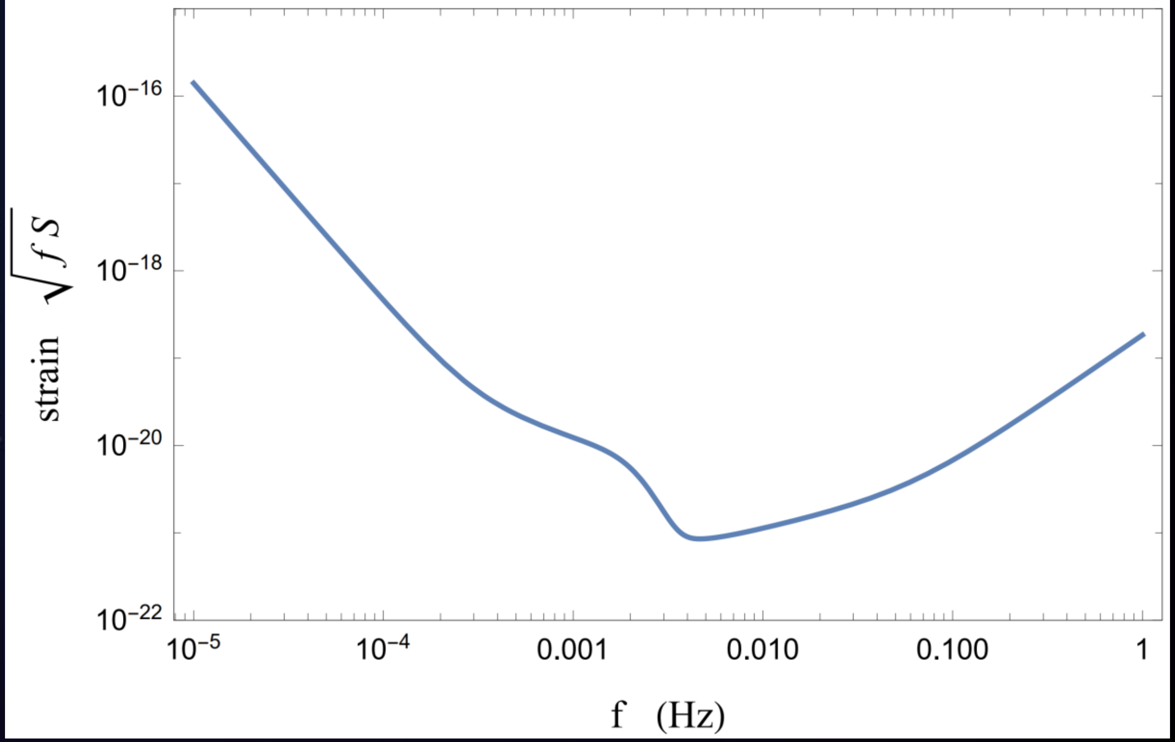
$$\tilde{H}_{2,2}(f) = \int_{-\infty}^{\infty} H_{2,2}(t)\, e^{i2\pi f t}\, dt$$

- The overlap integral is weighted with the LISA sensitivity, use approximate sensitivity curve with spectral density S

$$overlap = \langle a|b\rangle = \int_{-\infty}^{\infty} \frac{\tilde{a}\,\tilde{b}^* + \tilde{b}\,\tilde{a}^*}{S}\, df$$

- Use the fractional overlap with Teukolsky waveforms to benchmark kludge waveforms

$$fractional\ overlap = \frac{\left\langle H_{2,2}^{kludge} \middle| H_{2,2}^{Teuk} \right\rangle}{\sqrt{\left\langle H_{2,2}^{kludge} \middle| H_{2,2}^{kludge} \right\rangle \left\langle H_{2,2}^{Teuk} \middle| H_{2,2}^{Teuk} \right\rangle}}$$



Robson, Cornish & Liu 2018

Testing the weak-field approximation of kludges

- Test reliability of kludges as a function of eccentricity

- Kludges work very well for low eccentricities, not as well for high *e*

LISA template consequences vs *overlap*:

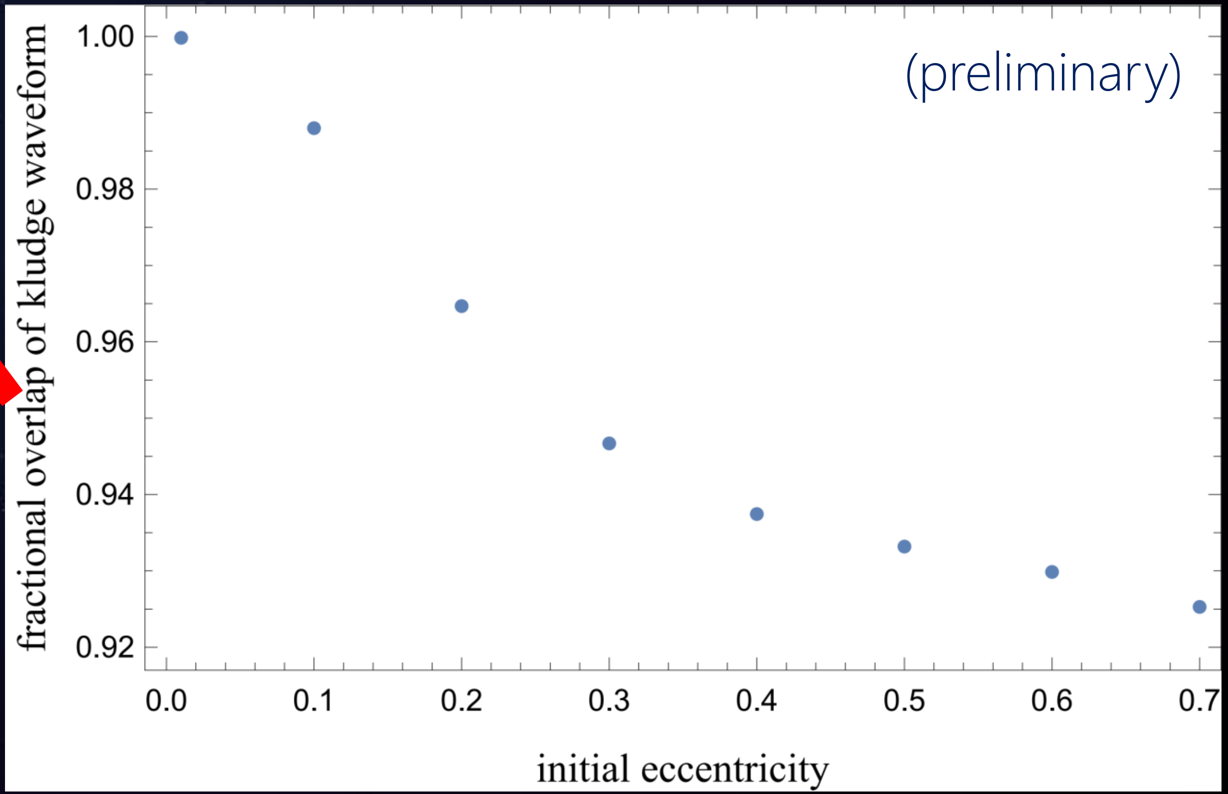
0.95 → ~15% decrease in LISA event rate

0.90 → ~30% decrease in LISA event rate

Babak, Fang, Gair, Glampedakis & Hughes (2008)

- These mis-matches should be amplified for prograde inspirals into Kerr black holes! (venture deeper into the strong field regime)

fractional overlap $= \dfrac{\left\langle H^{kludge}_{2,2} \middle| H^{Teuk}_{2,2} \right\rangle}{\sqrt{\left\langle H^{kludge}_{2,2} \middle| H^{kludge}_{2,2} \right\rangle \left\langle H^{Teuk}_{2,2} \middle| H^{Teuk}_{2,2} \right\rangle}}$



Improving speed of waveforms: direct frequency domain

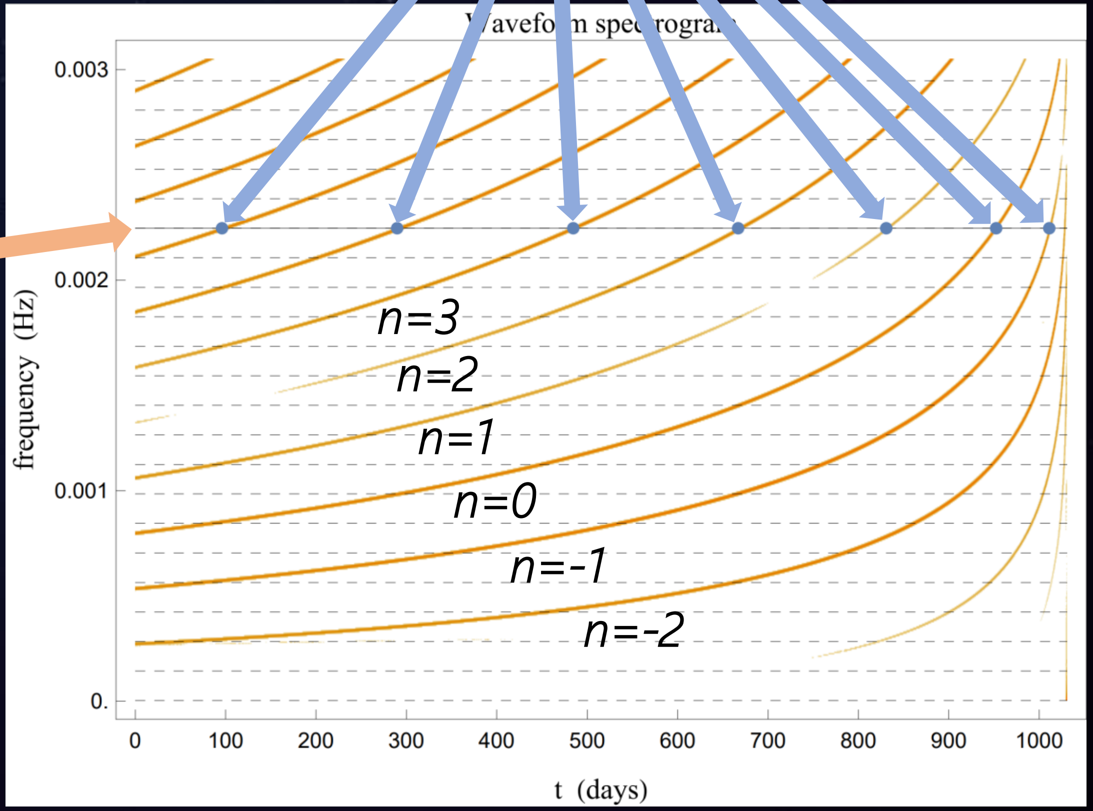
Speed cost: must re-sample inverse NIT trajectory at high resolution (~η^{-1} samples)

But the precomputed Teukolsky waveform data is already stored in the frequency domain!

(2) for each frequency bin, find the time when every harmonic intersects that frequency

$$H^{Teuk}_{2,2}(t) = \sum_n C_{2,2,n}\, e^{-i(n\Omega_r + 2\Omega_\varphi)t}$$

(3) print the Fourier coefficients for that frequency



(1) discretize frequencies into bins

This strategy should circumvent the inverse NIT and avoid high resolution re-sampling

LISA data analysis already occurs in the frequency domain!

Conclusions and future work

- LISA templates require waveforms with highly accurate phases: *need self-forced inspirals*
- LISA data analysis also requires high speed: *optimize equations of motion with NIT*
- Investigation: *Are kludge waveforms sufficient for pairing with self-forced inspirals?*

  Conclusion: *Yes for low eccentricity, probably need Teukolsky for high eccentricity*
- Challenge: *Although NIT inspirals are rapid, time domin waveform generation is too slow*

  Solution: *Calculate waveforms directly in the frequency domain?*
- Future work:

  Develop more general inspiral models (2nd order self-force, Kerr self-force, etc.)

  Pair these more general inspiral models with interpolated Kerr Teukolsky waveforms