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Time Domain Method for the Green Function in Schwarzschild Spacetime

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Green Functions in Black Hole Spacetimes

- The 1st Order self-force can be decomposed as

$$F^{\alpha}_{\text{self}}(z) = F^{\alpha}_{\text{inst}} + F^{\alpha}_{\text{hist}}$$

- The historical part relates to the effects from the part of the worldline *inside* the past light cone

$$h \sim (\text{local terms}) + \lim_{\epsilon \to 0} \int_{-\infty}^{\tau^- - \epsilon} G_{\alpha\beta\alpha'\beta'}(z(\tau), z(\tau'))u^{\alpha'} u^{\beta'} d\tau'$$

- G is the retarded Green Function

Mino, Sasaki,Tanaka*, 1997, Phys. Rev. D* **55**
Quinn, Wald*, 1997, Phys. Rev. D* **56**
Detweiler, Whiting, *2003, Phys. Rev. D* **67**

Green Functions in Black Hole Spacetimes

- For now, we restrict ourselves to a *scalar* self force, in Schwarzschild

$$(\Box_{(4)} - \zeta R)\Phi(x) = -4\pi\rho(x)$$

- This is just a wave equation in curved spacetime, so

$$\Phi(x) = \int G(x,x')\rho(x')\sqrt{-g}\; d^4x'$$

$$\Rightarrow F^{\alpha}_{\text{hist}}(z) = q^2 g^{\alpha\beta} \lim_{\epsilon\to 0^+} \int_{-\infty}^{\tau-\epsilon} \nabla_\beta G(z(\tau), z(\tau'))d\tau'$$

- So we try to solve for a Green function, which will satisfy:

$$\left(\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) - \zeta R\right)G(x,x') = -4\pi\frac{\delta_4(x-x')}{\sqrt{-g}}$$

Time Domain Numerical Method

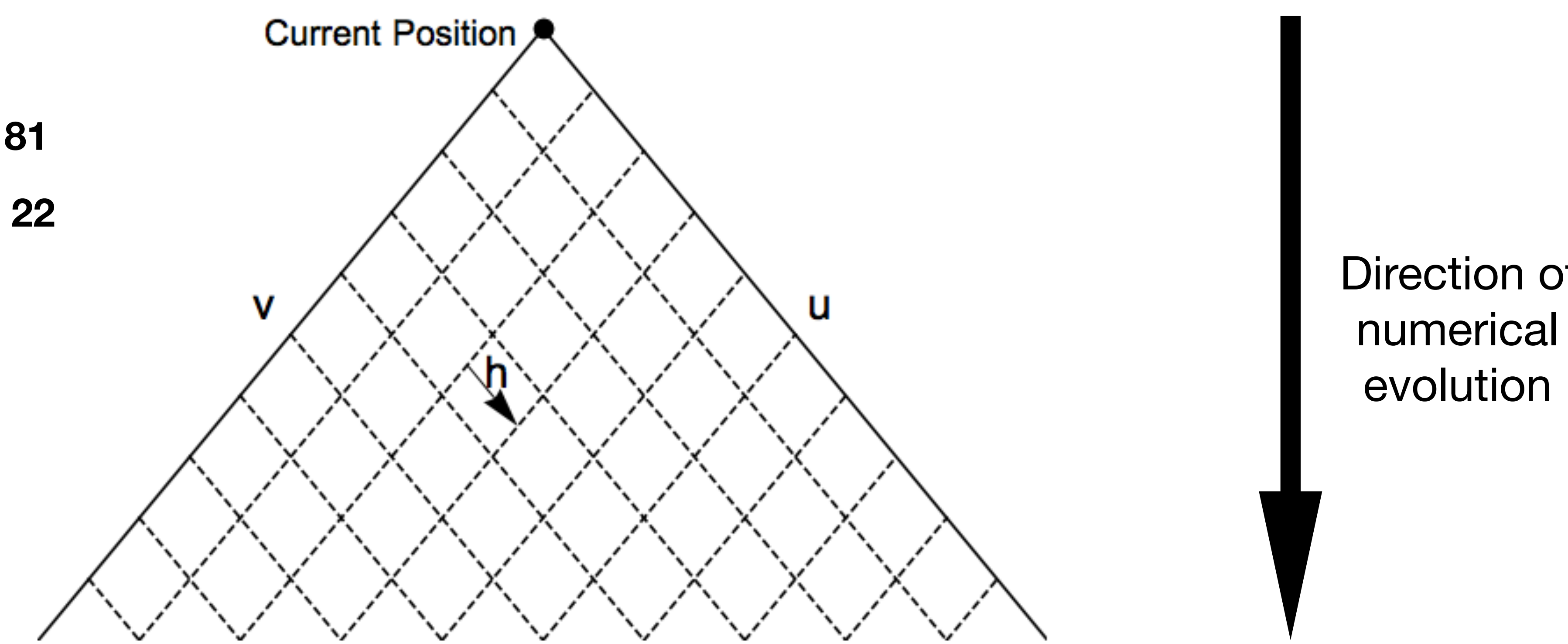
- Inside the past light cone, reduces to a 2D wave equation
- Grid in null coordinates, (u, v)

$$-4\partial_{uv}^2 G_l - \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) G_l = 0$$

Barack, Sago, *2010, Phys Rev D*, **81**

Lousto, *2005, Class. Quant. Grav.* **22**

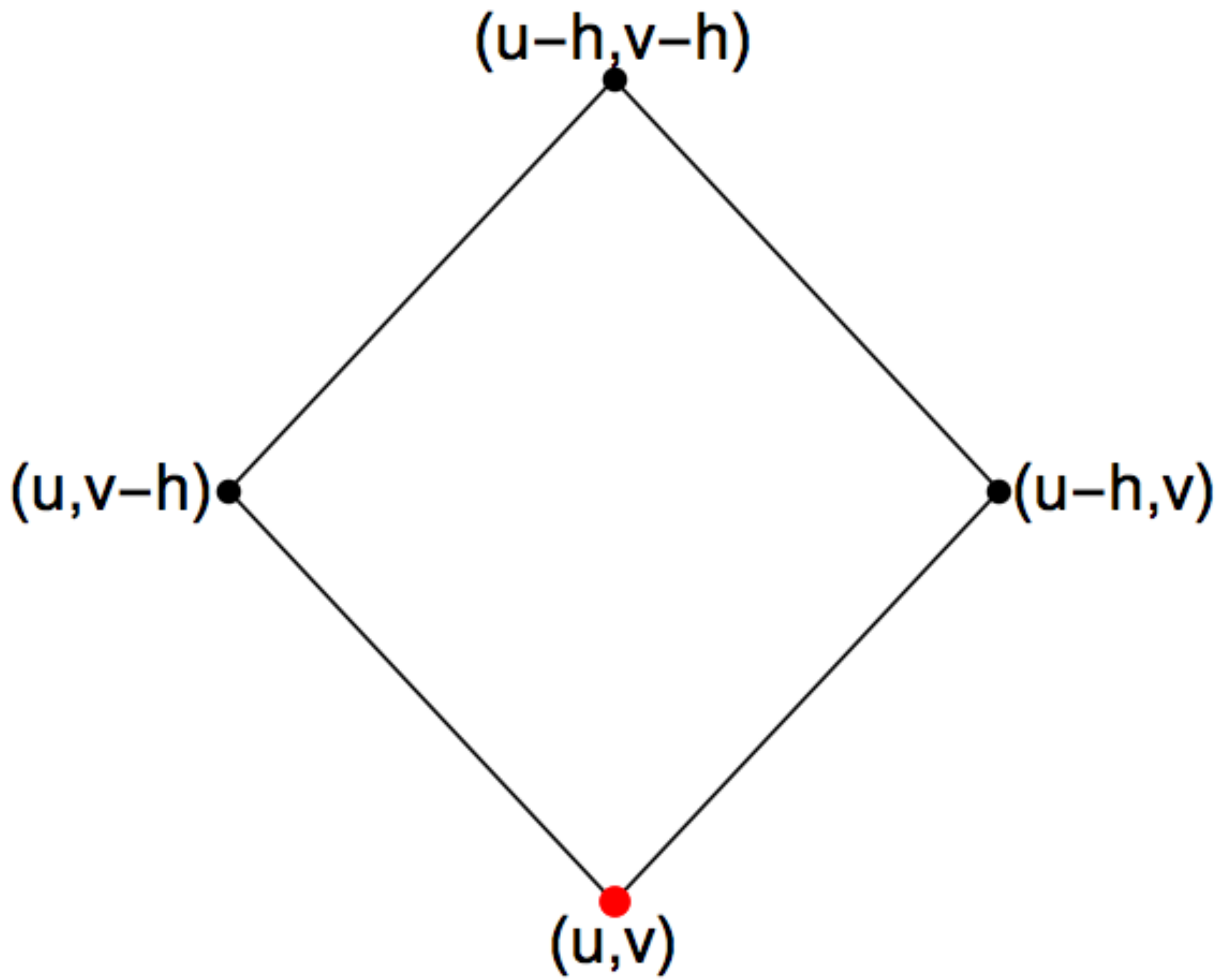
Haas, *2007, Phys. Rev. D* **75**



Past Light Cone

Time Domain Numerical Method

$$-4\partial_{uv}^2 G_l - \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) G_l = 0$$



Points needed for 2nd Order in h

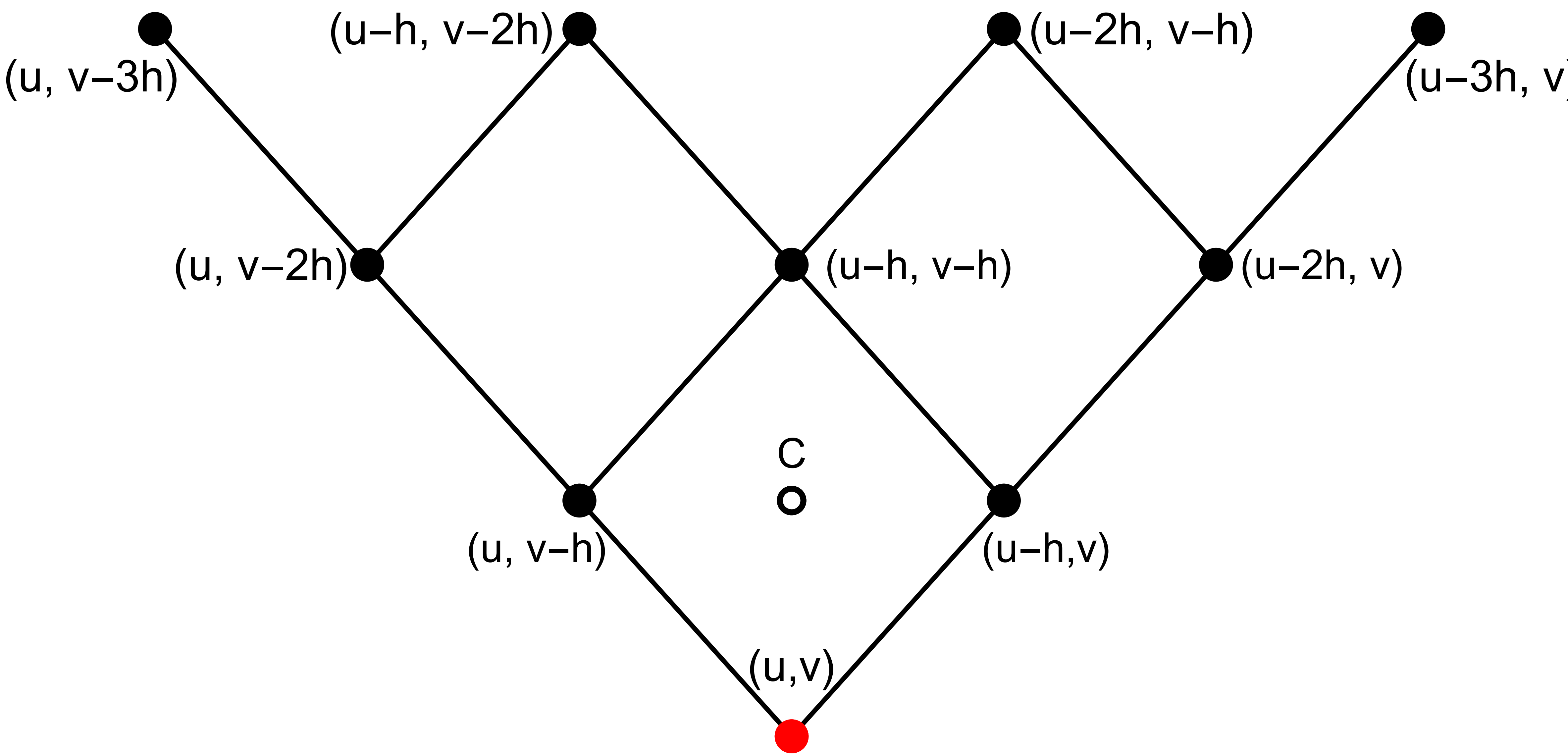
Time Domain Numerical Method

$$-4\partial_{uv}^2 G_l - \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) G_l = 0$$

$$G_l(u,v) = -G_l(u-h, v-h) + [G_l(u-h, v) + G_l(u, v-h)]\left(1 - \frac{h^2}{2} W(u,v)\right)$$

Time Domain Numerical Method

$$-4\partial_{uv}^2 G_l - \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) G_l = 0$$



Points needed for 4th Order in h

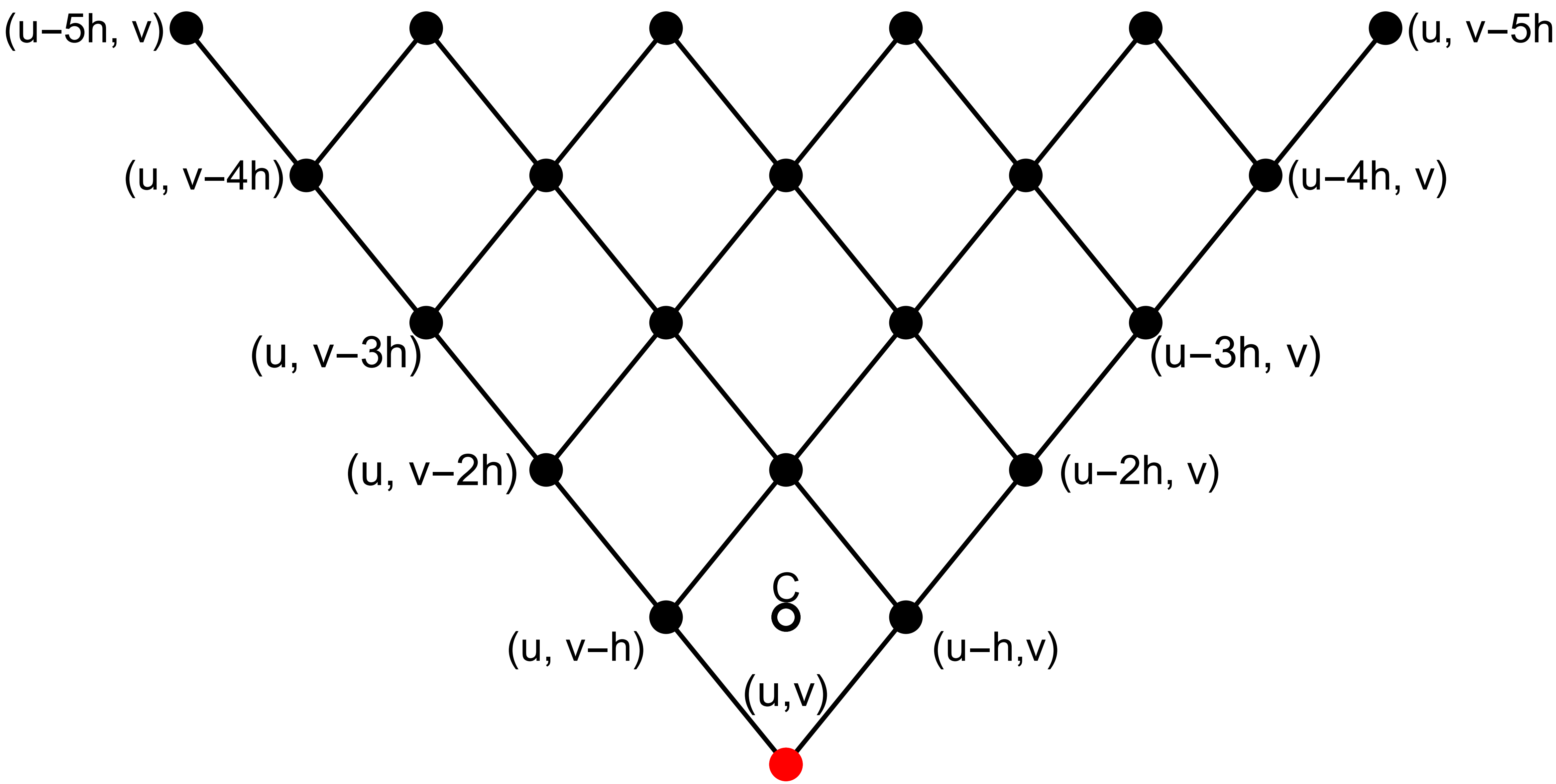
Time Domain Numerical Method

$$-4\partial_{uv}^2 G_l - \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) G_l = 0$$

$$\begin{aligned}
G_l(u,v) \quad = \quad & -G_l(u-h,v-h) + G_l(u,v-h) + G_l(u-h,v) \\
& -\frac{h^2}{24}\left(2W(u,v)\left(-G_l(u-h,v-h) + (G_l(u-h,v) + G_l(u,v-h))\left(1 - \frac{h^2}{2}W(u,v)\right)\right)\right. \\
& +10(W(u-h,v-h)G_l(u-h,v-h) + W(u,v-h)G_l(u,v-h) + W(u-h,v)G_l(u-h,v)) \\
& -4(W(u-2h,v)G_l(u-2h,v) + W(u,v-2h)G_l(u,v-2h)) \\
& +(W(u-3h,v)G_l(u-3h,v) + W(u,v-3h)G_l(u,v-3h) \\
& \left. -W(u-h,v-2h)G_l(u-h,v-2h) - W(u-2h,v-h)G_l(u-2h,v-h))\right)
\end{aligned}$$

Time Domain Numerical Method

$$-4\partial_{uv}^2 G_l - \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) G_l = 0$$



Points needed for 6th Order in h

Time Domain Numerical Method

$$-4\partial^2_{uv}G_l - \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right)G_l = 0$$

$$\begin{aligned}
G_l(u,v) \quad = \quad & -G_l(u-h,v-h) + G_l(u,v-h) + G_l(u-h,v) \\
& -\frac{h^2}{1440}\Big(108W(u,v)G_l^{(4)}(u,v) + 371(W(u,v-3h)G_l(u,v-3h) + W(u-3h,v)G_l(u-3h,v)) \\
& -154(W(u-4h,v)G_l(u-4h,v) + W(u,v-4h)G_l(u,v-4h)) \\
& +116(W(u-3h,v-h)G_l(u-3h,v-h) + W(u-h,v-3h)G_l(u-h,v-3h)) \\
& +40W(u-2h,v-2h)G_l(u-2h,v-2h) \\
& +27(W(u-5h,v)G_l(u-5h,v) + W(u,v-5h)G_l(u,v-5h)) \\
& -19(W(u-4h,v-h)G_l(u-4h,v-h) + W(u-h,v-4h)G_l(u-h,v-4h)) \\
& -5(W(u-3h,v-2h)G_l(u-3h,v-2h) + W(u-2h,v-3h)G_l(u-2h,v-3h)) \\
& +627(W(u-h,v)G_l(u-h,v) + W(u,v-h)G_l(u,v-h)) \\
& +1032W(u-h,v-h)G_l(u-h,v-h) \\
& -504(W(u-2h,v)G_l(u-2h,v) + W(u,v-2h)G_l(u,v-2h)) \\
& -329(W(u-2h,v-h)G_l(u-2h,v-h) + W(u-h,v-2h)G_l(u-h,v-2h))\Big))
\end{aligned}$$

Time Domain Numerical Method

$$-4\partial_{uv}^2 G_l - \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M(1-s^2)}{r^3}\right) G_l = 0$$

$$\begin{aligned}
G_l(u,v) \quad = \quad & -G_l(u-h,v-h) + G_l(u,v-h) + G_l(u-h,v) \\
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& +116(W(u-3h,v-h)G_l(u-3h,v-h) + W(u-h,v-3h)G_l(u-h,v-3h)) \\
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& -19(W(u-4h,v-h)G_l(u-4h,v-h) + W(u-h,v-4h)G_l(u-h,v-4h)) \\
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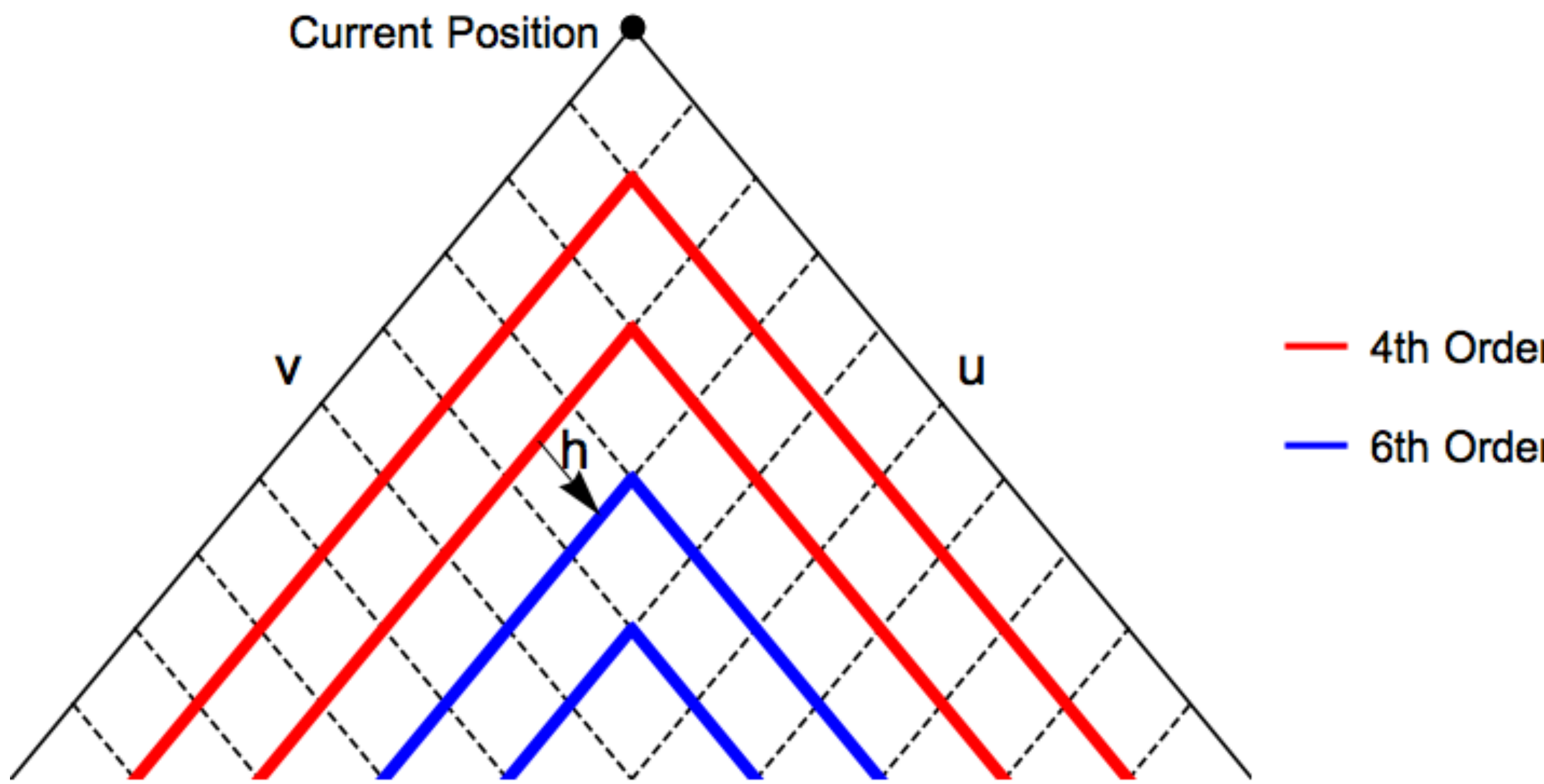
& -329(W(u-2h,v-h)G_l(u-2h,v-h) + W(u-h,v-2h)G_l(u-h,v-2h))\Big))
\end{aligned}$$

Initial Conditions

- For higher order, we need data on more than one ‘previous’ null surface



- We use a standard ansatz from BHPT, the Hadamard series

$$G_l = \Theta(-\sigma) \sum_{n=0}^{\infty} U_n \sigma^n \qquad \sigma(x, x') = -\frac{1}{2}(\Delta\tau)^2 = -\frac{1}{2}\Delta u \Delta v$$

Décanini, Folacci, *2006, Phys Rev D*, **73**

Initial Conditions

- Can derive transport equations for the U_n's

$$U_0 = \Delta(x, x') = 1$$

$$-fV_l + 2(\Delta u \partial_u U_1 + \Delta v \partial_v U_1) + 2U_1 = 0$$

$$\Box U_n - fV_l U_n + 2(n+1)(\Delta u \partial_u U_{n+1} + \Delta v \partial_v U_{n+1}) + 2(n+1)^2 U_{n+1} = 0$$

- Coefficients independent of t (in Schwarzschild)

- For 4th order, need at least U_5, for 6th order U_7, etc.

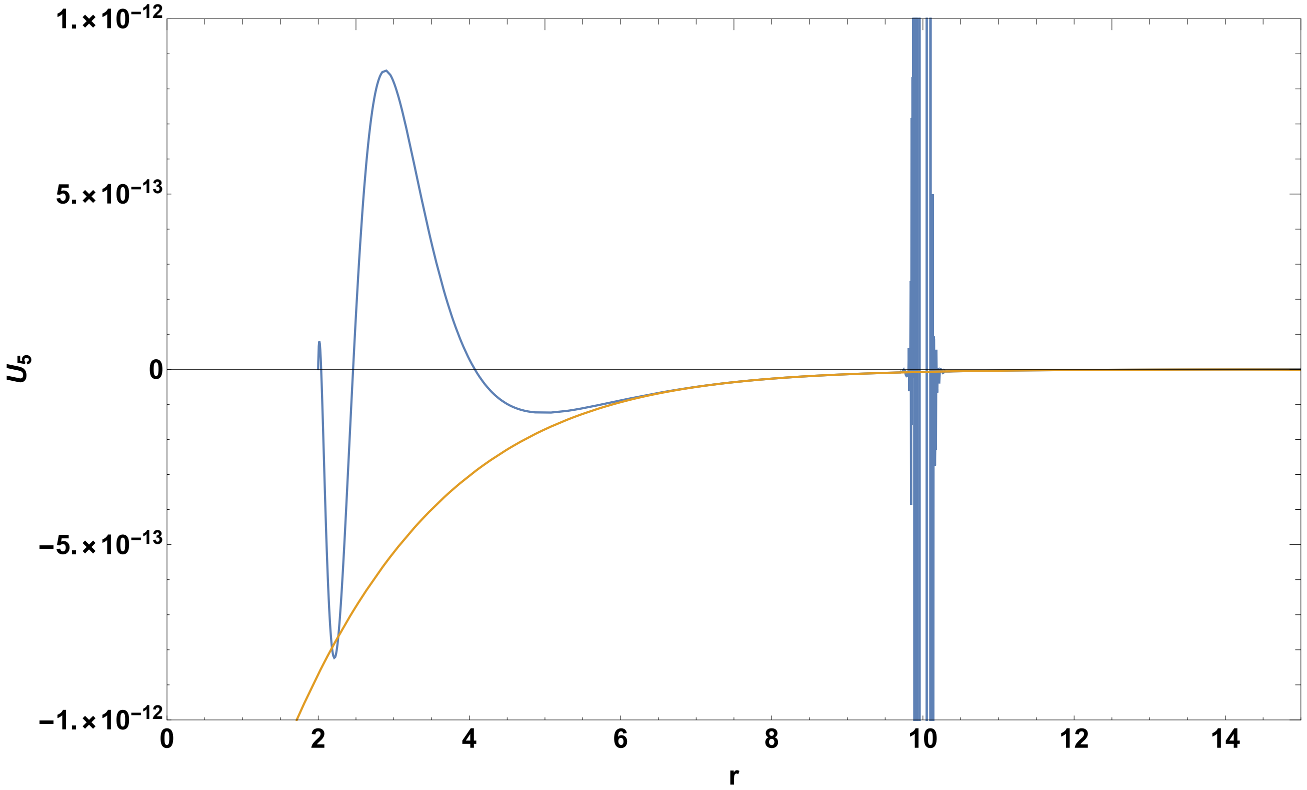
Initial Conditions

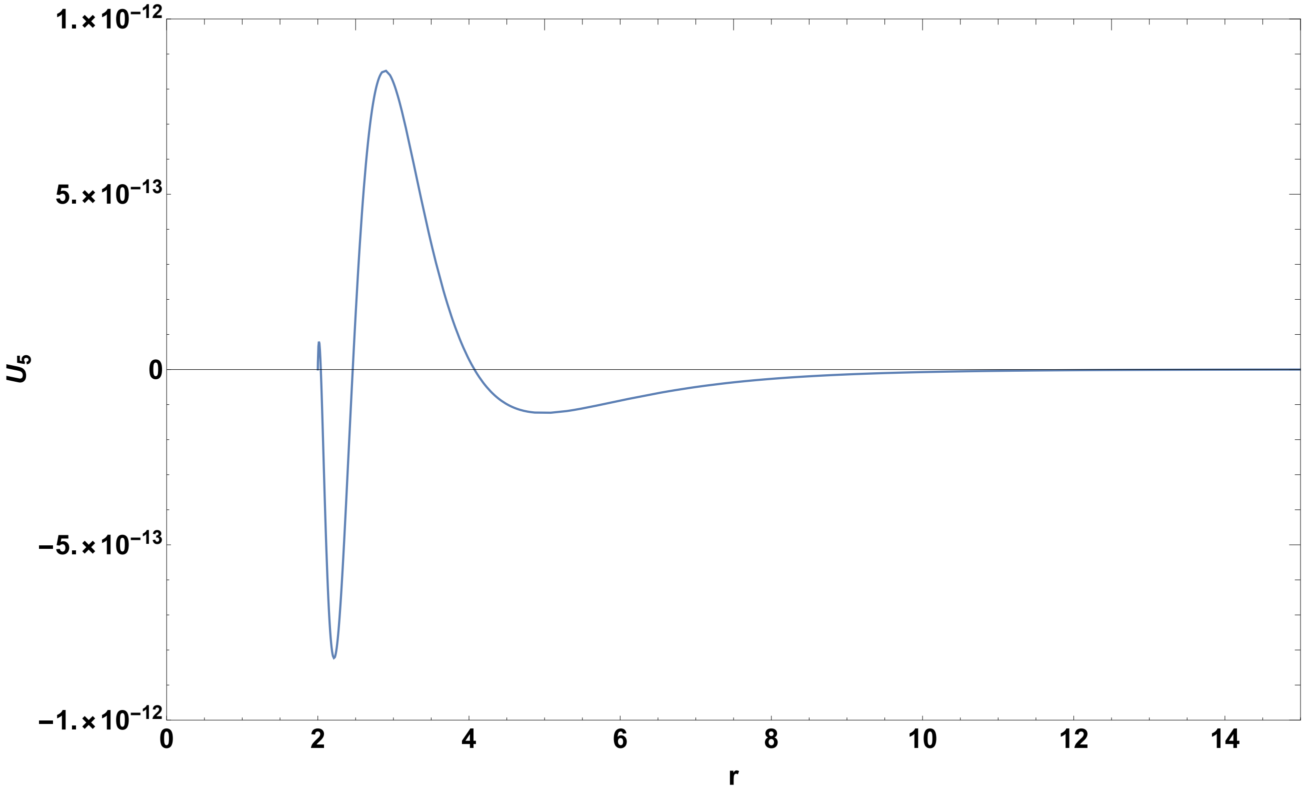
$$U_1(r, r_0) = \frac{(r - r_0)(-M(r + r_0)(s^2 - 1) + r r_0 l(l+1))}{2r^2 r_0^2 (r^* - r_0^*)}$$

$$\begin{aligned} U_2(r, r_0) \quad &= \quad \frac{(r - r_0)^2(-M(r + r_0)(s^2 - 1) + r r_0 l(l+1))^2}{16 r^4 r_0^4 (r^* - r_0^*)^2} \\ &+ \frac{(r - r_0)(-M(r + r_0)(s^2 - 1) + l(l+1) r r_0)}{4r^2 r_0^2 (r^* - r_0^*)^3} \\ &- \frac{(-2M + r0)(l(l+1) r_0 - 2M(s^2 - 1))}{4 r_0^4 (r^* - r_0^*)^2} \\ &- \frac{1}{8(r^* - r_0^*)^2} \left(1 - \frac{2M}{r}\right) \left(\frac{l(l+1)}{r^2} - \frac{2M(s^2 - 1)}{r^3}\right) \\ &+ \frac{1}{8(r^* - r_0^*)^2} \left(1 - \frac{2M}{r_0}\right) \left(\frac{l(l+1)}{r_0^2} - \frac{2M(s^2 - 1)}{r_0^3}\right) \end{aligned}$$

Initial Conditions

- We must treat the value of the initial conditions carefully near coincidence





Green Function Derivatives

- For the self-force, also require the derivatives of the Green Function: $\partial_u G$, $\partial_v G$
- For example, for a scalar charge:

$$F_\mu = q \int_{-\infty}^{\tau^-} \nabla_\mu G(z(\tau), z(\tau'))d\tau'$$

- Can use finite differencing or compute directly
- Latter requires values of both derivatives and Green function on previous points in the domain

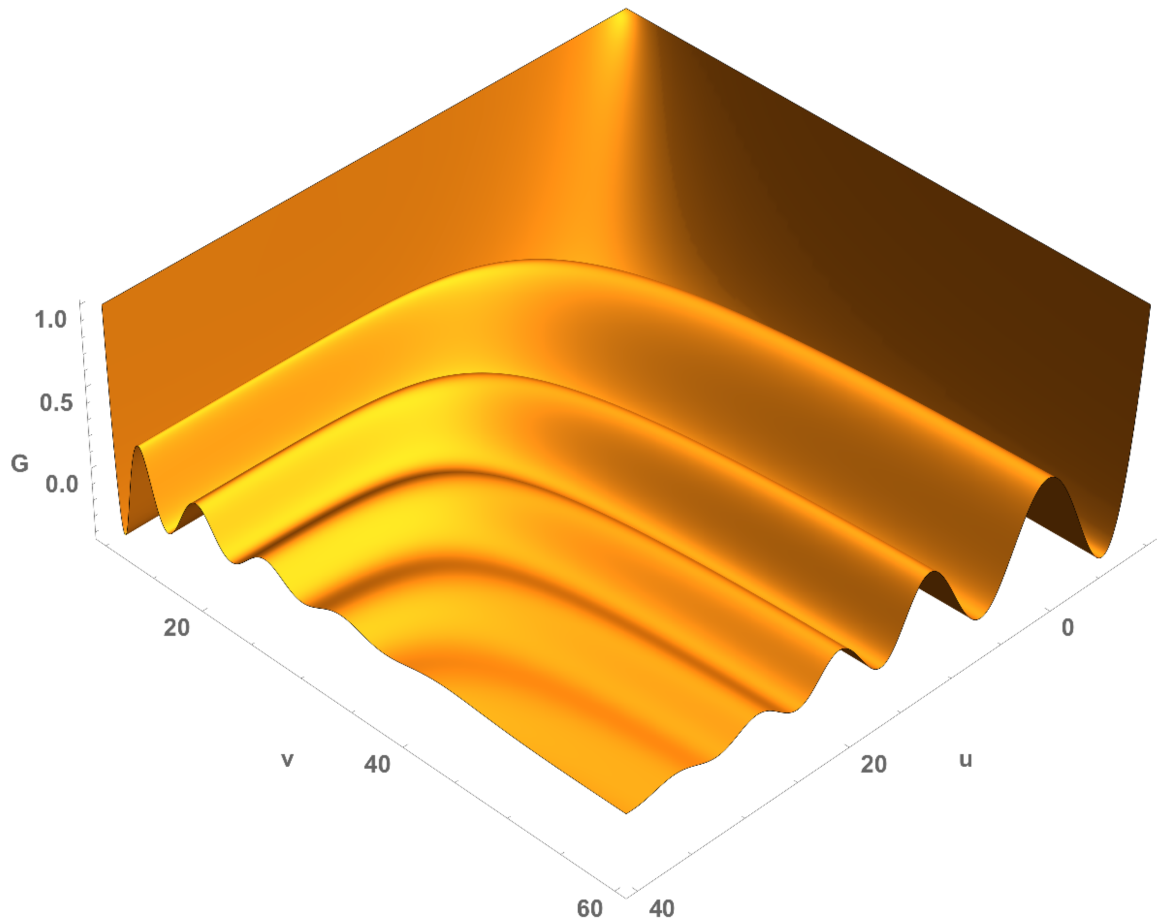
Green Function Derivatives

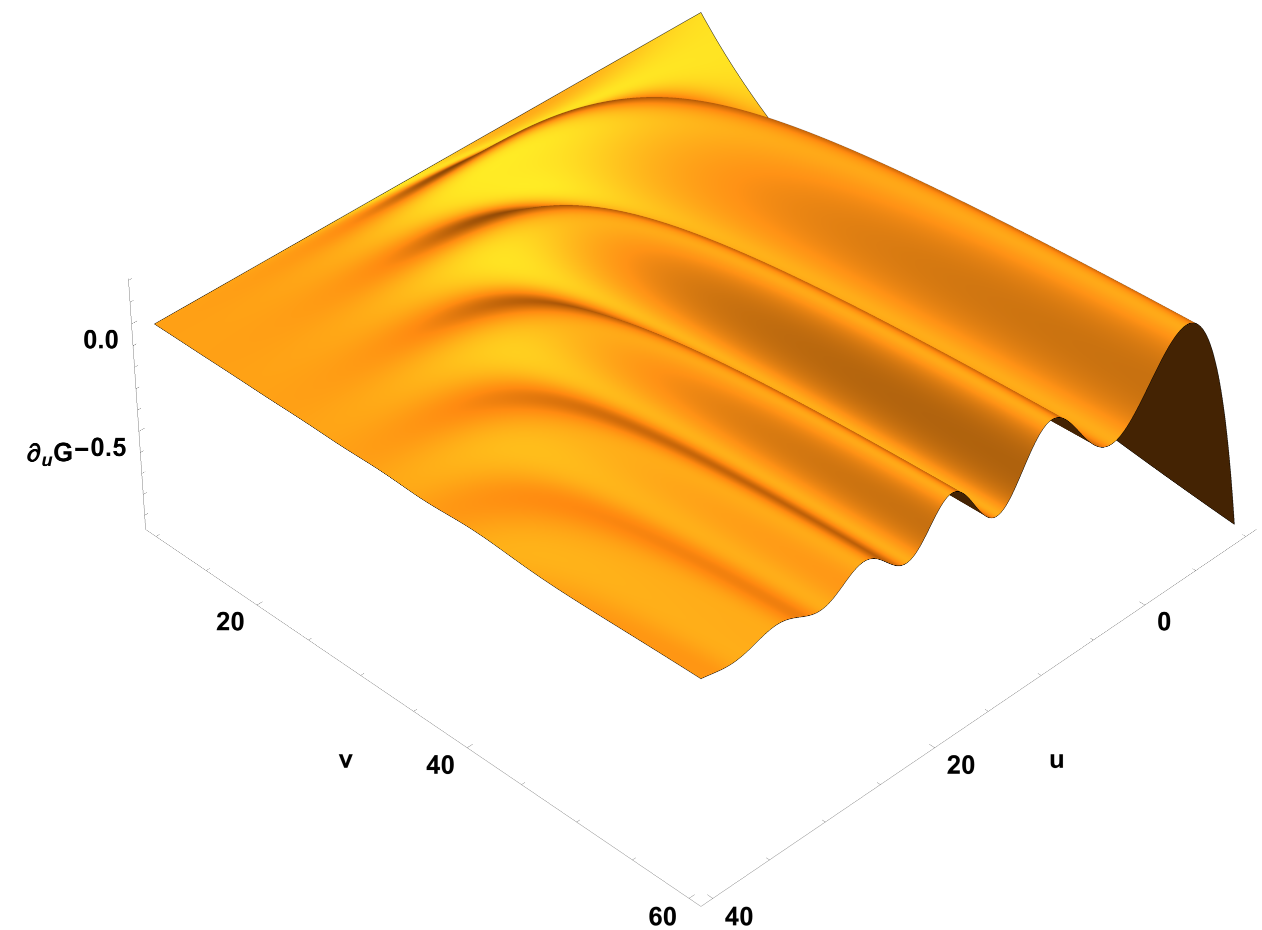
- For direct computation, start by differentiating the wave equation

$$-4\partial^2_{uv}(\partial_u G_l) - \partial_u\left(\left(1-\frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2}+\frac{2M(1-s^2)}{r^3}\right)\right)G_l$$
$$-\left(1-\frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2}+\frac{2M(1-s^2)}{r^3}\right)\partial_u G_l = 0$$

- The first term is the same as before, the second and third are both of familiar form
- We then differentiate the Hadamard series to obtain the necessary initial data
- This requires finding derivatives of the U_n's, which we have expressions for
- Note that now 2nd order will require U_1, 4th order U_6, 6th order U_8, etc.

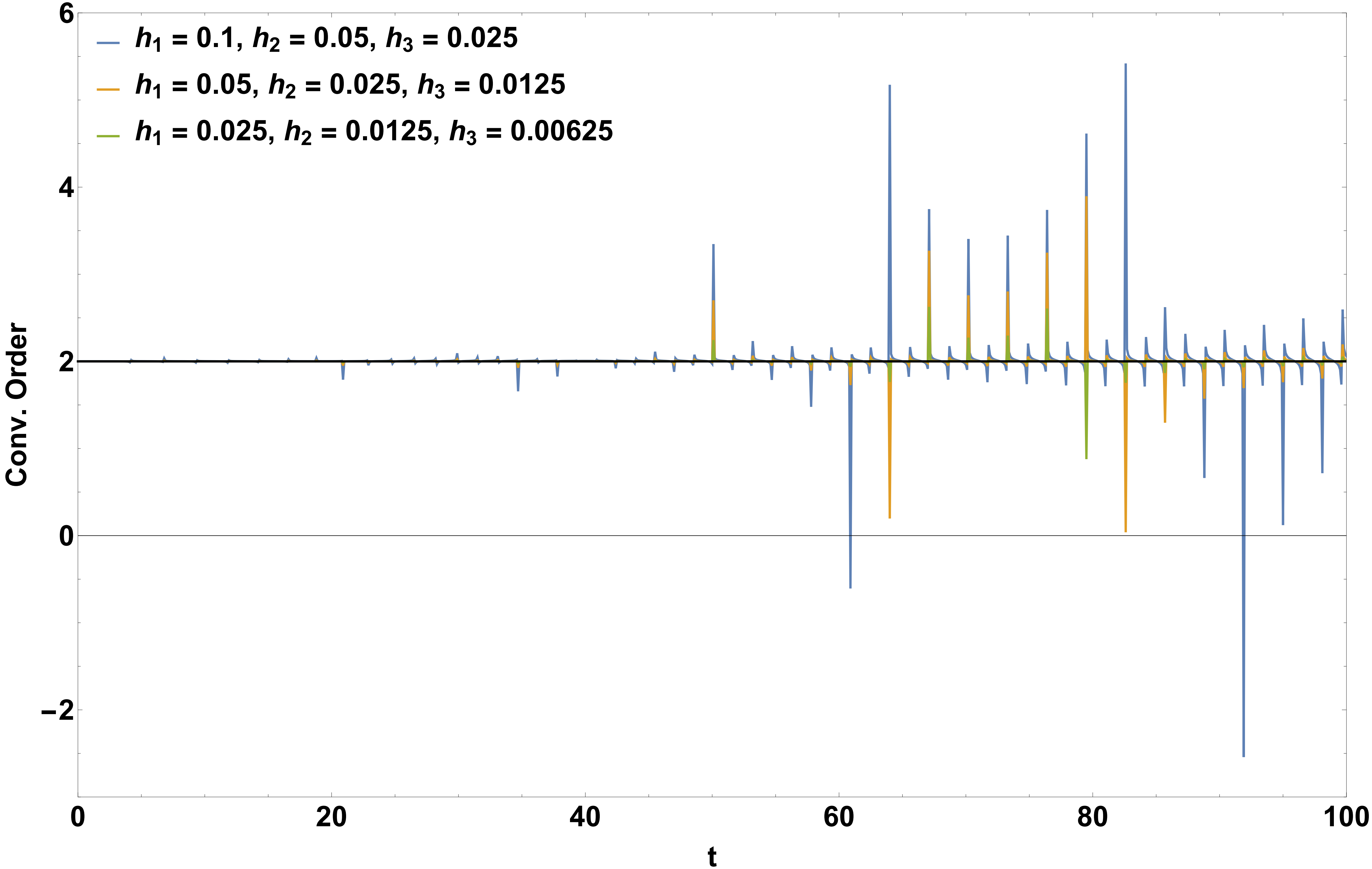
Results





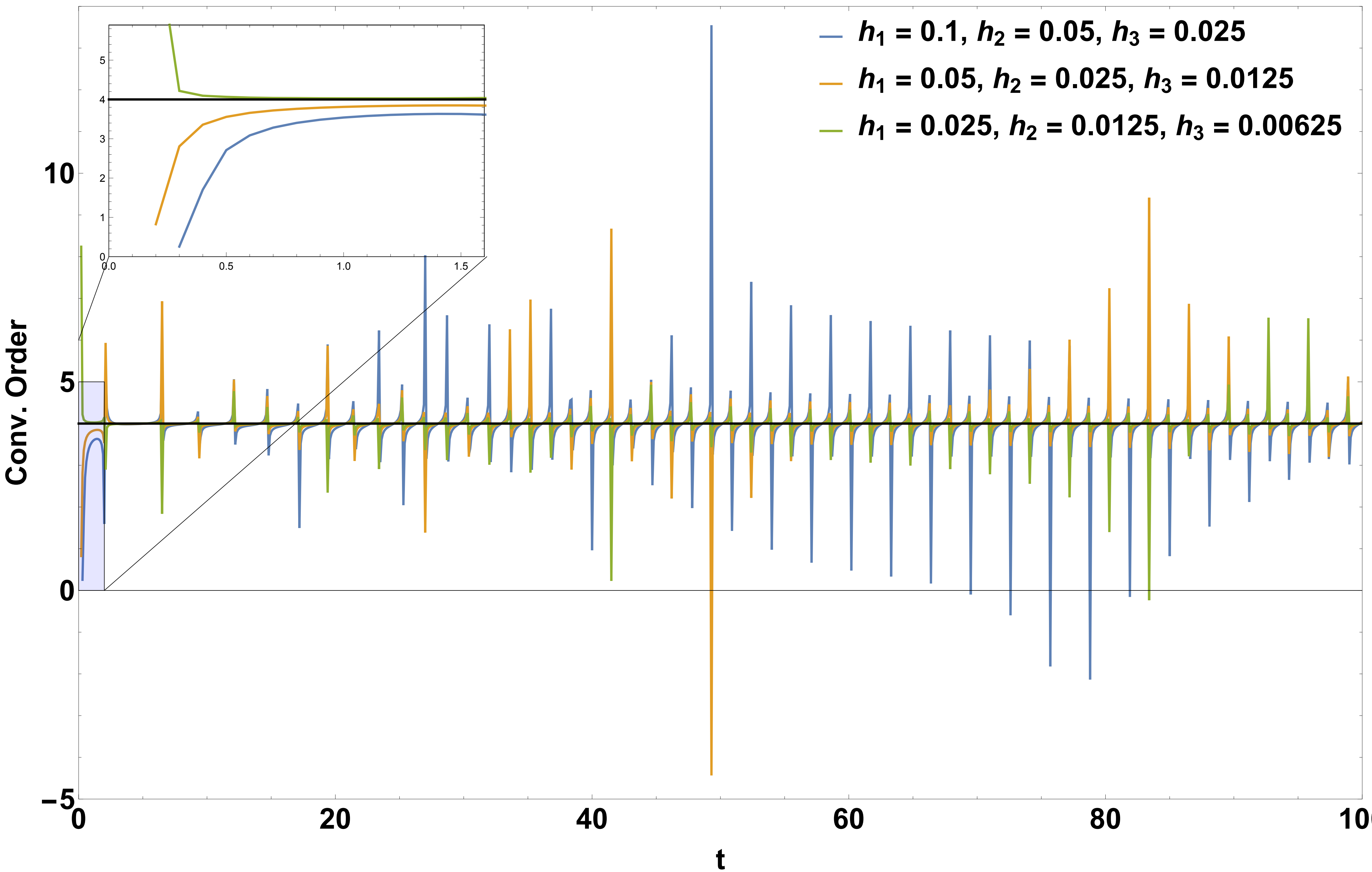
- Mathematica implementation of 2nd and 4th order algorithms for G and both derivatives
- Algorithms valid for scalar, electromagnetic and gravitational perturbations to background
- Extension to higher orders straightforward (6th started)
- C code implementation of the above

Results



h_1 = 0.1, h_2 = 0.05, h_3 = 0.025
h_1 = 0.05, h_2 = 0.025, h_3 = 0.0125
h_1 = 0.025, h_2 = 0.0125, h_3 = 0.00625
Conv. Order
t
6
4
2
0
−2
0
20
40
60
80
100

Results



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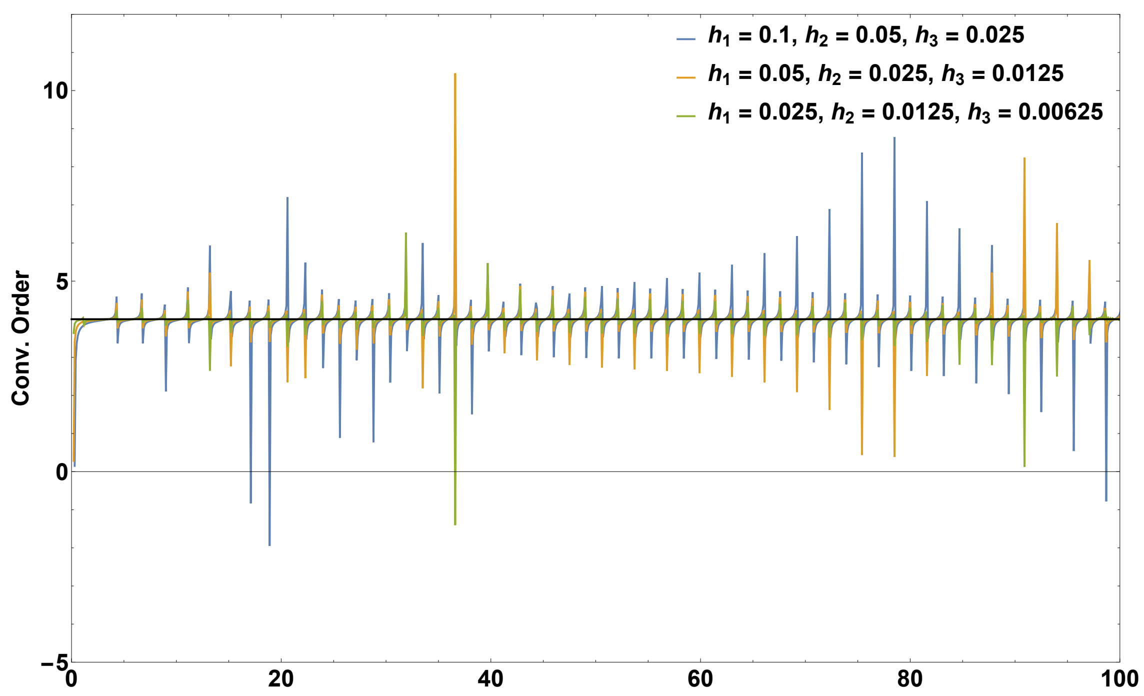
h_1 = 0.05, h_2 = 0.025, h_3 = 0.0125
h_1 = 0.025, h_2 = 0.0125, h_3 = 0.00625
Conv. Order
t

Results



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$h_1 = 0.05$, $h_2 = 0.025$, $h_3 = 0.0125$
$h_1 = 0.025$, $h_2 = 0.0125$, $h_3 = 0.00625$
Conv. Order
10
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0
−5
0
20
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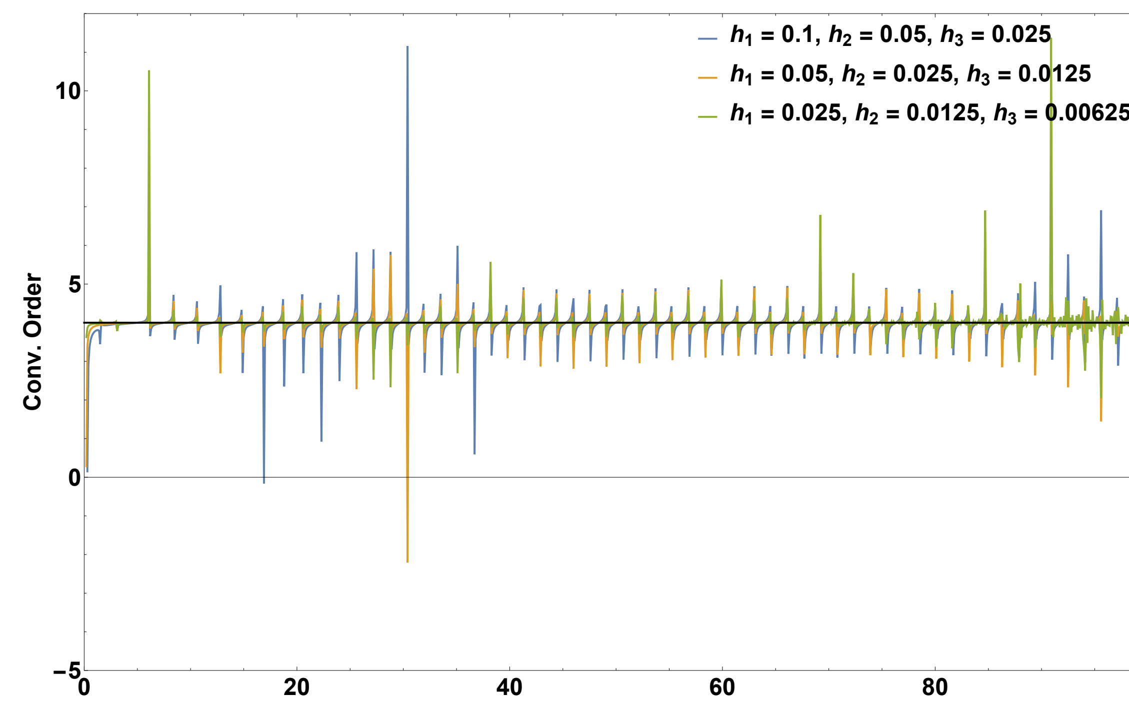
60
80
100



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Conv. Order
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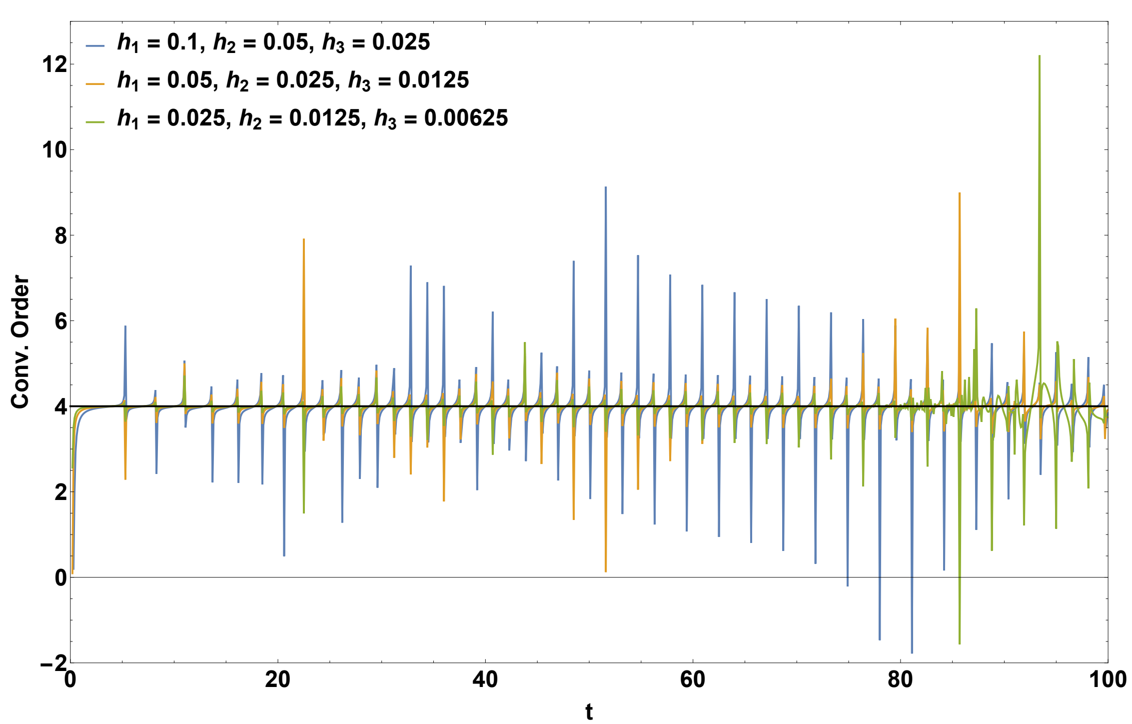

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$h_1 = 0.05$, $h_2 = 0.025$, $h_3 = 0.0125$
$h_1 = 0.025$, $h_2 = 0.0125$, $h_3 = 0.00625$
Conv. Order
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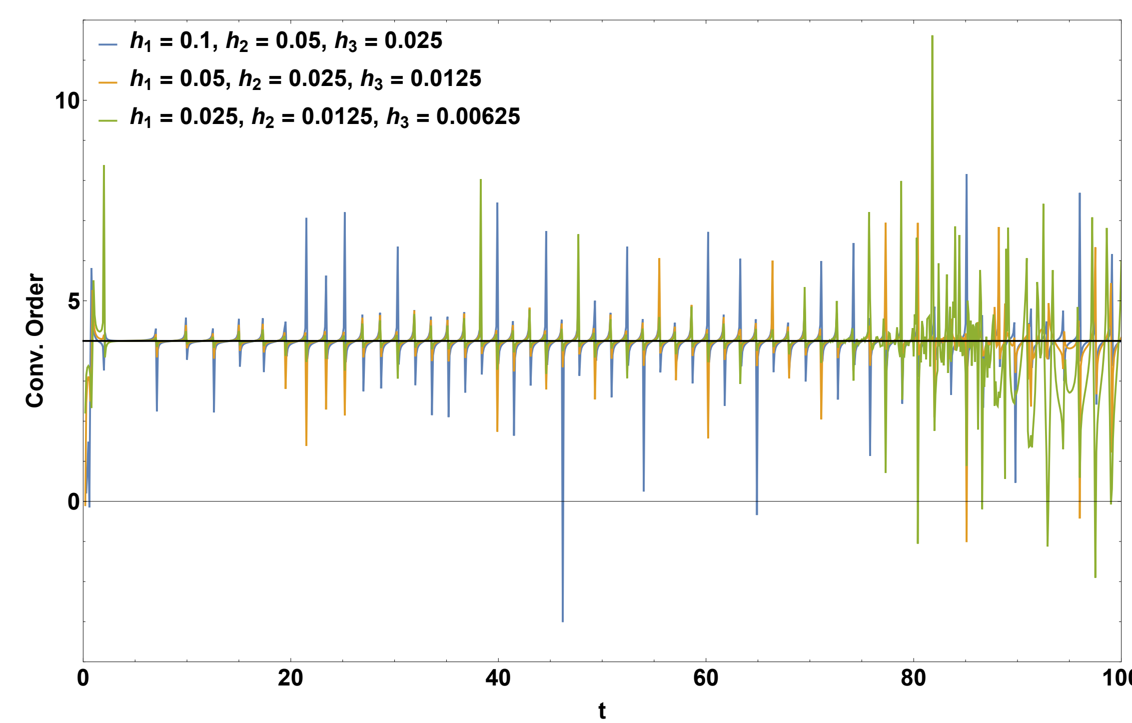
20
40
60
80
100
t



$h_1 = 0.1$, $h_2 = 0.05$, $h_3 = 0.025$
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Conv. Order
10
5
0
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20
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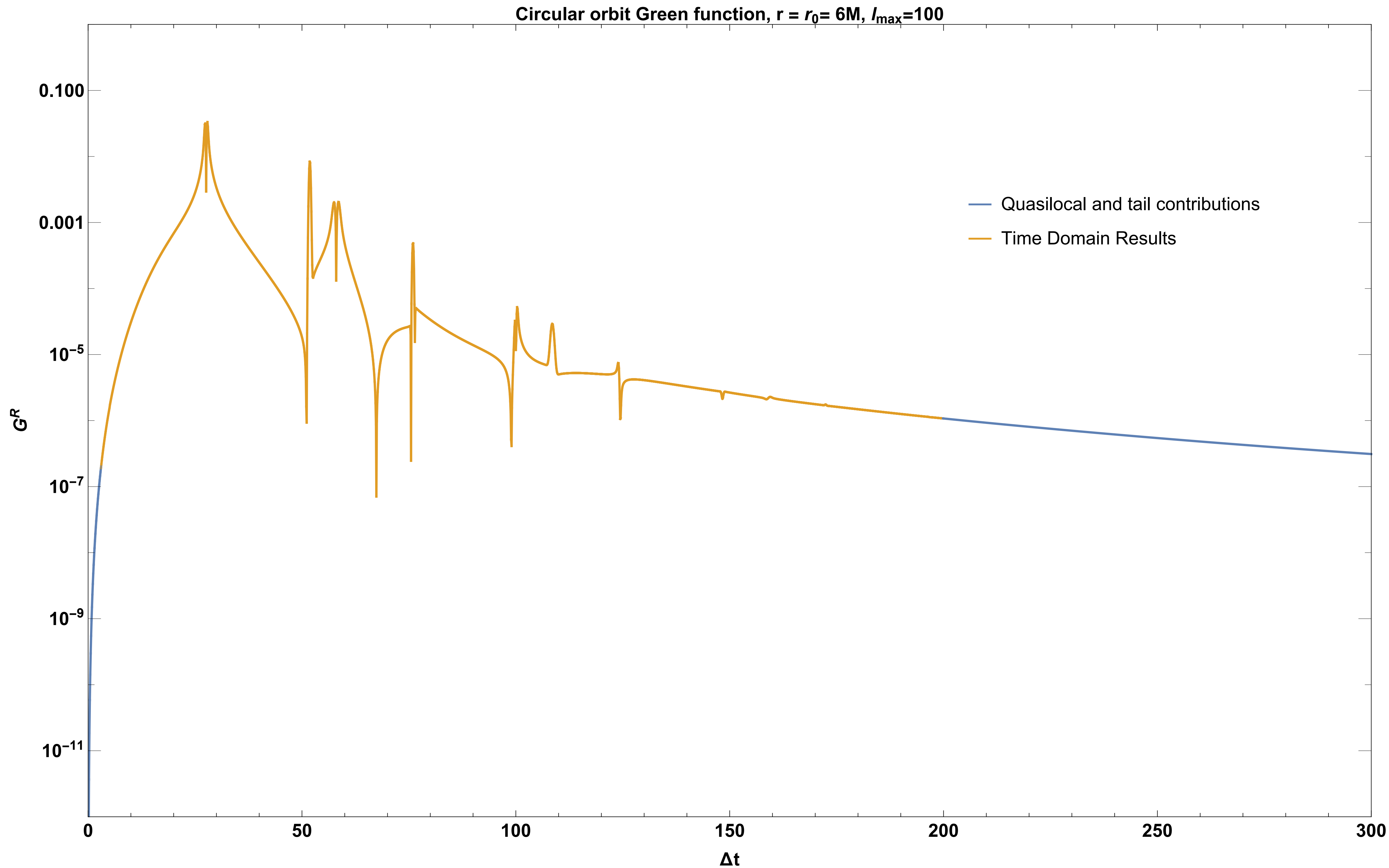
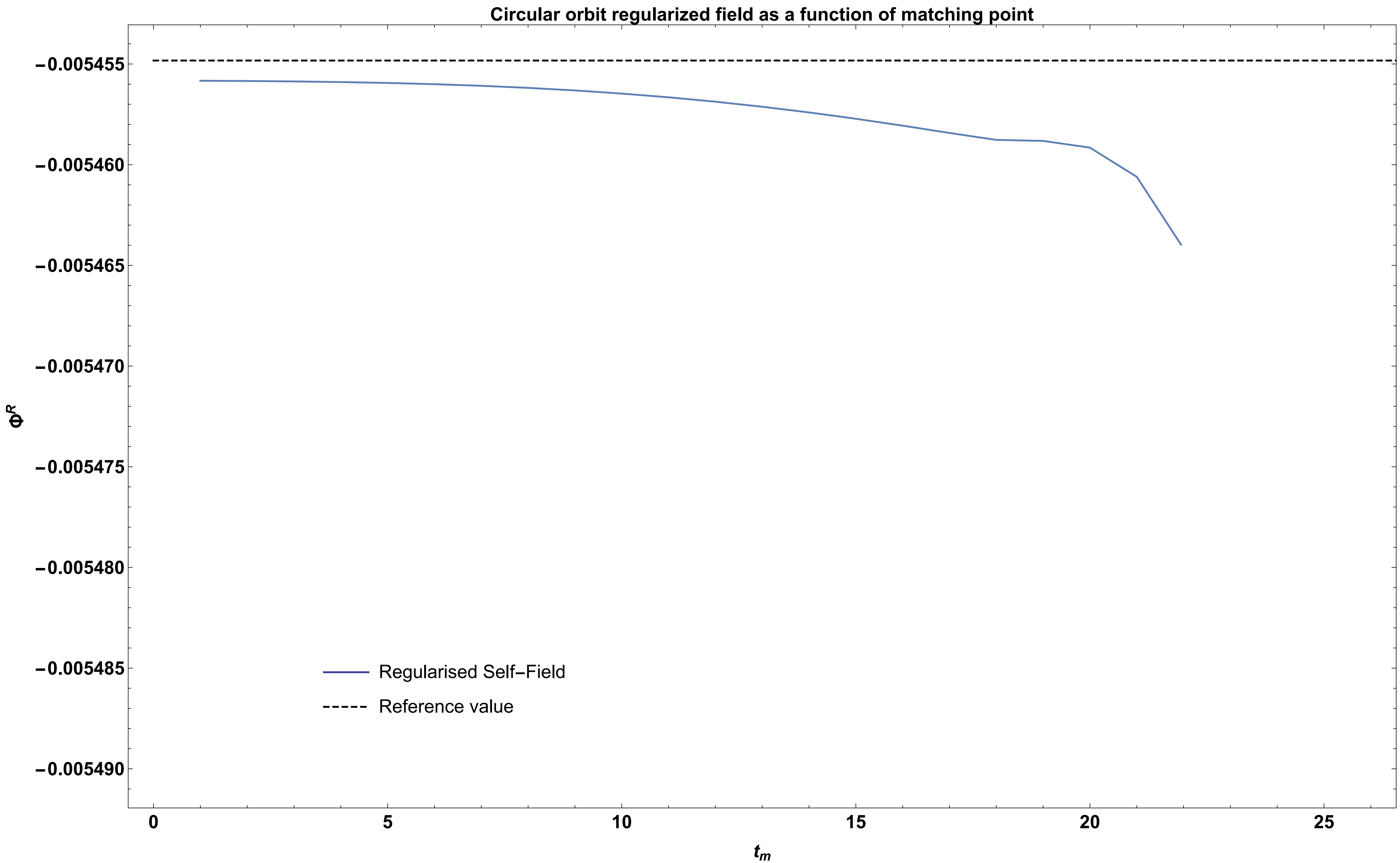
60
80
100
t

Results



Casals, Nolan, Ottewill, Wardell, *In prep.*

Results



Casals, Nolan, Ottewill, Wardell, *In prep.*
Diaz-Rivera, Messaritaki, Whiting, Detweiler, *2004, Phys. Rev. D* **70**

Future Work

- Scalar self-force
- Gravitational self-force
- Higher order schemes (in progress)
- Variety of orbits, bound and unbound
- Mixed second derivatives - Derivatives with respect to both the base point and field point (in progress)
- Input into Surrogate Model(s)
- Extension to Kerr

To name a few things…

Thank you
Questions?

Bonus Slide 1
From 4D to 2D

- Simplification to 2D is done through a number of straightforward steps

1. Tortoise Coordinate

$$r_* = r + 2M \log\left(\frac{r}{2M} - 1\right) \qquad \frac{dr}{dr_*} = 1 - \frac{2M}{r} = f$$

$$(*) \to -\frac{\partial^2 G}{\partial t^2} + \frac{\partial^2 G}{\partial r_*^2} + \frac{2f}{r}\frac{\partial G}{\partial r_*} + \frac{f}{r^2}\nabla^2_{\Omega_2} G = \frac{-4\pi}{r^2}\delta_2(x^A - x^{A'})\delta_{\Omega_2}(x^a, x^{a'})$$

2. Conformal Transformation

$$\hat{g}_{\mu\nu}(t, r_*, \theta, \phi) = \frac{g_{\mu\nu}(t, r_*, \theta, \phi)}{r^2} \qquad G = \frac{1}{rr'}\hat{G}$$

$$(*) \to -\frac{\partial^2 \hat{G}}{\partial t^2} + \frac{\partial^2 \hat{G}}{\partial r_*^2} + \frac{f}{r^2}\left(\nabla^2_{\Omega_2} - \frac{2M}{r}\right)\hat{G} = -4\pi\delta_2(x^A - x^{A'})\delta_{\Omega_2}(x^a, x^{a'})$$

3. Spherical Harmonic Decomposition

$$\hat{G}(x, x') = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)G_l(x^A, x^{A'})P_l(\cos\gamma)$$

$$(*) \to -\frac{\partial^2 G_l}{\partial t^2} + \frac{\partial^2 G_l}{\partial r_*^2} - f\left(\frac{l(l+1)}{r^2} + \frac{2M(1-s^2)}{r^3}\right)G_l = -4\pi\delta_2(x^A - x^{A'})$$

4. Null Coordinates

$$u = t - r_* \qquad v = t + r_* \qquad g_{\mu\nu}(u, v) = -\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(*) \to -4\frac{\partial^2 G_l}{\partial u \partial v} - f\left(\frac{l(l+1)}{r^2} + \frac{2M(1-s^2)}{r^3}\right)G_l = -4\pi\delta_2(x^A - x^{A'})$$

Bonus Slide 2
Kerr

- A number of significant issues posed by transition to Kerr:
  - Cannot fully separate angular coordinates
  - Initial conditions more complex
  - Reformulation of the problem

- But essentially the same steps as for Schwarzschild

- First of the above has been solved previously for similar problems

- Third may require exploring a number of coordinate systems

- Starting with Bardeen-Press-Teukolsky, then moving on to Teukolsky

Bonus Slide 3
Other To-Dos

- Mixed second derivatives - Derivatives with respect to both the base point and field point
  - Necessary for computing stress-energy
  - Also possibly useful for 2nd-Order Osculating Elements
- Experiment with hyperboloidal coordinates
  - Possibly obtain the global Green function
- Hyperbolic scatterings
  - Potential comparisons with Post-Minkowskian methods
- Zoom-whirls
  - Building on discussions from Monday
- Arbitrary precision using GSL
  - Improves C codes robustness to numerical roundoff errors
- Applications to QFT in curved spacetime
- Self-consistent evolution