

A practical guide to second-order self-force calculations

Adam Pound, Niels Warburton
Jeremy Miller, Barry Wardell
+many helpful discussions with Leor Barack

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- ❖ This talk will give a practical guide to second-order calculations
- ❖ We will point out areas where contributions are needed

Gravitational Self-force

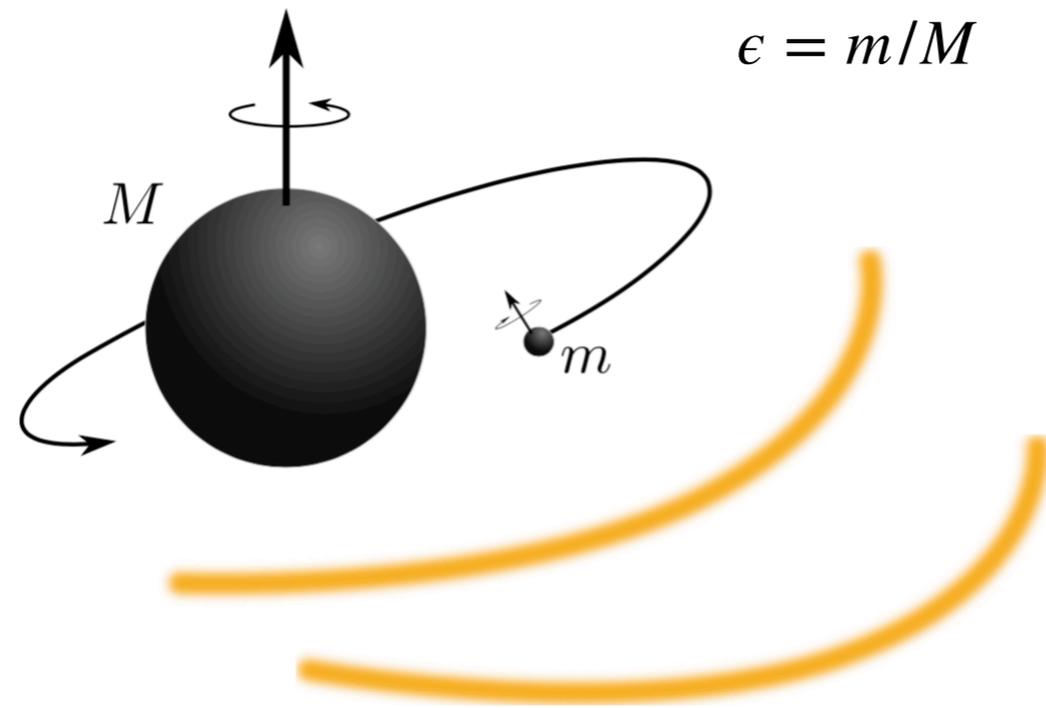


Image credit: A. Pound

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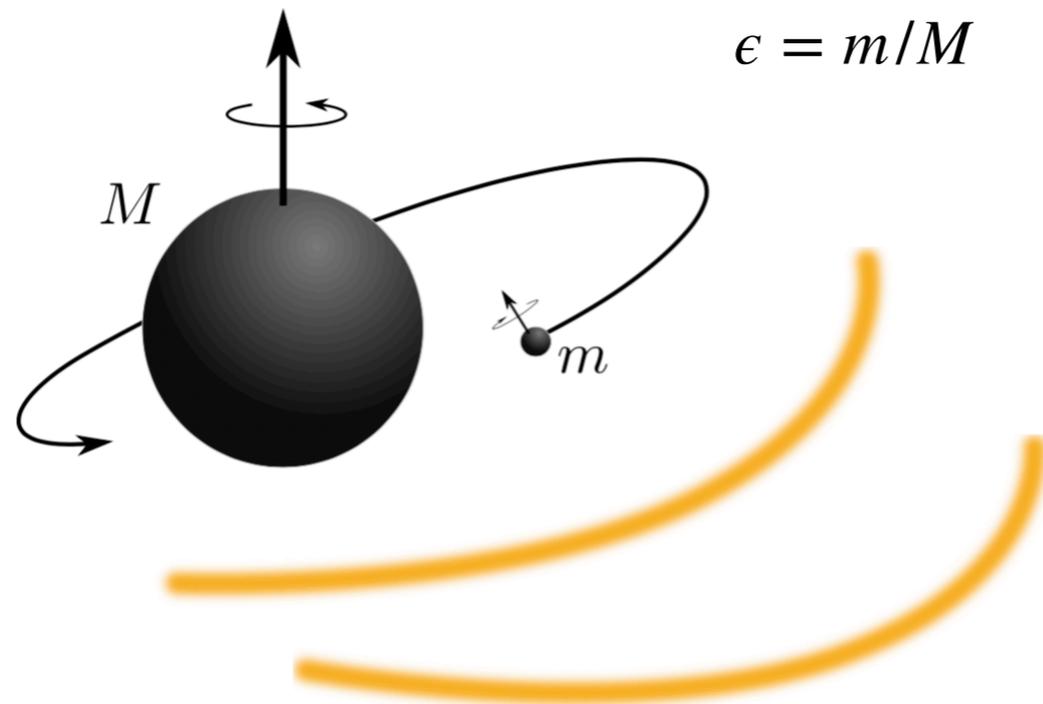


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Write metric as perturbative expansion about a background

$$g_{\mu\nu} = g_{\mu\nu}^0 + \epsilon h_{\mu\nu}^1 + \epsilon^2 h_{\mu\nu}^2 + \mathcal{O}(\epsilon^3)$$

Substitute expansion into Einstein equation

$$G_{\mu\nu}[g] = 8\pi T_{\mu\nu}$$

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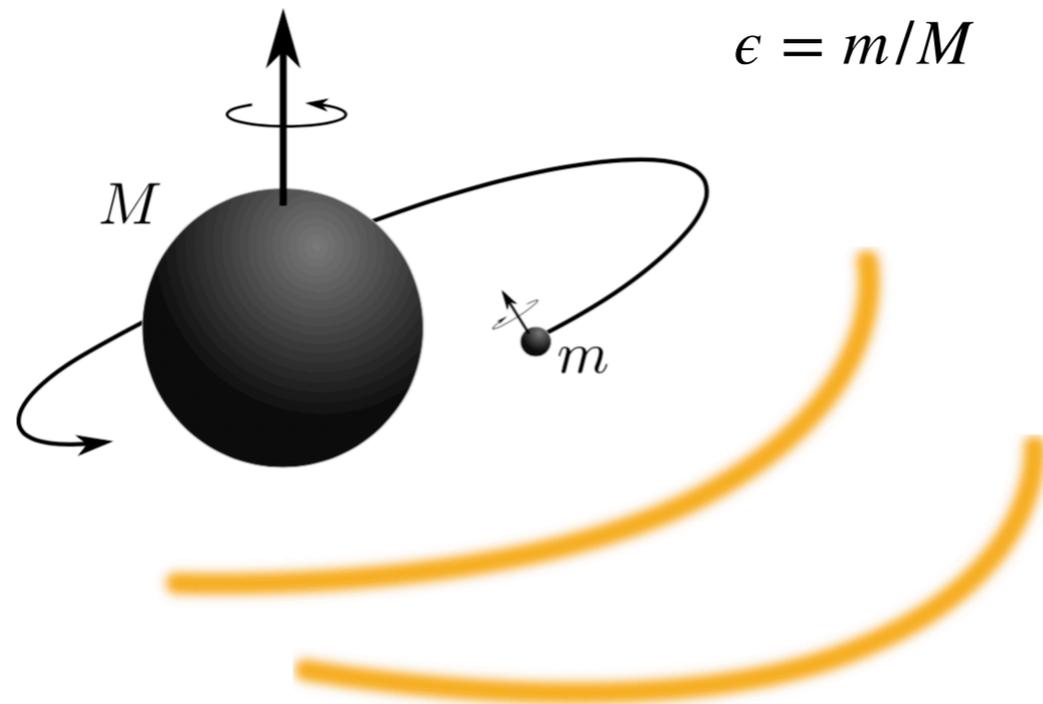


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Expanding out,

$$G_{\mu\nu}[g] = G_{\mu\nu}[g^0] + \epsilon G_{\mu\nu}^1[h^1] + \epsilon^2 \left(G_{\mu\nu}^1[h^2] + G_{\mu\nu}^2[h^1, h^1] \right) + \mathcal{O}(\epsilon^3)$$

Obtain equations at each order in ϵ , which we can solve for $h_{\mu\nu}^1, h_{\mu\nu}^2, \dots$, along with equations of motion for a worldline. Goal: compute $h_{\mu\nu}^1, h_{\mu\nu}^2, z^\mu$.

$$\frac{Dz^\alpha}{d\tau} = \epsilon F_1^\alpha + \epsilon^2 F_2^\alpha$$

Why go to second-order?

On an inspiral timescale $t \sim M^2/m$, the phase of the gravitational wave has an expansion (excluding resonances):

$$\phi = \epsilon^{-1} \phi_0 + \phi_1 + \mathcal{O}(\epsilon)$$

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Adiabatic order

From the orbit averaged piece of first-order self-force $\langle F_1^\alpha \rangle$

$\langle F_1^\alpha \rangle$ can be related to the **fluxes**, thus avoiding a local calculation of the self-force

Good enough for **detection** and rough parameter estimation for **astrophysics of EMRIs**

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Post-Adiabatic order

Three contributions:

- First-order, oscillatory dissipative self-force $F_1^{\alpha(diss,osc)}$
- First-order conservative self-force $F_1^{\alpha(cons)}$
- **Second-order** orbit averaged self-force $\langle F_2^\alpha \rangle$

Needed for **precision tests of GR**

Potential application to **IMRIs**

Non-linear perturbation theory

Expand Einstein tensor and make regular/singular split in second-order perturbation

$$G_{\mu\nu}[g] = G_{\mu\nu}[g^0] + \epsilon G_{\mu\nu}^1[h^1] + \epsilon^2 \left(G_{\mu\nu}^1[h^2] + G_{\mu\nu}^2[h^1, h^1] \right) + \mathcal{O}(\epsilon^3)$$

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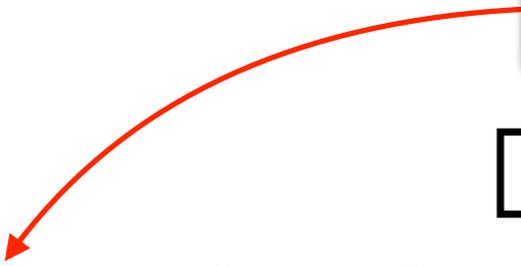
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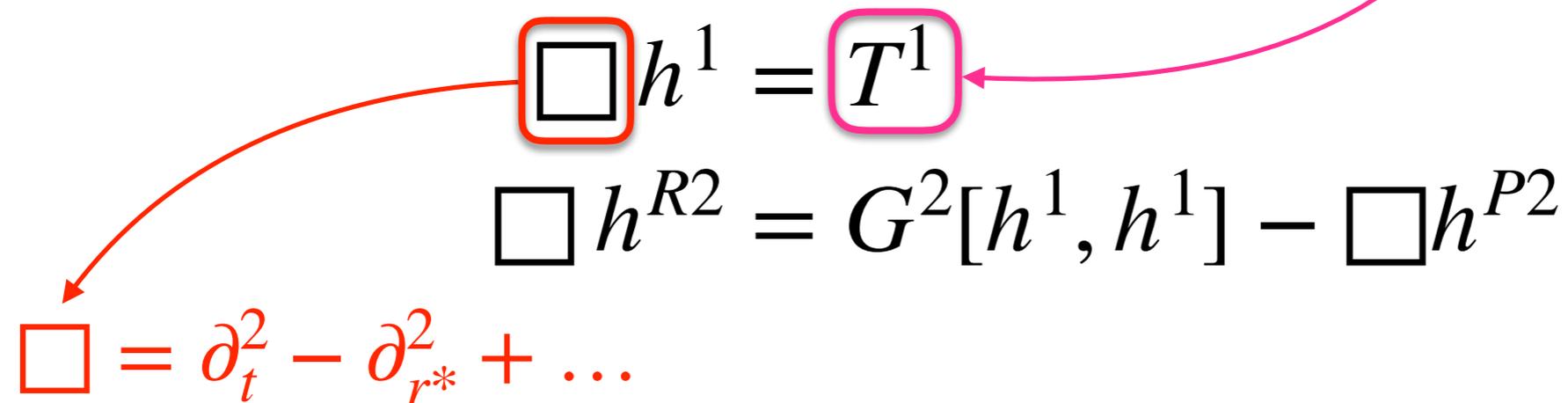
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- Pound (2012)*
- Gralla (2012)

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Equations of motion take the form:

$$\frac{D^2 z^\mu}{d\tau} = \epsilon F^{1\mu}[h^{R1}] + \epsilon^2 F^{2\mu}[h^{R2}]$$

*Pound, Phys. Rev. Lett. 109, 051101 (2012)

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Expand the box operator:

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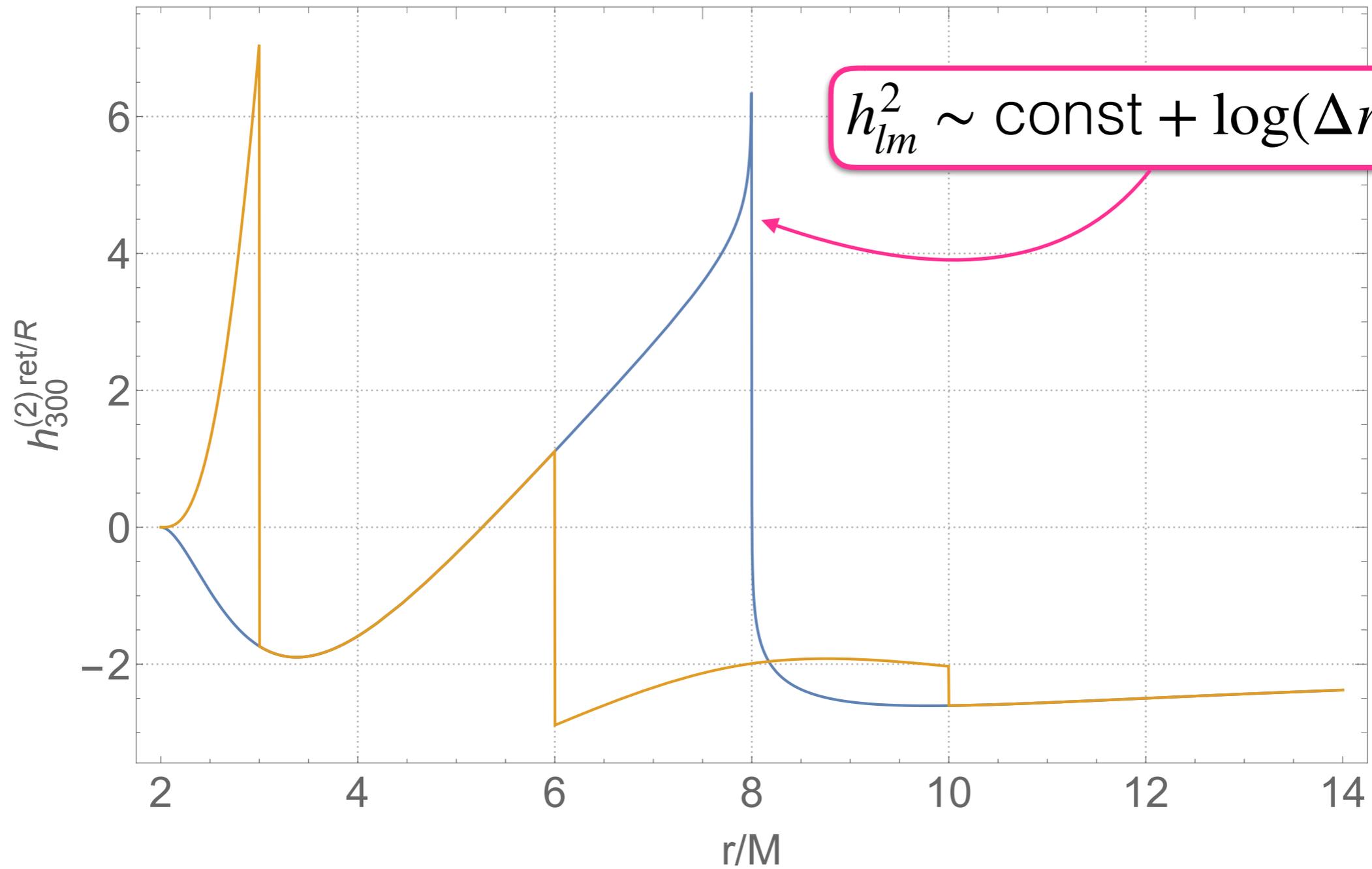
General form of the two-timescale expanded field equations:

$$\begin{aligned} \square_{\omega}^0 h^1 &= T^1 \\ \square_{\omega}^0 h^{R2} &= G_{\omega}^2[h^1, h^1] - \square_{\omega}^0 h^{P2} - \square_{\omega}^1 h^1 \end{aligned}$$

Adam will discuss the two-timescale expanded EoM

First results: monopole of 2nd-order metric perturbation

Particle on circular orbit of Schwarzschild black hole



First application: Gravitational binding energy

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Total energy in the spacetime, defined as the Bondi mass on $u=\text{const}$ slice we match to at large radius

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Expanding out for small mass ratio

$$E_{\text{bind}} = \hat{\mathcal{E}}_0(y) - 1 + \epsilon E_{SF} + O(\epsilon^2)$$

$$E_{SF} = \hat{M}_B^{(2)} - \hat{M}_{\text{BH}}^{(2)} + \hat{\mathcal{E}}_0(y) - 1 + \frac{\delta\hat{M}(1 - 6y)y}{3(1 - 3y)^{3/2}}$$

We directly compute each of these quantities from the monopole piece of the first and **second** order metric perturbations on the horizon and at scri.

$$\hat{M}_B^{(2)} = \lim_{r \rightarrow \infty} \frac{1}{4} \left(\bar{h}_{0,0}^{2(1)} + \bar{h}_{0,0}^{2(6)} \right) Y_{0,0}$$

$$\delta\hat{M} = - \frac{y}{(1 - 3y)^{1/2}}$$

$$\hat{M}_{\text{irr}}^{(2)} \sim \lim_{r \rightarrow 2M} \left(\bar{h}_{0,0}^{2(i)} + \partial_{\tilde{w}} \bar{h}_{0,0}^{1(j)} + \bar{h}_{0,0}^{1(i)} \bar{h}_{0,0}^{1(j)} \right)$$

$$\hat{\mathcal{E}}_0(y) = \frac{1 - 2y}{(1 - 3y)^{1/2}}$$

First application: Gravitational binding energy

Binding energy can also be computed from the **first-law of binary black hole mechanics** using first-order self-force data*

$$\delta M + \Omega \delta J = z_1 \delta m_1 + z_2 \delta m_2$$

Rewriting this in terms of binding energy as a function of y , we get

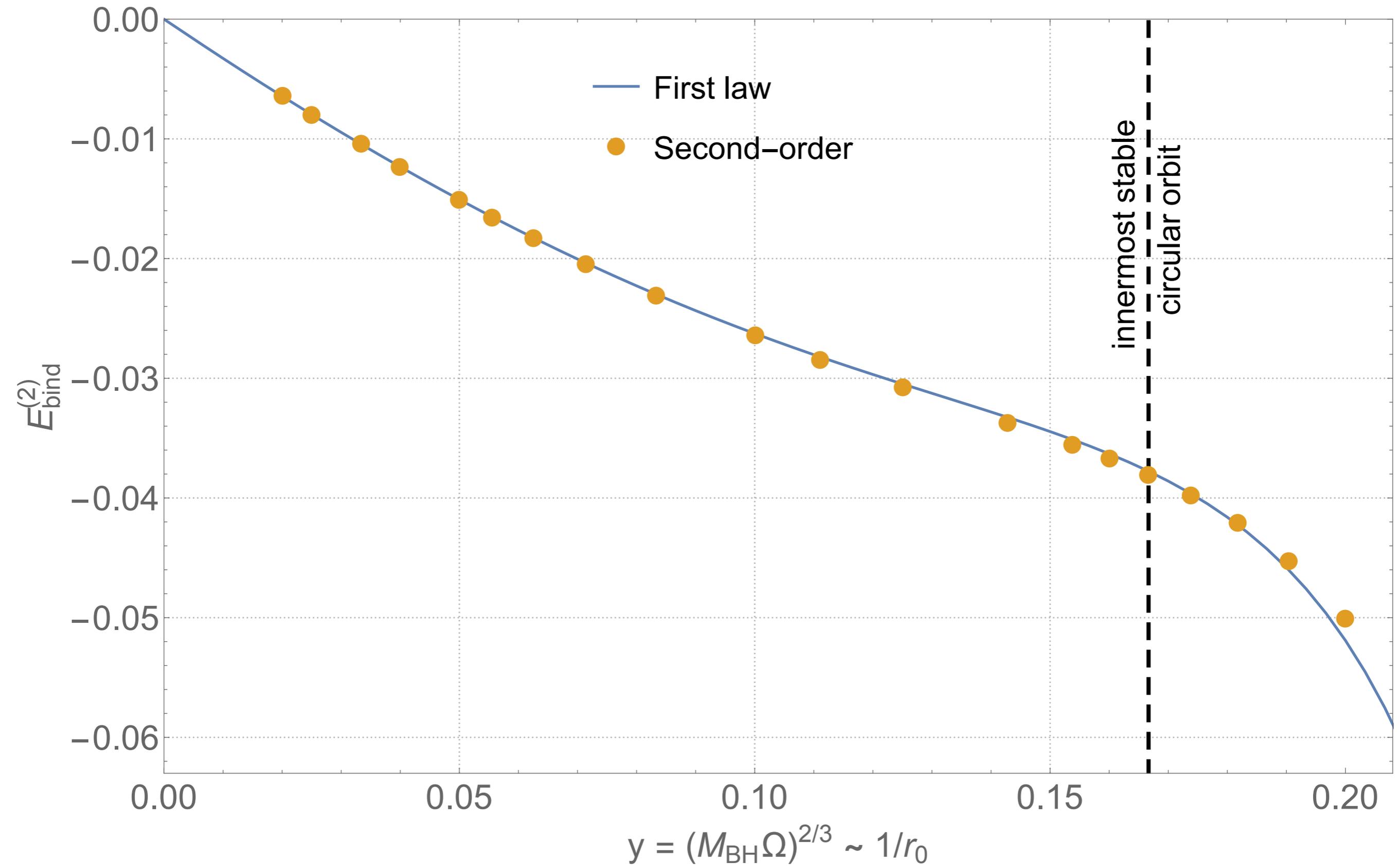
$$E_{SF}^{\text{1st law}} = \frac{1}{2} z_{SF}(y) - \frac{y}{3} \frac{dz_{SF}}{dy} - 1 + \sqrt{1 - 3y} + \frac{y}{6} \frac{5 - 12y}{(1 - 3y)^{3/2}}$$

Just as the energy balance law relates second-order fluxes to the first-order local SF, the FLBM relates a binary's second-order energy, as defined at infinity, to the first-order, local Detweiler redshift z_{SF} .

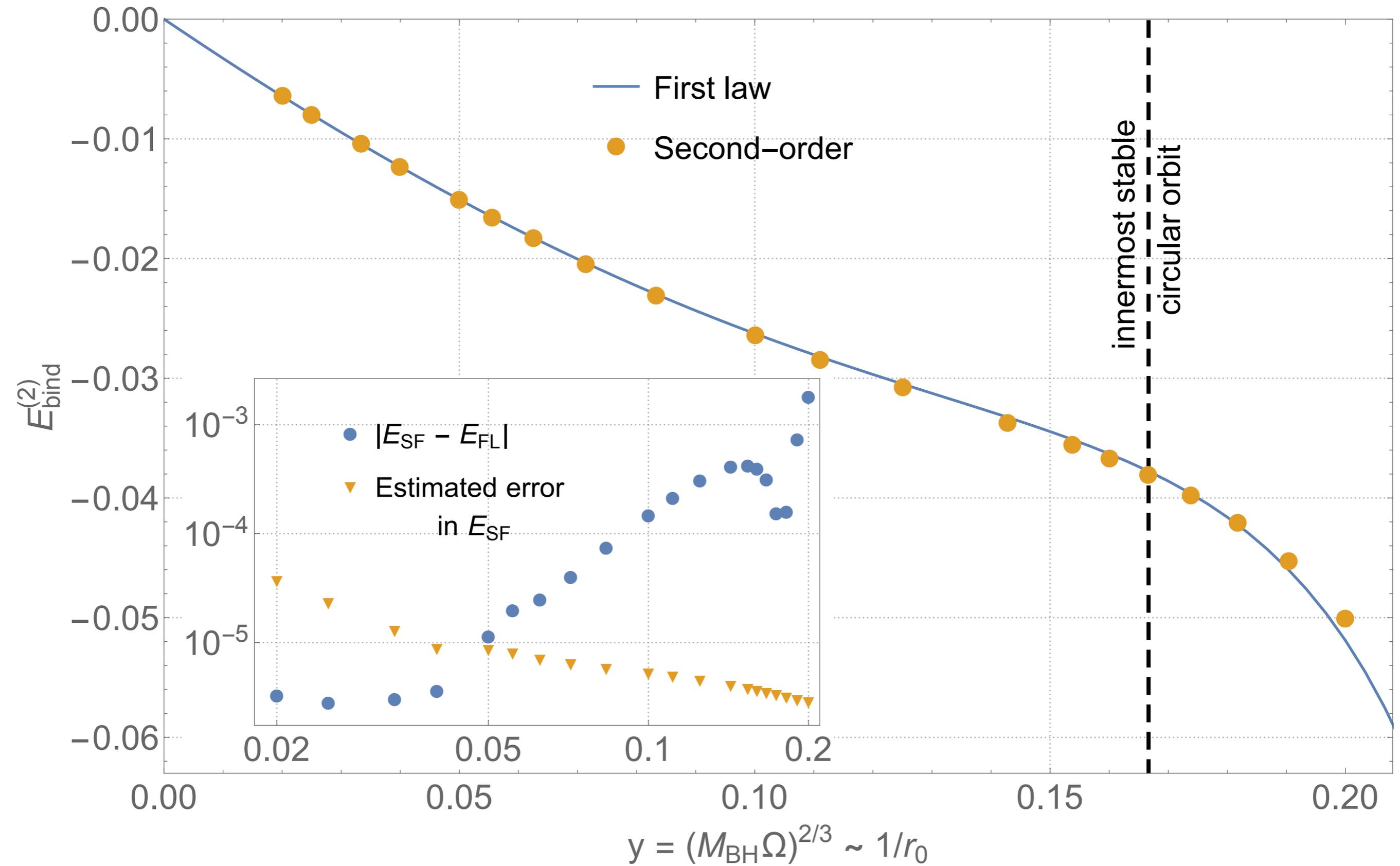
Important to note that the first-law assumes a conservative, helically symmetric, asymptotically flat spacetime.

*Le Tiec, Barausse, Buonanno, Phys. Rev. Lett. 108:131103 (2012)

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- Compact: only defined near worldline

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- Derivatives w.r.t. slow time:

$$\frac{dh^1}{d\tilde{t}} \sim \frac{dr_0}{d\tilde{t}} \frac{dh^1}{dr_0} \sim \left(\frac{dE}{dr_0} \right)^{-1} \dot{\mathcal{E}} \frac{dh^1}{dr_0}$$

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Apply retarded boundary conditions

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Explicitly
given by:

$$\begin{aligned} G_{\omega}^2[h, h] \equiv & -\frac{1}{2} h^{\mu\nu} (2h_{\mu(\alpha;\beta)\nu} - h_{\alpha\beta;\mu\nu} - h_{\mu\nu;\alpha\beta}) \\ & + \frac{1}{4} h^{\mu\nu}{}_{;\alpha} h_{\mu\nu;\beta} + \frac{1}{2} h^{\mu}{}_{\beta}{}^{;\nu} (h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}) \\ & - \frac{1}{2} \bar{h}^{\mu\nu}{}_{;\nu} (2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}) \end{aligned}$$

Challenges constructing this term:

- divergent at the particle: as $(\Delta r)^{-2}$ in Lorenz gauge
- mode coupling with finite number of first-order modes
- numerical noise near the horizon
- decays too slowly at the horizon and infinity

Non-linear perturbation theory: behaviour at the particle

mode decomposition

$$h^1 \sim (\Delta r)^{-1}$$

$$\implies h_{lm}^1 \sim \text{const} + \Delta r + \dots$$

$$h_{lm}^1(r_0) = h_{lm}^{ret1}(r_0) - h_{lm}^{P1}(r_0)$$

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Must use effective-source regularization

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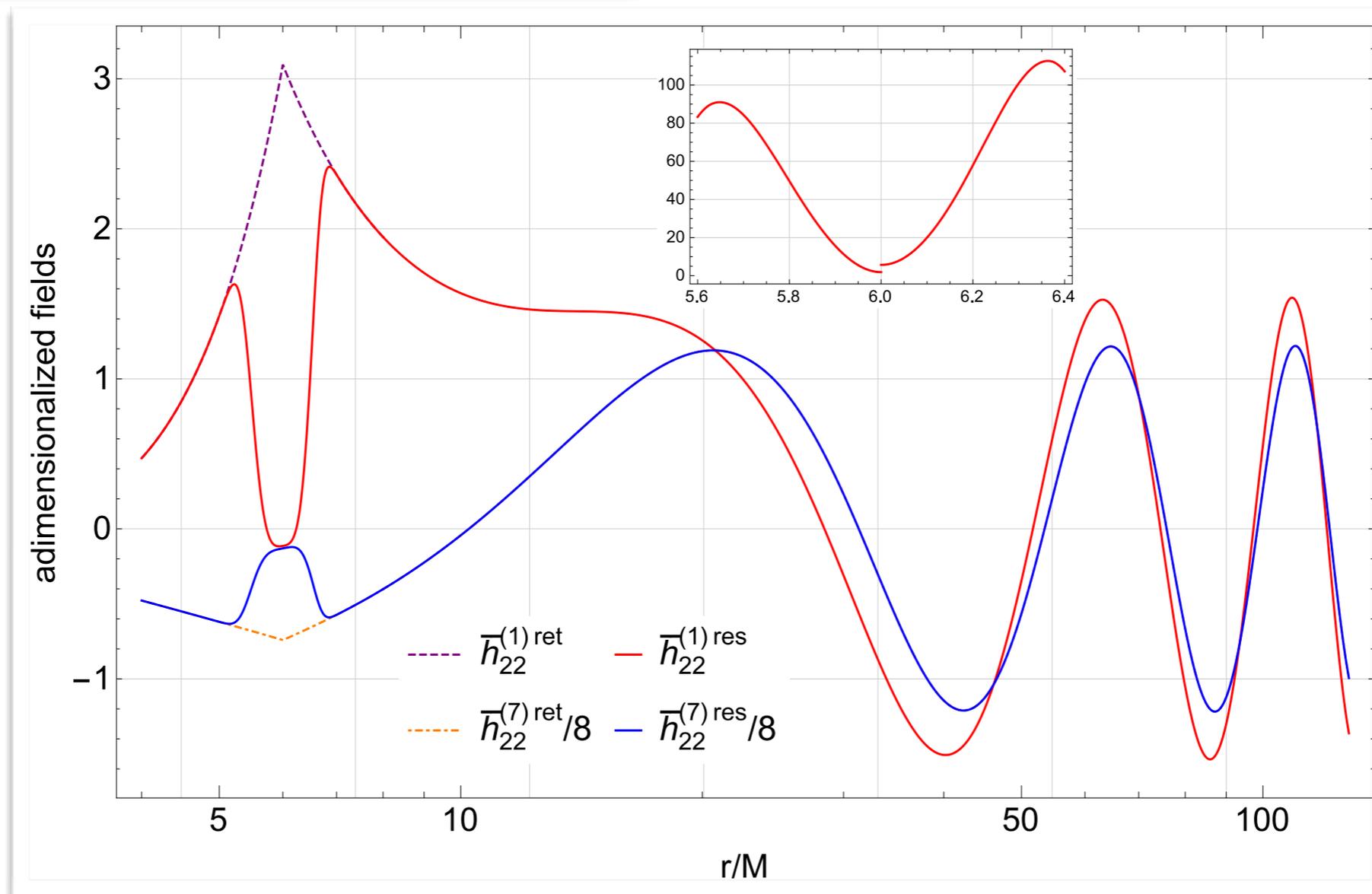
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Non-linear perturbation theory: mode coupling

$$G_{\mu\nu}^2[h^1, h^1] = \sum_{ilm} G_{ilm}(r; r_0) e^{-im\Omega t} Y_{\mu\nu}^{ilm}(r, \theta^A)$$

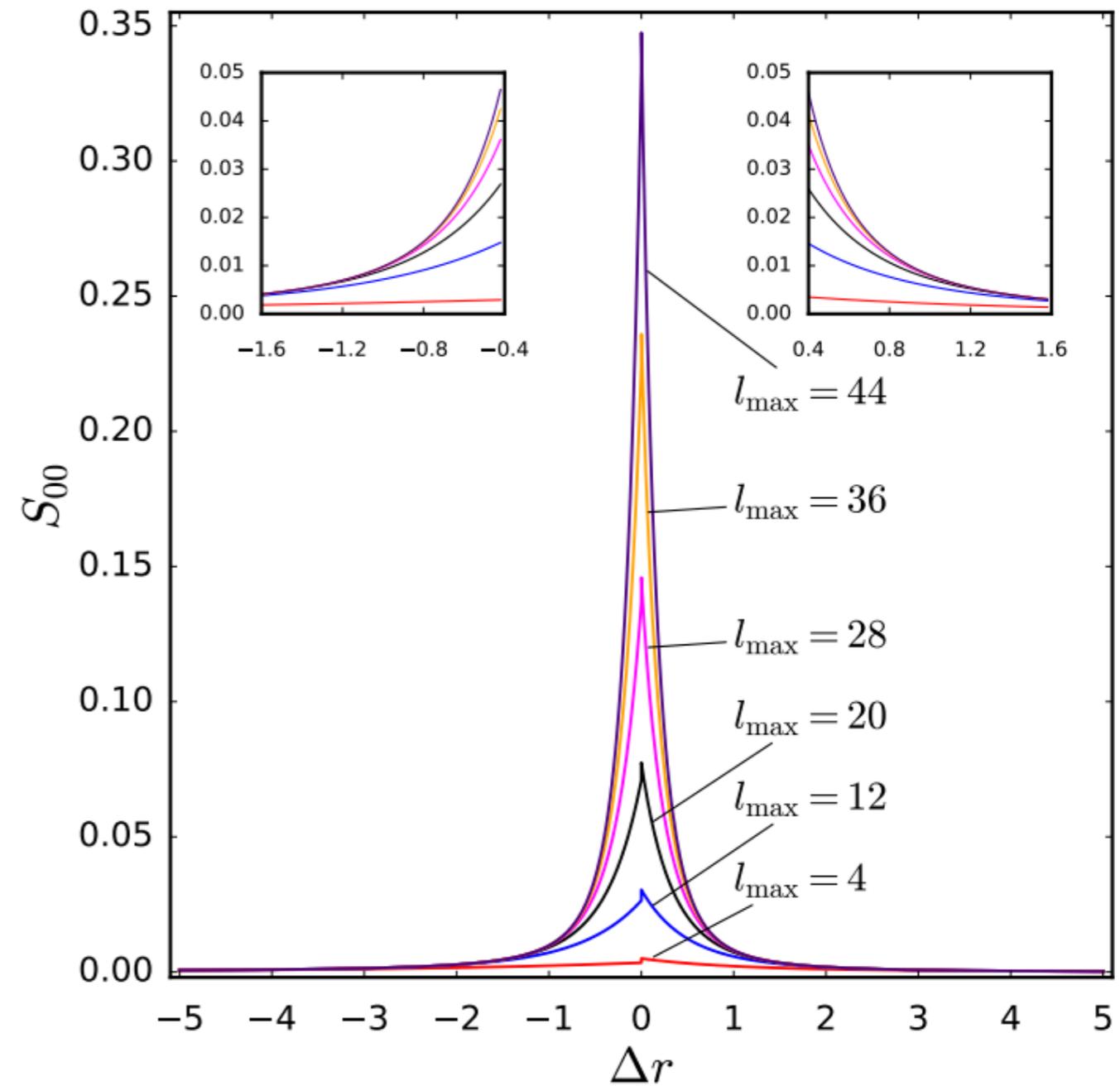
$$G_{ilm}^2 = \sum_{\substack{i'l'm' \\ i''l''m''}} \mathcal{D}_{ilm}^{i'l'm'i''l''m''} [h_{i'l'm'}^1, h_{i''l''m''}^1]$$

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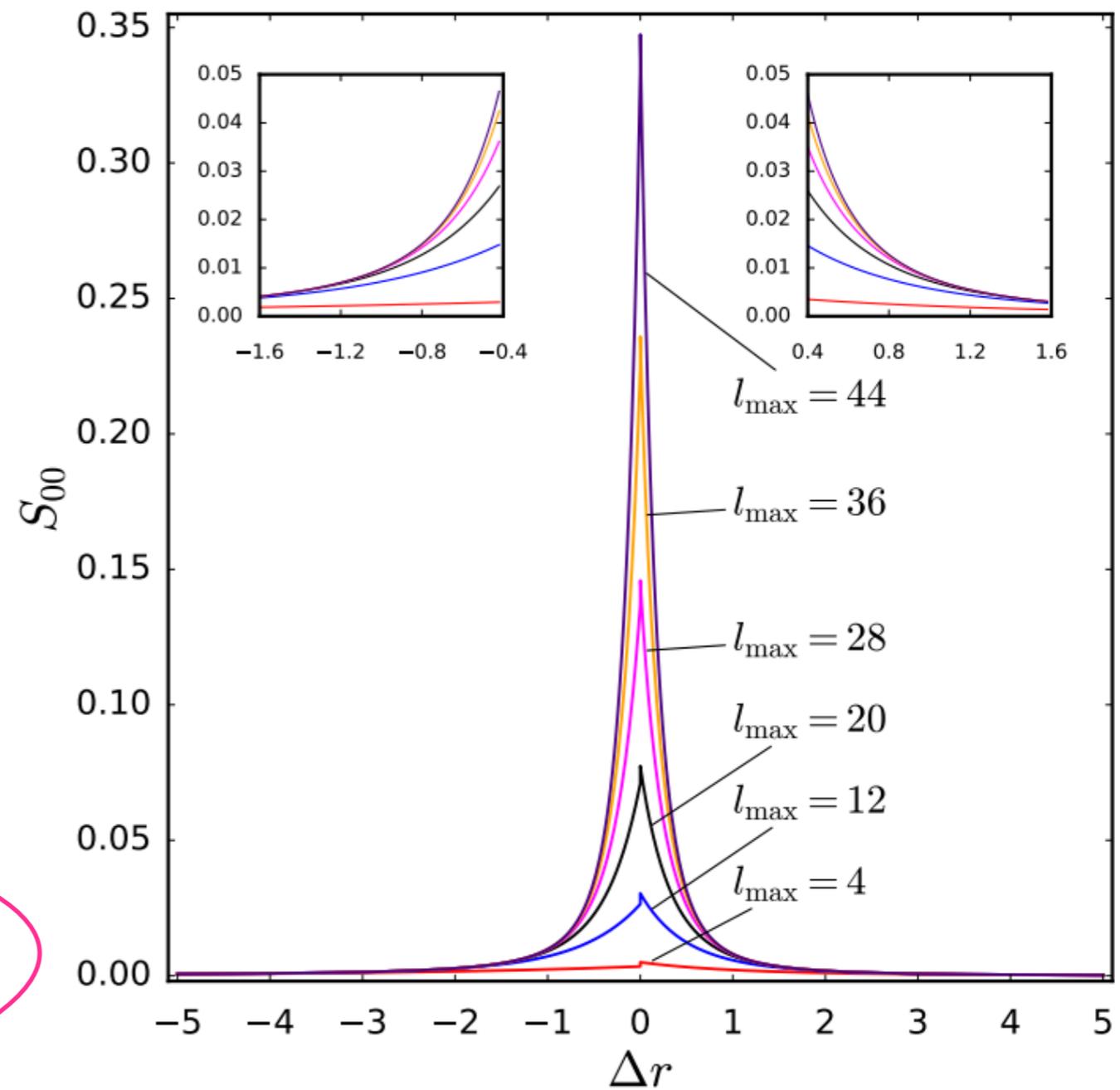
Solution:

$$h^1 = h^{R1} + h^{P1}$$

$$G^2[h^1, h^1] = G^2[h^{P1}, h^{P1}] + 2G^2[h^{P1}, h^{R1}] + G^2[h^{R1}, h^{R1}]$$

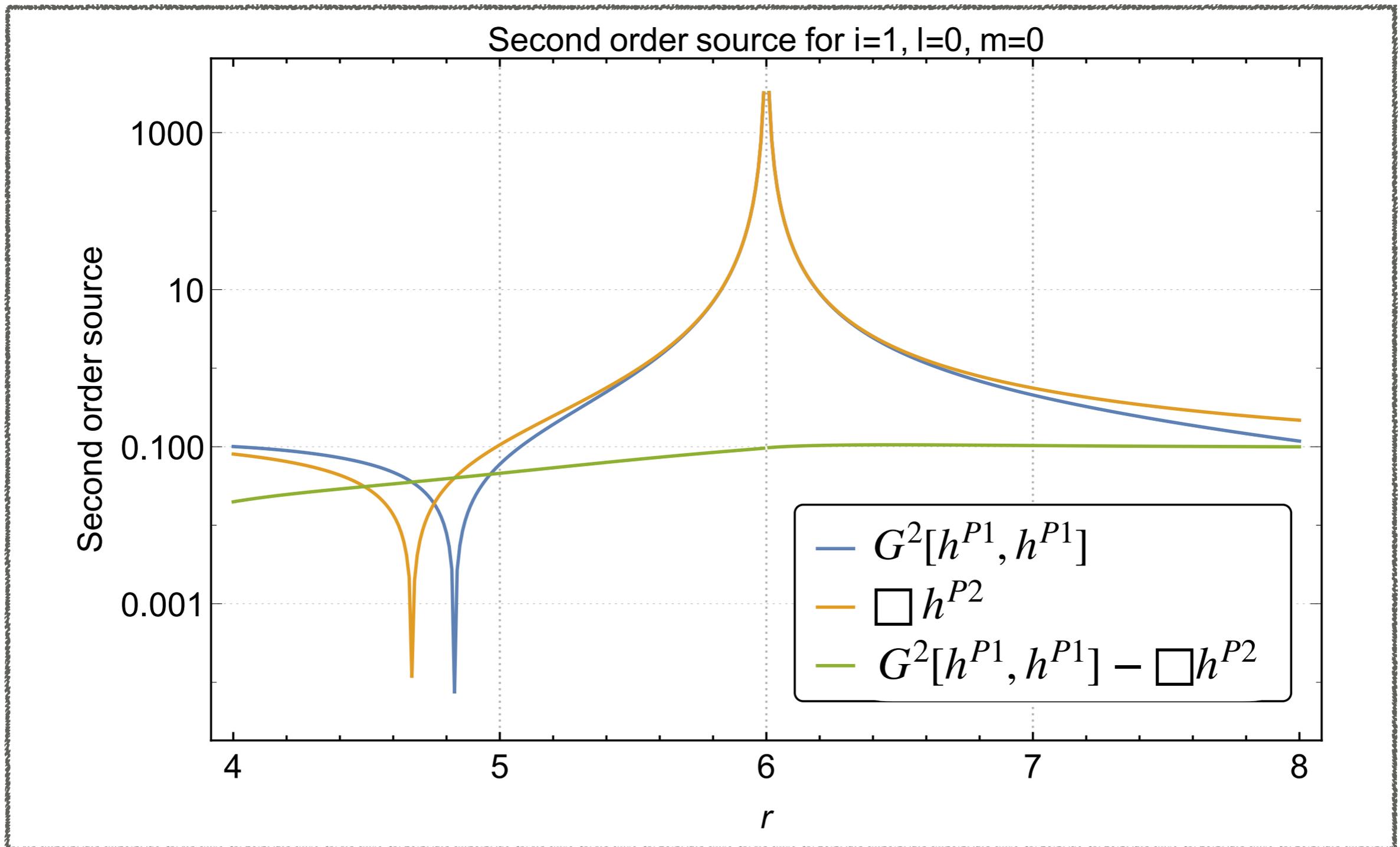
Can tackle this term analytically

(but have to use **exact** mode decomposition of h^{P1} for other terms done using mode coupling - no shortcuts)



Non-linear perturbation theory

effective source near the worldline



Many details now set out in 10+ papers:

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Nonlinear gravitational self-force. I. Field outside a small body

Adam Pound¹

¹ *School of Mathematics, University of Southampton, Southampton, United Kingdom, SO17 1BJ*
(Dated: September 5, 2012)

A small extended body moving through an external spacetime $g_{\alpha\beta}$ creates a metric perturbation $h_{\alpha\beta}$, which forces the body away from geodesic motion in $g_{\alpha\beta}$. The foundations of this effect, called the gravitational self-force, are now well established, but concrete results have mostly been limited to linear order. Accurately modeling the dynamics of compact binaries requires proceeding to nonlinear orders. To that end, I show how to obtain the metric perturbation outside the body

Nonlinear gravitational self-force: second-order equation of motion

Adam Pound

*Mathematical Sciences and STAG Research Centre,
University of Southampton, Southampton, United Kingdom, SO17 1BJ*
(Dated: May 25, 2017)

When a small, uncharged, compact object is immersed in an external background spacetime, at zeroth order in its mass it moves as a test particle in the background. At linear order, its own gravitational field alters the geometry around it, and it moves instead as a test particle in a certain effective metric satisfying the linearized vacuum Einstein equation. In the letter [Phys. Rev. Lett. 109, 051101 (2012)], using a method of matched asymptotic expansions, I showed that the same statement holds true at second order: if the object's leading-order spin and quadrupole moment vanish, then through second order in its mass it moves on a geodesic of a certain smooth, locally causal vacuum metric defined in its local neighbourhood. Here I present the complete details of the derivation of that result. In addition, I extend the result, which had previously been derived in

Many details now set out in 10+ papers:

Nonlinear gravitational self-force. I. Field outside a small body

Adam Pound¹
¹School of Mathematics, University of Southampton, Southampton, United Kingdom, SO17 1BJ
(Dated: September 5, 2012)

A body moving through an external spacetime $g_{\alpha\beta}$ creates a metric perturbation $h_{\alpha\beta}$. The foundations of this effect, known as the gravitational self-force, are now well established, but concrete results have mostly been derived for a test particle in a certain background spacetime. In this paper, we show how to obtain the metric perturbation outside the body

Nonlinear gravitational self-force: second-order equation of motion

Adam Pound
Mathematical Sciences and STAG Research Centre,
University of Southampton, Southampton, United Kingdom, SO17 1BJ
(Dated: May 25, 2017)

A compact object is immersed in an external background spacetime, at a test particle in the background. At linear order, its own metric perturbation $h_{\alpha\beta}$ is small, and it moves instead as a test particle in a certain background spacetime. In the letter [Phys. Rev. Lett. 108, 111101 (2012)], I showed that the same asymptotic expansions, I showed that the same leading-order spin and quadrupole moment geodesic of a certain smooth, locally present the complete details of which have previously been derived in

A practical, covariant puncture for second-order self-force

Adam Pound and Jeremy Miller
Mathematical Sciences, University of Southampton, Southampton, United Kingdom
(Dated: May 5, 2014)

Accurately modeling an extreme-mass-ratio inspiral requires knowledge of the second-order gravitational self-force on the inspiraling small object. Recently, numerical puncture schemes formulated to calculate this force, and their essential analytical ingredients have been derived from first principles. However, the *puncture*, a local representation of the small object's self-field, in each of these schemes has been presented only in a local coordinate system centered on the small object, while a numerical implementation will require the puncture in coordinates covering the entire numerical domain. In this paper we provide an explicit covariant self-field as a local expansion in

Conservative effect of the second-order gravitational self-force on quasicircular orbits in Schwarzschild spacetime

Adam Pound
Mathematical Sciences, University of Southampton, Southampton, United Kingdom, SO17 1BJ
(Dated: October 14, 2014)

A compact object moving on a quasicircular orbit about a Schwarzschild black hole gradually spirals inward due to the dissipative action of its gravitational self-force. But in addition to driving the inspiral, the self-force has a conservative piece. Within a second-order self-force formalism, I derive a second-order generalization of Detweiler's redshift variable, which provides a gauge-invariant measure of conservative effects on quasicircular orbits. I sketch a frequency-domain numerical

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A body moving through an external spacetime $g_{\alpha\beta}$ creates a metric perturbation. The foundations of this effect, which are now well established, by modeling the dynamics of the body away from geodesic motion in $g_{\alpha\beta}$ show how to obtain the results have mostly been outside the body requires proceeding

second-order equation of motion

Second-order perturbation theory: problems on large scales

Adam Pound

Mathematical Sciences, University of Southampton, Southampton, United Kingdom, SO17 1BJ
(Dated: October 20, 2015)

In general-relativistic perturbation theory, a point mass accelerates away from geodesic motion due to its gravitational self-force. Because the self-force is small, one can often approximate the motion as geodesic. However, it is well known that self-force effects accumulate over time, making the geodesic approximation fail on long timescales. It is less well known that this failure at large times translates to a failure at large distances as well. At second perturbative order, two large-distance pathologies arise: spurious secular growth and infrared-divergent retarded integrals. Both stand in the way of practical computations of second-order self-force effects.

Utilizing a simple flat-space scalar toy model, I develop methods to overcome these obstacles. The

A practical, covariant puncture for second-order self-force

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Applying the effective-source approach to frequency-domain self-force calculations

Niels Warburton¹ and Barry Wardell^{2,1}

¹School of Mathematical Sciences and Complex & Adaptive Systems Laboratory, University College Dublin, Belfield, Dublin 4, Ireland

²Department of Astronomy, Cornell University, Ithaca, NY 14853, USA

(Dated: 28th October 2014)

The equations of motion of a point particle interacting with its own field are defined in terms of a certain regularized self-field. Two of the leading approaches to solving these equations are the mode-sum and effective-source approaches. We generalize these approaches by generalizing traditional frequency-domain self-force calculations to the effective-source approach. For a toy scalar-field puncture field from which the regularized residual field can be calculated. In addition to its application in our effective-

Applying the effective-source approach to Lorenz-gauge self-force

Barry Wardell

¹Department of Astronomy, Cornell University, Ithaca, NY 14853, USA

²School of Mathematical Sciences and Complex & Adaptive Systems Laboratory, University College Dublin, Belfield, Dublin 4, Ireland

³MIT Kavli Institute for Astrophysics and Space Research,

Massachusetts Institute of Technology, Cambridge, MA 02139, USA

With a view to developing a formalism that will be applicable at second perturbative order, we devise a new practical scheme for computing the gravitational self-force experienced by a point mass moving in a curved background spacetime. Our method works in the frequency domain and employs the effective-source approach, in which a distributional source for the retarded metric perturbation is replaced with an effective source for a certain regularized self-field. A key ingredient of the calculation is the analytic determination of an appropriate puncture field from which the effective source and regularized residual field can be calculated. In addition to its application in our effective-

Second-order perturbation theory: the problem of infinite mode coupling

Jeremy Miller,¹ Barry Wardell,^{2,3} and Adam Pound¹

¹Mathematical Sciences and STAG Research Centre, University of Southampton, Southampton, SO17 1BJ, United Kingdom

²School of Mathematical Sciences and Complex & Adaptive Systems Laboratory, University College Dublin, Belfield, Dublin 4, Ireland

³Department of Astronomy, Cornell University, Ithaca, NY 14853, USA

(Dated: August 25, 2016)

Second-order self-force computations, which will be essential in modeling extreme-mass-ratio inspirals, involve two major new difficulties that were not present at first order. One is the problem of large scales, discussed in [Phys. Rev. D 92, 104047 (2015)]. Here we discuss the second difficulty, which occurs instead on small scales: if we expand the field equations in spherical harmonics, then because the first-order field contains a singularity, we require an arbitrarily large number of first-order modes to accurately compute even a single second-order mode. This is a generic feature

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A practical, covariant puncture for second-order self-force calculations

Adam Pound and Jeremy Miller

Second-order self-force calculation of gravitational binding energy

Adam Pound,¹ Barry Wardell,² Niels Warburton,² and Jeremy Miller¹

¹*School of Mathematics and STAG Research Centre, University of Southampton, Southampton, United Kingdom, SO17 1BJ*

²*School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland*
(Dated: June 30, 2018)

Self-force theory is currently the leading method of modeling extreme-mass-ratio inspirals (EMRIs), key sources for the gravitational wave detectors LISA. It is well known that for an accurate EMRI model, *second-order self-force* calculations are essential. However, these calculations are beset by obstacles. In this letter, we present a new method for computing second-order self-force effects on quasicircular orbits about a Schwarzschild black hole. As a demonstration, we calculate the gravitational binding energy of these binaries.

In preparation

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Applying the effective-source approach to Lorenz-gauge gauge

Barry Wardell

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With a view to developing a formalism that will be applicable at second perturbative order, we devise a new practical scheme for computing the gravitational self-force experienced by a point mass moving in a curved background spacetime. Our method works in the frequency domain and employs the effective-source approach, in which a distributional source for the retarded metric perturbation is replaced with an effective source for a certain regularized self-field. A key ingredient of the calculation is the analytic determination of an appropriate puncture field from which the effective source and regularized residual field can be calculated. In addition to its application in our effective-

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Second-order perturbation theory: problems on large scales

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In general-relativistic perturbation theory, the motion of a small body due to its gravitational self-force is modeled as geodesic. However, in the geodesic approximation, the self-force is neglected. This approximation fails at large distances, where distance pathologies arise: spur

Two-timescale evolution of the perturbed Einstein equations: method for calculating the second-order self-force

Jeremy Miller¹ and Adam Pound¹

¹Mathematical Sciences and STAG Research Centre,
University of Southampton, Southampton, SO17 1BJ, United Kingdom
(Dated: April 29, 2019)

The gravitational self-force accelerates an object away from geodesic motion, as it radiates energy away through gravitational waves. Second-order self-force calculations are crucial for accurate models of gravitational waves. A computation of the second-order self-force requires knowledge of how the object evolves slowly - self-force effects are substantial only over the inspiral timescale but negligible during a single orbit. A technique for describing the inspiral accommodating both short-term and long-term influences is the *two timescale expansion*, using a *fast time* and a *slow time* variable to characterize the two different timescales.

In preparation

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Adam Pound,¹ Barry Wardell,² Niels Warburton,² and Jeremy Miller¹

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Tensor-harmonic decomposition of the second-order Einstein equation in Schwarzschild spacetime

Adam Pound

Mathematical Sciences and STAG Research Centre,
University of Southampton, Southampton, SO17 1BJ, United Kingdom

(Dated: October 22, 2018)

High-accuracy gravitational-wave modeling demands going beyond traditional, first-order perturbation theory. However, practical spacetimes have thus far been particularly motivated by the need for second-order self-force. I present a practical tensor-harmonic decomposition of the second-order Einstein equations in a Schwarzschild background. The results are valid in any gauge, but I give particular emphasis to the Lorenz gauge.

In preparation

Second-order self-force calculation of gravitational binding energy

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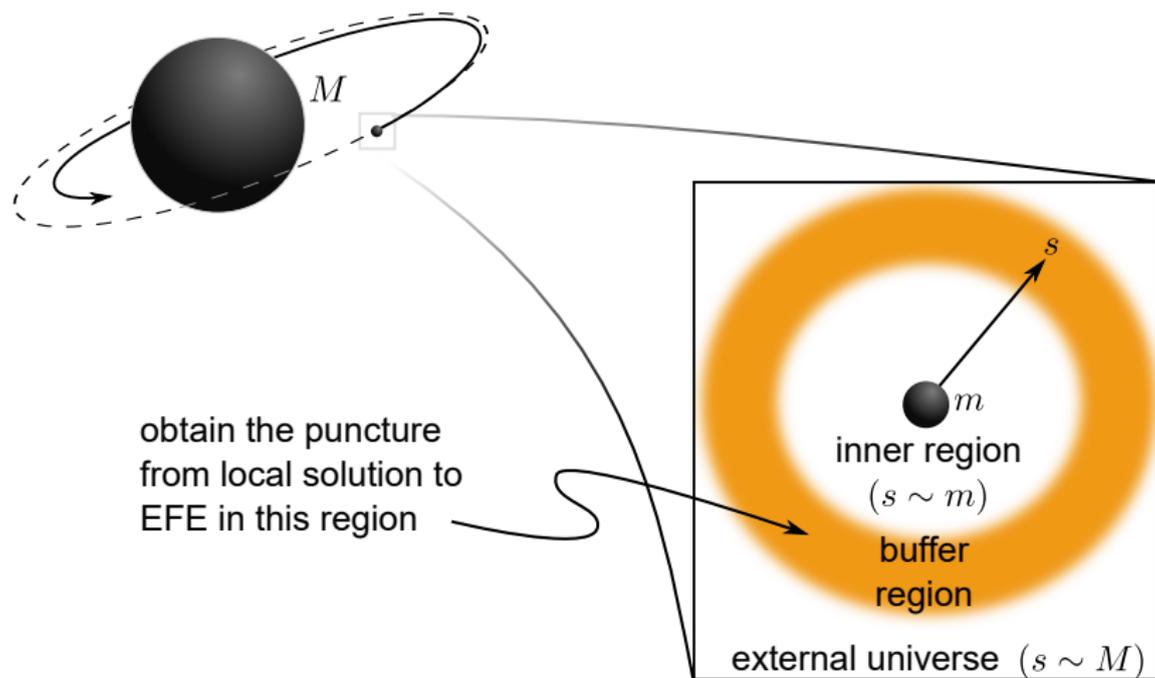
$$\square_{\omega}^0 h_{\omega}^{R2} = G_{\omega}^2[h^1, h^1] - \square_{\omega}^0 h_{\omega}^{P2} - \square_{\omega}^1 h_{\omega}^1$$

with retarded boundary conditions

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Basic idea



Form of puncture

- In most gauges, the puncture has the form

$$h_{\mu\nu}^{P1} \sim \frac{m}{s} + O(s^0)$$
$$h_{\mu\nu}^{P2} \sim \frac{m^2}{s^2} + \frac{mh^{R1}}{s} + O(s^0)$$

- h^{P2} is available in Lorenz gauge, in arbitrary spacetime, through order s^1 , in covariant form [Pound and Miller 2014]
- Also available, in less ready-to-use form, in “P smooth” gauges [Gralla 2012] and “highly regular” gauges [Pound 2017]

Expansion of worldline

- The worldline z^μ is where the puncture diverges
- Quasicircular orbit: $z^\alpha = (t, r_p(\tilde{t}, \epsilon), \pi/2, \phi_p(\tilde{t}, \epsilon))$, where

$$r_p = r_0(\tilde{t}) + \epsilon r_1(\tilde{t}) + \dots$$

$$\frac{d\phi_p}{dt} := \Omega = \Omega_0(\tilde{t}) + \epsilon \Omega_1(\tilde{t}) + \dots$$

- Plug expansion into the puncture:

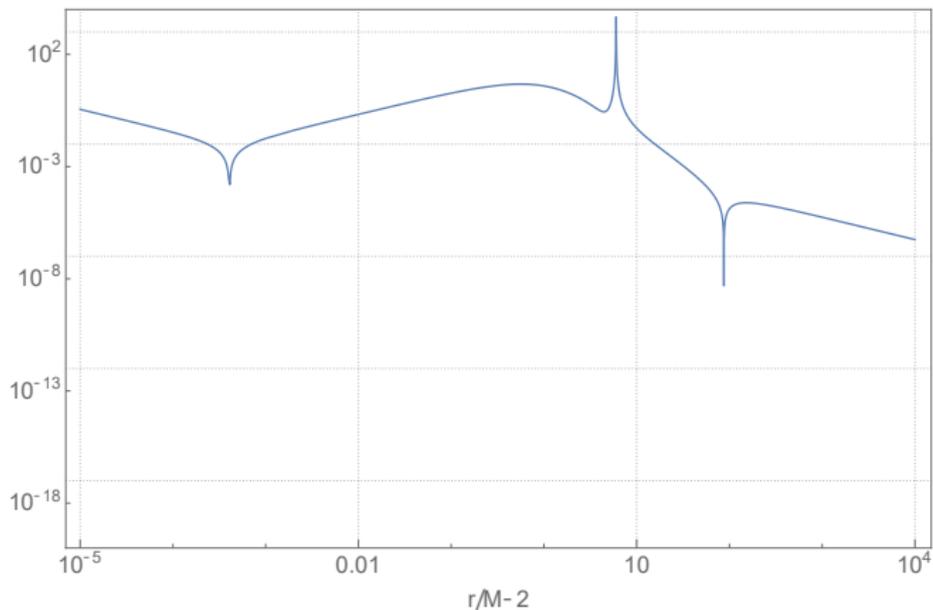
$$\Rightarrow h^{P2} \rightarrow h^{P2} + \frac{\delta h^{P1}}{\delta r_p} r_1 + \frac{\delta h^{P1}}{\delta \dot{r}_p} \dot{r}_0 + \frac{\delta h^{P1}}{\delta \Omega} \Omega_1$$

- The two-timescale puncture is also available in covariant form for generic bound orbits

Demonstration in Lorenz gauge

For quasicircular orbits in Schwarzschild,

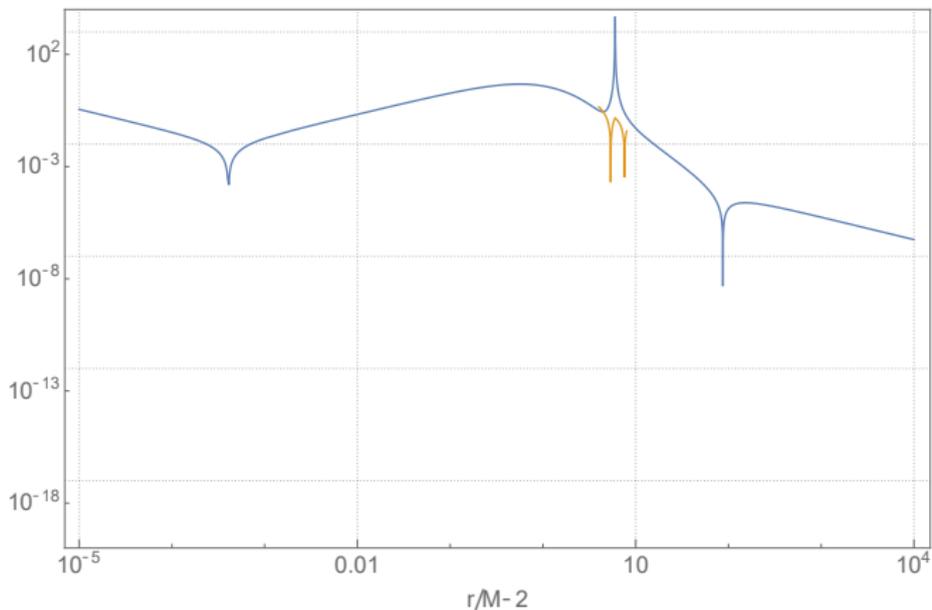
$$h_{\omega lm}^{P2} \sim m^2 \log|r - r_0| + (mh^{\mathcal{R}1} + r_1 + \dot{r}_0)|r - r_0|$$



Demonstration in Lorenz gauge

For quasicircular orbits in Schwarzschild,

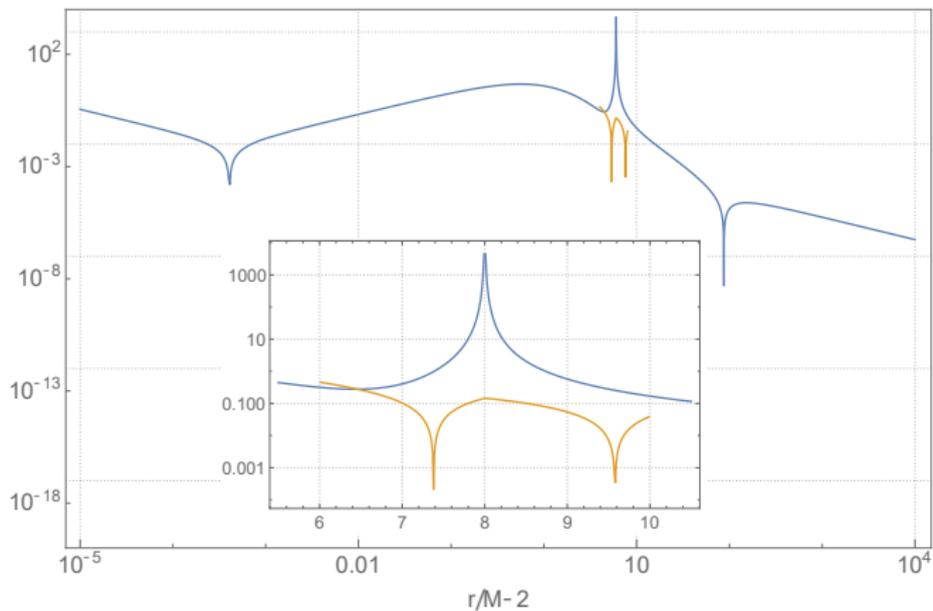
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Extension to other gauges or variables

- Almost all calculations of first-order self-force have been built on Lorenz-gauge $h_{\mu\nu}^{S1}$, even if they calculate the retarded field in a different gauge
- Can do the same at second order:
 - ▶ work in locally Lorenz gauge: $h_{\mu\nu}^{P2}$ in Lorenz, $h_{\mu\nu}^{R2}$ in whatever's convenient
 - ▶ transform from $h_{\mu\nu}^{P2,Lor}$ to some desired gauge
 - ▶ calculate punctures for Teukolsky (or Regge-Wheeler, Zerilli-Moncrief, etc.) variables: $\psi^{P2} = \psi[h^{P2,Lor}]$

$$\square_{\omega}^0 h_{\omega}^{R2} = G_{\omega}^2[h^1, h^1] - \square_{\omega}^0 h_{\omega}^{P2} - \square_{\omega}^1 h_{\omega}^1$$

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Computing parametric derivative of h^1

- $\square_\omega^1 h_\omega^1 \sim \partial_{\tilde{t}} h_\omega^1 + \Omega_1 h_\omega^1$
- can convert to derivatives wrt parameters: e.g.,

$$h_\omega^1(\tilde{t}) = h_\omega^1(r_0(\tilde{t}), \delta M(\tilde{t}), \delta J(\tilde{t}))$$

$$\Rightarrow \partial_{\tilde{t}} h_\omega^1 = \frac{dr_0}{d\tilde{t}} \partial_{r_0} h_\omega^1 + \frac{d\delta M}{d\tilde{t}} \partial_{\delta M} h_\omega^1 + \frac{d\delta J}{d\tilde{t}} \partial_{\delta J} h_\omega^1$$

- to find $\partial_{r_0} h_\omega^1$, differentiate $\square_\omega^0 h_\omega^1 = T_\omega^1$ wrt r_0 :

$$\partial_{r_0}(\square_\omega^0 h_\omega^1) = \partial_{r_0} T_\omega^1$$

$$\Rightarrow \square_\omega^0(\partial_{r_0} h_\omega^1) = \partial_{r_0} T_\omega^1 - (\partial_{r_0} \square_\omega^0) h_\omega^1$$

- solve for $\partial_{r_0} h_\omega^1$ with your favorite frequency-domain method

Computing parametric derivative of h^1

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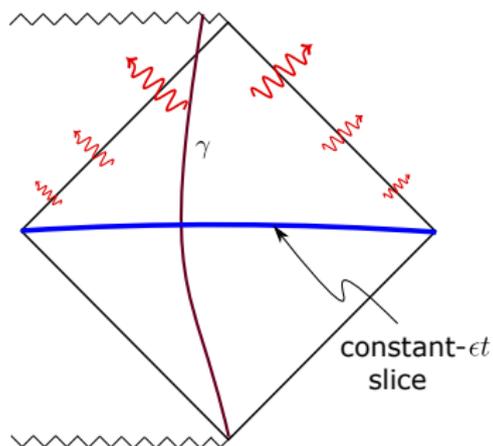
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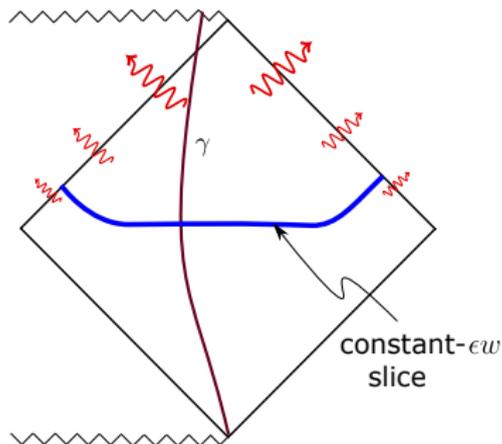
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Dependence on choice of slow time



$$h_{\omega}^1 \sim \frac{e^{\pm i\omega r^*}}{r} \text{ near bd'ries}$$

$$\Rightarrow \partial_{r_0} h_{\omega}^1 \sim \frac{i\omega' r^* e^{\pm i\omega r^*}}{r}$$

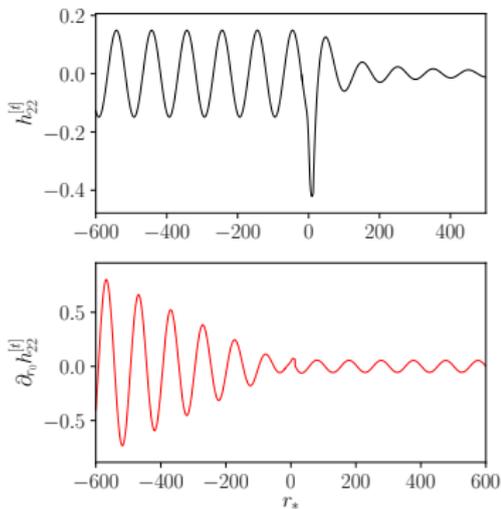


$$h_{\omega}^1 \sim \frac{A(r_0)}{r} \text{ near bd'ries}$$

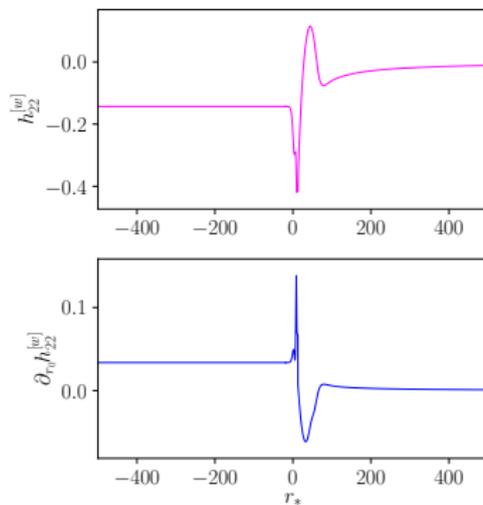
$$\Rightarrow \partial_{r_0} h_{\omega}^1 \sim \frac{A'(r_0)}{r}$$

Demonstration in Lorenz gauge

slow time ϵt



slow time ϵw



Extension to other gauges and variables

- Exactly the same idea applies
- e.g., two-timescale expansion of Teukolsky (or Zerilli-Moncrief, etc.) equation,

$$\hat{\square}_{\omega}^0 \psi_{\omega}^{R2} = S_{\omega}[h^1, h^1] - \hat{\square}_{\omega}^0 \psi_{\omega}^{P2} - \hat{\square}_{\omega}^1 \psi_{\omega}^1,$$

$$\square_{\omega}^0 h_{\omega}^{R2} = G_{\omega}^2[h^1, h^1] - \square_{\omega}^0 h_{\omega}^{P2} - \square_{\omega}^1 h_{\omega}^1$$

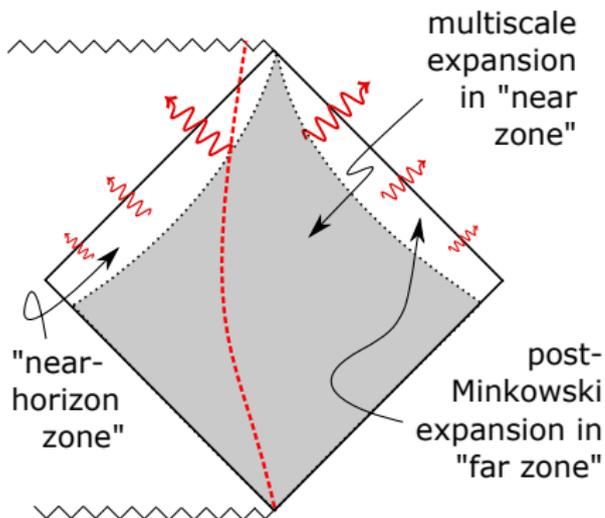
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Matched expansions

[Pound, Moxon, Flanagan, Hinderer, Yamada, Isoyama, Tanaka]



- We want to impose retarded BCs on full, physical metric perturbation
- This *does not* trivially imply simple BCs for two-timescale fields because the two-timescale expansion breaks down near horizon and infinity

- To find BCs for two-timescale fields, obtain other (time-domain) approximations near \mathcal{H}^+ and \mathcal{I}^+

Boundary conditions

- Use post-Minkowski expansion near \mathcal{I}^+ and a near-horizon expansion near \mathcal{H}^+
- Obtain solutions by using leading-order retarded (time domain) Green's functions
- post-Minkowski solution looks like

$$h^{PM} = h^{\mathcal{P},PM} + h^{\mathcal{R},PM}$$

$h^{\mathcal{P},PM}$ = particular solution, $h^{\mathcal{R},PM}$ = outgoing wave

- Analogous near horizon

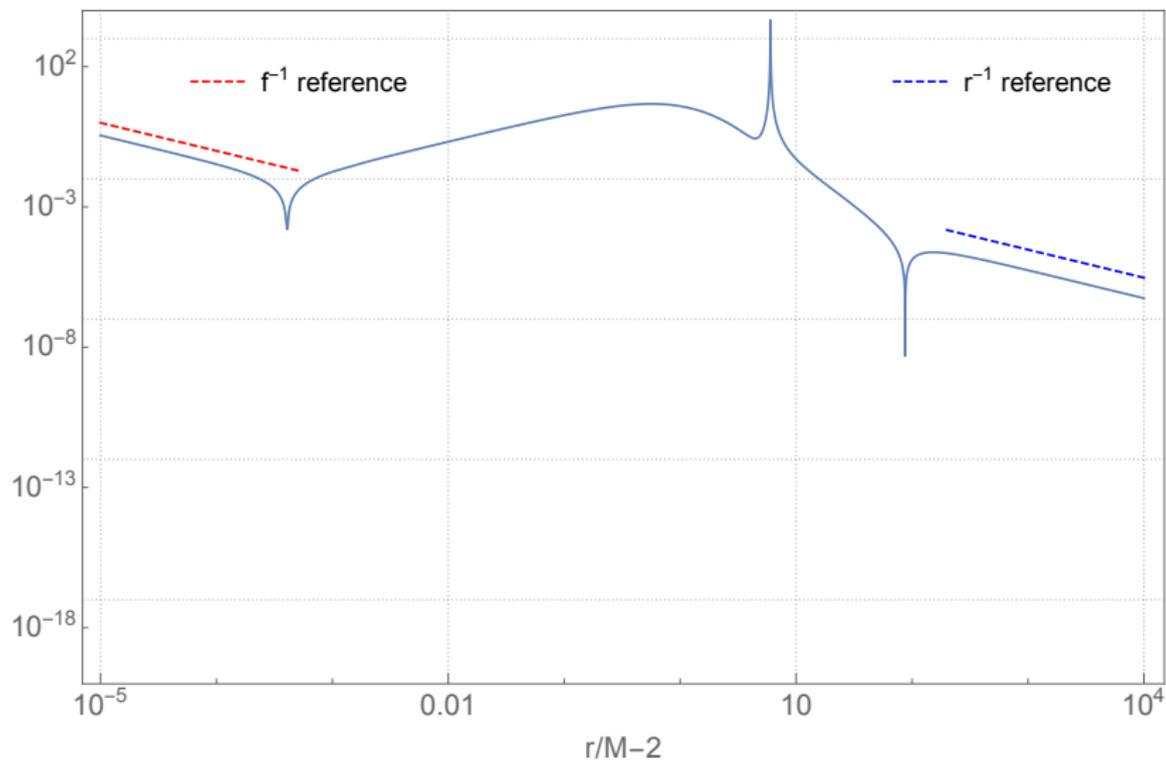
Boundary punctures

- BC for two-timescale solution h^{TT} is $\text{FZ}[h^{TT}] = \text{NZ}[h^{PM}]$
- To enforce BC, use $\text{NZ}[h^{\mathcal{P},PM}]$ as a puncture at large r , impose standard outgoing, retarded BC on residual field

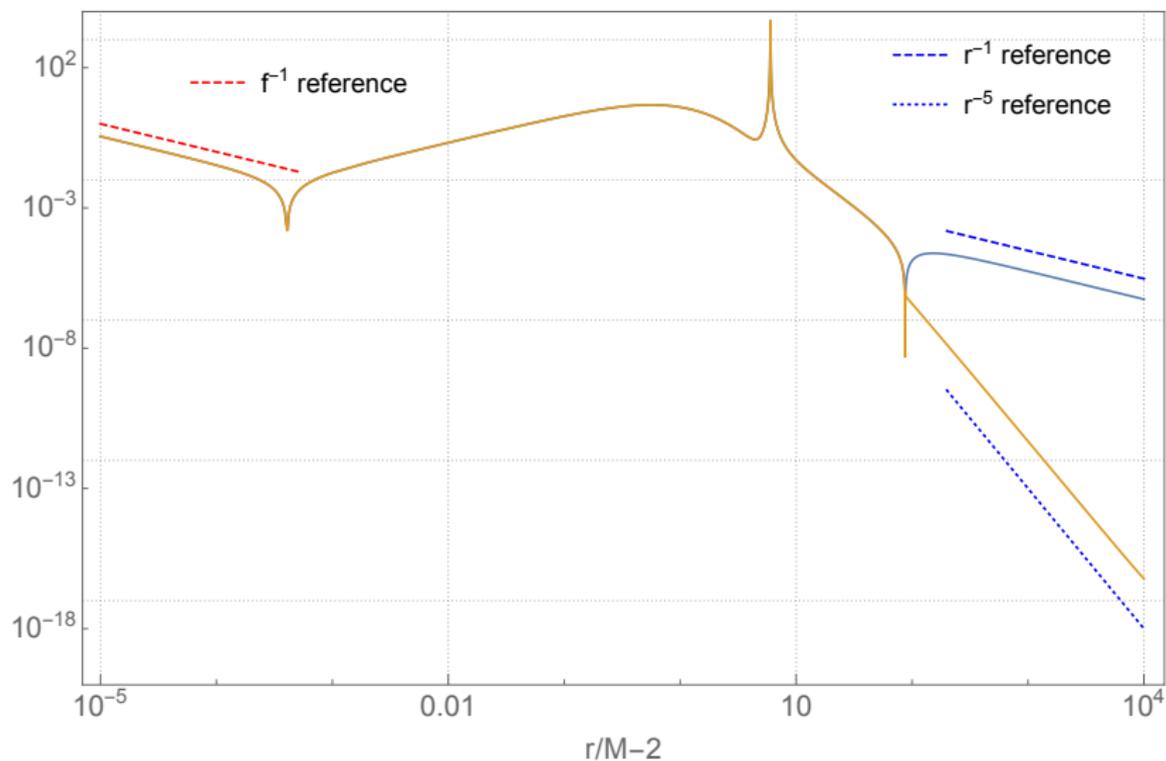
$$h^{\mathcal{R},TT} = h^{TT} - \text{NZ}[h^{\mathcal{P},PM}]$$

- Numerically computed $h^{\mathcal{R},TT}$ determines $h^{\mathcal{R},PM}$
- Analogous near horizon

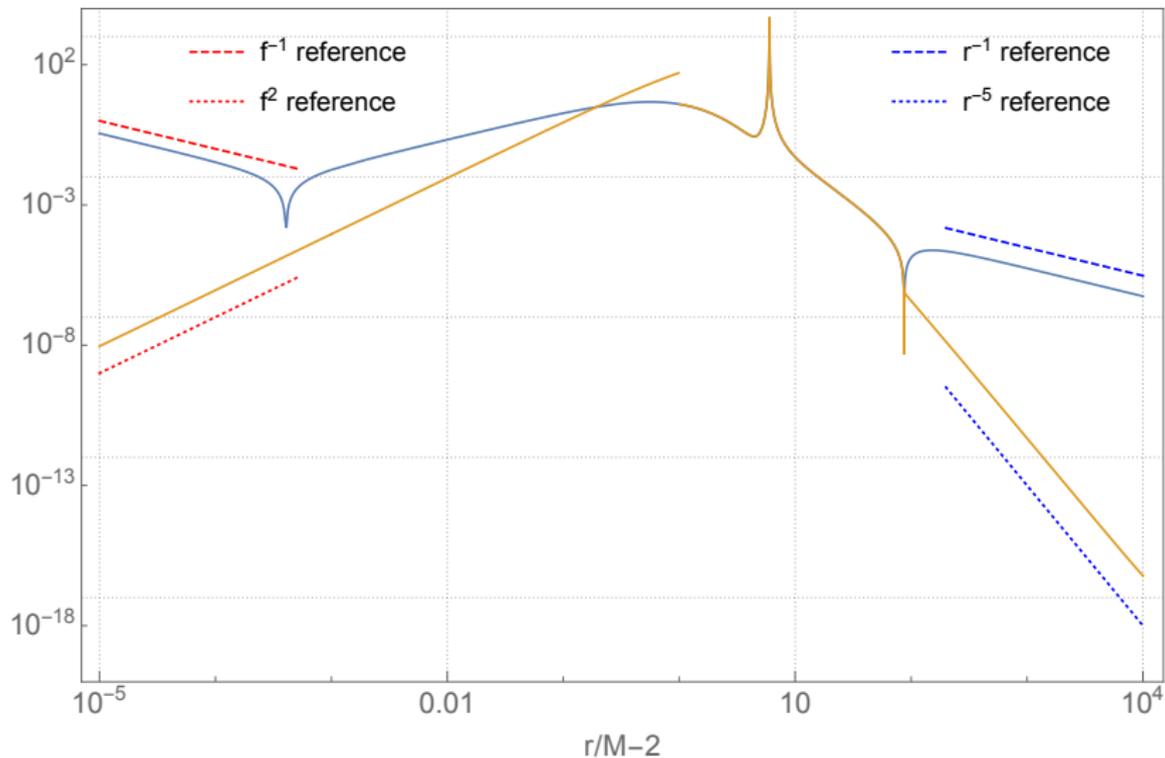
Demonstration in Lorenz gauge



Demonstration in Lorenz gauge



Demonstration in Lorenz gauge



Extension to other gauges and variables

- The boundary punctures may not be strictly necessary in some cases
 - ▶ nonstationary modes with \tilde{w} as slow time
 - ▶ in “nicer” gauges
- When needed, punctures can be obtained from Lorenz-gauge punctures
 - or directly, using same methods as in Lorenz gauge (obtaining approximate solutions in time domain near boundaries)

Reminder: quasicircular orbits

$$r_p = r_0(\tilde{t}) + \epsilon r_1(\tilde{t}) + O(\epsilon^2)$$

$$\Omega = \Omega_0(\tilde{t}) + \epsilon \Omega_1(\tilde{t}) + O(\epsilon^2)$$

$$\phi_p = \int \Omega dt = \frac{1}{\epsilon} \int (\Omega_0 + \epsilon \Omega_1 + \dots) d\tilde{t}$$

$$h \sim \sum_{nlm} \epsilon^n h_{\omega_m l m}^n(\tilde{t}, r) e^{-im\phi_p(\tilde{t})} Y_{lm}$$

$$(\omega_m = m\Omega_0)$$

Reminder: governing equations

$$\square h^1 = T^1$$

$$\square h^{R2} = G^2[h^1, h^1] - \square h^{P2}$$

$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon f_1^\mu[h^{R1}] + \epsilon^2 f_2^\mu[h^{R2}]$$

$$h^{P1} \sim \frac{m}{|x^\alpha - z^\alpha|}$$

$$h^{P2} \sim \frac{m^2}{|x^\alpha - z^\alpha|^2} + \frac{mh^{R1}}{|x^\alpha - z^\alpha|}$$

Reminder: governing equations

$$\square_{\omega}^0 h_{\omega}^1 = T_{\omega}^1$$

$$\square h^{R2} = G^2[h^1, h^1] - \square h^{P2}$$

$$\frac{D^2 z^{\mu}}{d\tau^2} = \epsilon f_1^{\mu}[h^{R1}] + \epsilon^2 f_2^{\mu}[h^{R2}]$$

$$h^{P1} \sim \frac{m}{|x^{\alpha} - z^{\alpha}|}$$

$$h^{P2} \sim \frac{m^2}{|x^{\alpha} - z^{\alpha}|^2} + \frac{mh^{R1}}{|x^{\alpha} - z^{\alpha}|}$$

Reminder: governing equations

$$\square_{\omega}^0 h_{\omega}^1 = T_{\omega}^1$$

$$\square_{\omega}^0 h_{\omega}^{R2} = G_{\omega}^2[h^1, h^1] - \square_{\omega}^0 h_{\omega}^{P2} - \square_{\omega}^1 h_{\omega}^1$$

$$\frac{D^2 z^{\mu}}{d\tau^2} = \epsilon f_1^{\mu}[h^{R1}] + \epsilon^2 f_2^{\mu}[h^{R2}]$$

$$h^{P1} \sim \frac{m}{|x^{\alpha} - z^{\alpha}|}$$

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Reminder: governing equations

$$\square_{\omega}^0 h_{\omega}^1 = T_{\omega}^1$$

$$\square_{\omega}^0 h_{\omega}^{R2} = G_{\omega}^2[h^1, h^1] - \square_{\omega}^0 h_{\omega}^{P2} - \square_{\omega}^1 h_{\omega}^1$$

$$\Omega_0 = \sqrt{M/r_0^3}$$

$$\frac{dr_0}{d\tilde{t}} \propto f_1^{\text{diss}}[h_{\omega}^{R1}]$$

$$\Omega_1 \propto f_1^{\text{cons}}[h_{\omega}^{R1}] + r_1$$

$$\frac{dr_1}{d\tilde{t}} \propto f_2^{\text{diss}}[h_{\omega}^{R2}]$$

$$h^{P1} \sim \frac{m}{|x^{\alpha} - z^{\alpha}|}$$

$$h^{P2} \sim \frac{m^2}{|x^{\alpha} - z^{\alpha}|^2} + \frac{mh^{R1}}{|x^{\alpha} - z^{\alpha}|}$$

Reminder: governing equations

$$\square_{\omega}^0 h_{\omega}^1 = T_{\omega}^1$$

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$$\Omega_0 = \sqrt{M/r_0^3}$$

$$\frac{dr_0}{d\tilde{t}} \propto f_1^{\text{diss}}[h_{\omega}^{R1}]$$

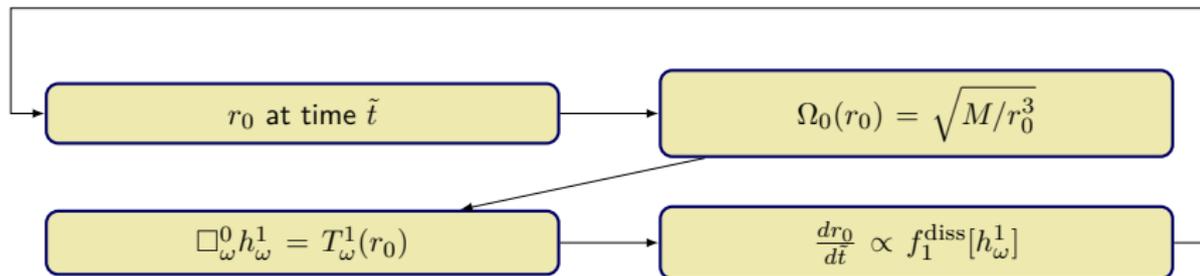
$$\Omega_1 \propto f_1^{\text{cons}}[h_{\omega}^{R1}] + r_1$$

$$\frac{dr_1}{d\tilde{t}} \propto f_2^{\text{diss}}[h_{\omega}^{R2}]$$

$$h_{\omega}^{P1} \sim \text{const.} + |r - r_0|$$

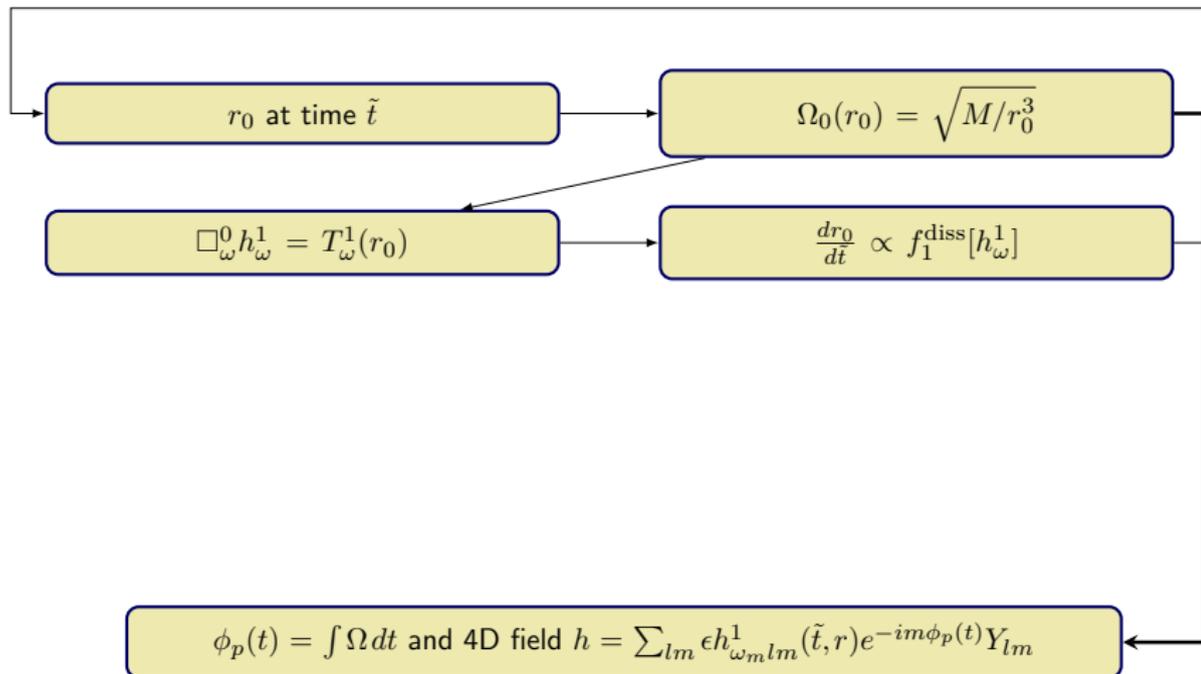
$$h_{\omega}^{P2} \sim m^2 \log|r - r_0| + \text{const.} + (mh^{R1} + r_1 + \dot{r}_0)|r - r_0|$$

Framework for computing a waveform

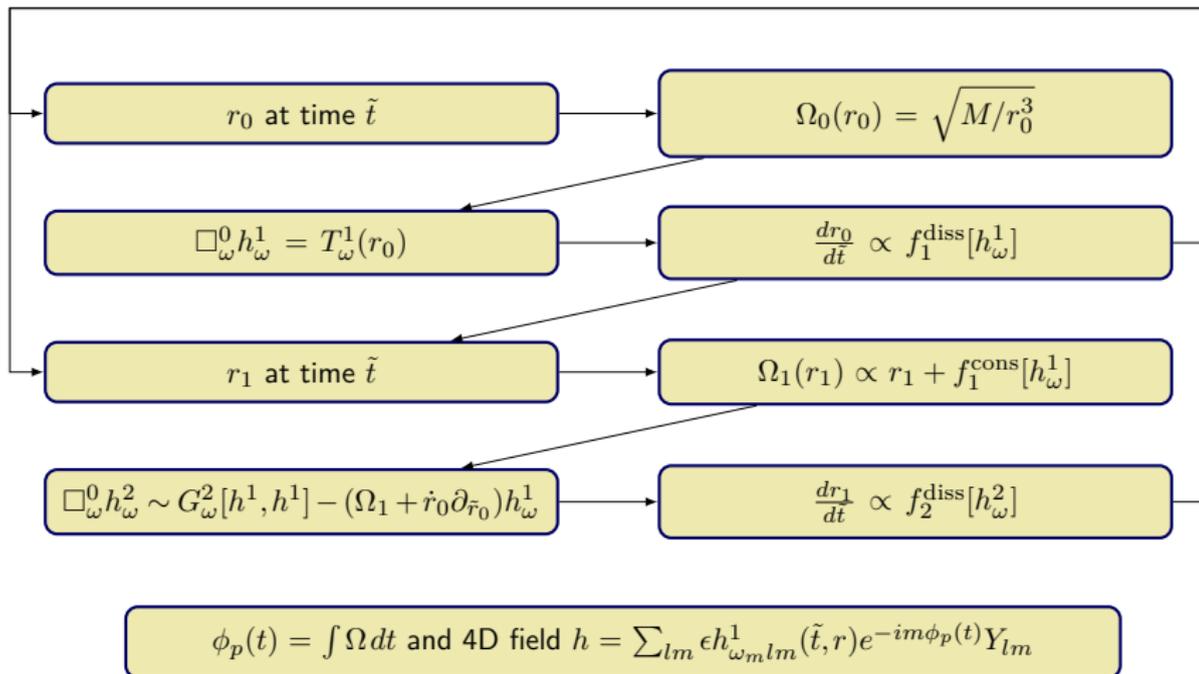


$$\phi_p(t) = \int \Omega dt \text{ and 4D field } h = \sum_{lm} \epsilon h_{\omega_m lm}^1(\tilde{t}, r) e^{-im\phi_p(t)} Y_{lm}$$

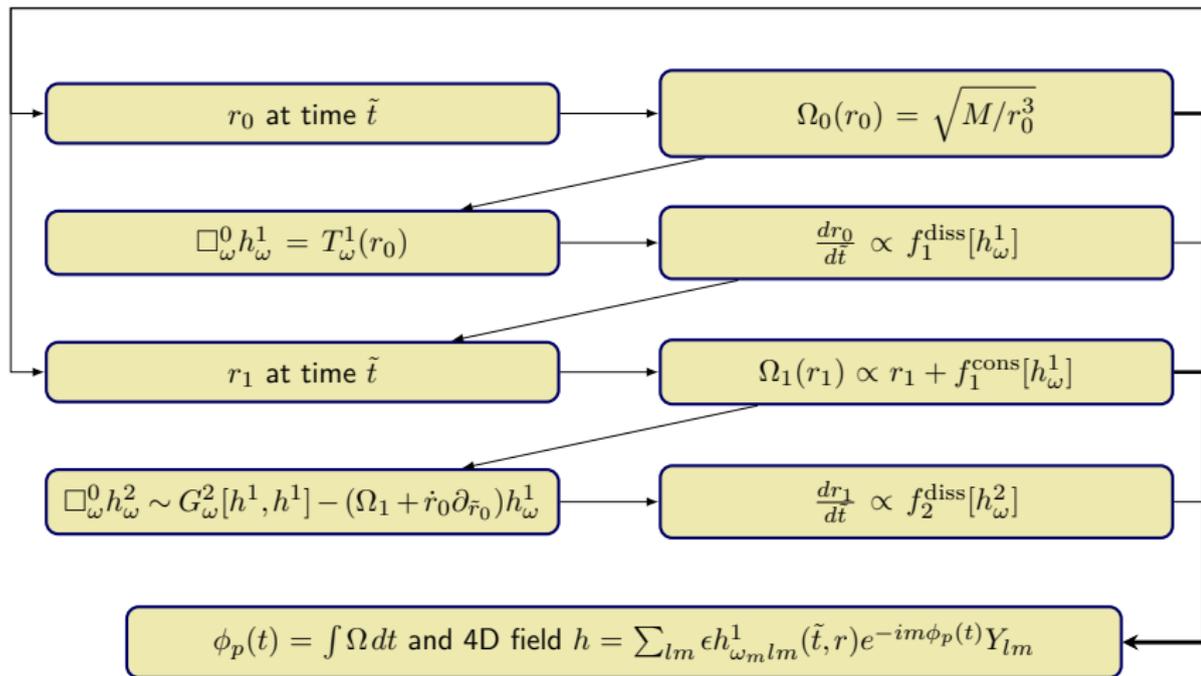
Framework for computing a waveform



Framework for computing a waveform

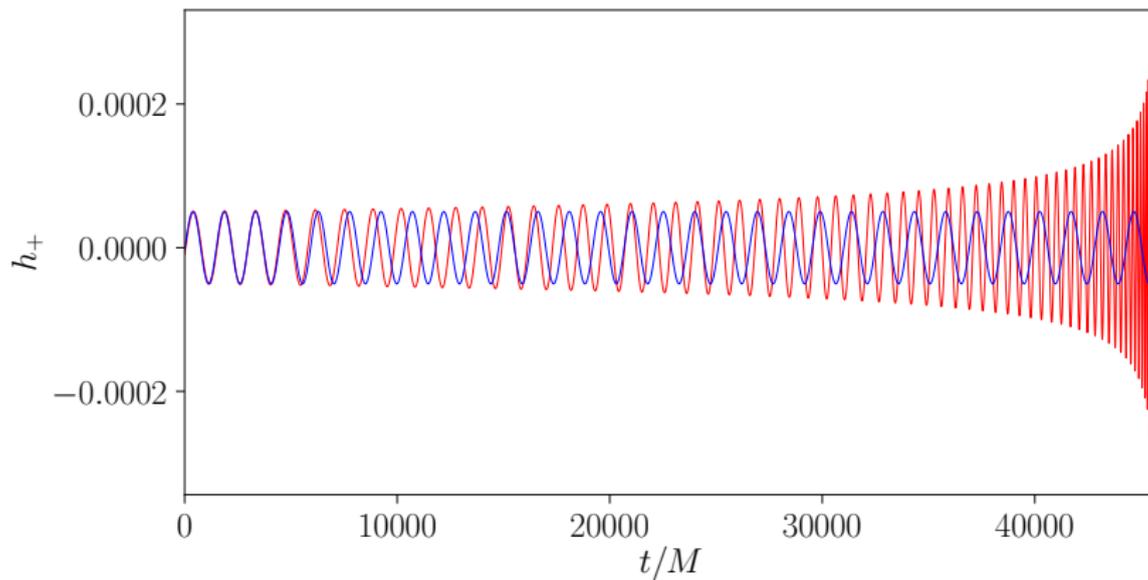


Framework for computing a waveform



Demonstration in Lorenz gauge

22 mode of adiabatic vs geodesic waveform



Notes on two-timescale framework

- solving the two-timescale equations gives us the post-adiabatic inspiral and waveform as an output
- the two-timescale expansion is currently the main framework under development for computing post-adiabatic waveforms — or for doing any second-order calculations at all

How you can contribute

- optimize numerical implementation of $G_\omega^2[h^1, h^1]$
- calculate explicit punctures h_ω^{P2} (in coordinates, decomposed into harmonics) for generic orbits in Schwarzschild and Kerr
- calculate derivatives of h_ω^1 with respect to orbital parameters for generic orbits
- calculate explicit punctures at horizon and infinity for generic orbits
- incorporate particle's spin into two-timescale framework
- develop near-resonance expansions to transition across resonances
- develop method of computing h_ω^n in Lorenz gauge in Kerr
- develop tools for computing ψ_ω^2 (e.g., coupling formula $h_{\omega lm}^1 \rightarrow S_{\omega lm}[h^1, h^1]$)
- develop method of analytically calculating second-order quantities using PN expansions