A practical guide to second-order self-force calculations

Adam Pound, Niels Warburton
Jeremy Miller, Barry Wardell
+many helpful discussions with Leor Barack

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- This talk will give a practical guide to second-order calculations
- We will point out areas where contributions are needed

Gravitational Self-force

$\epsilon = m/M$

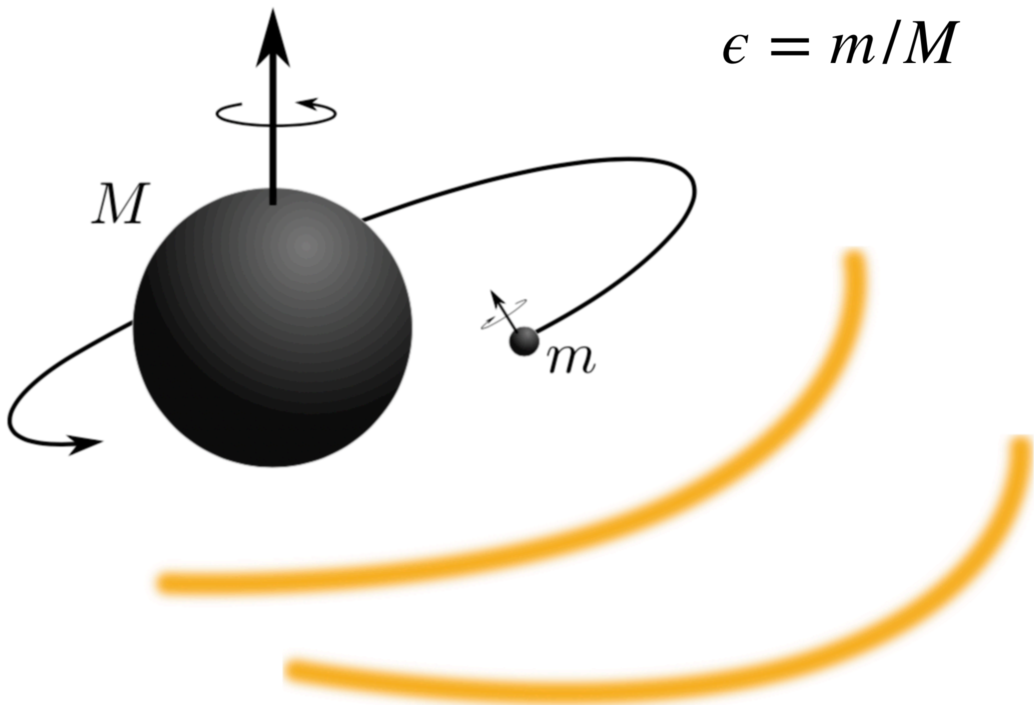

Image credit: A. Pound

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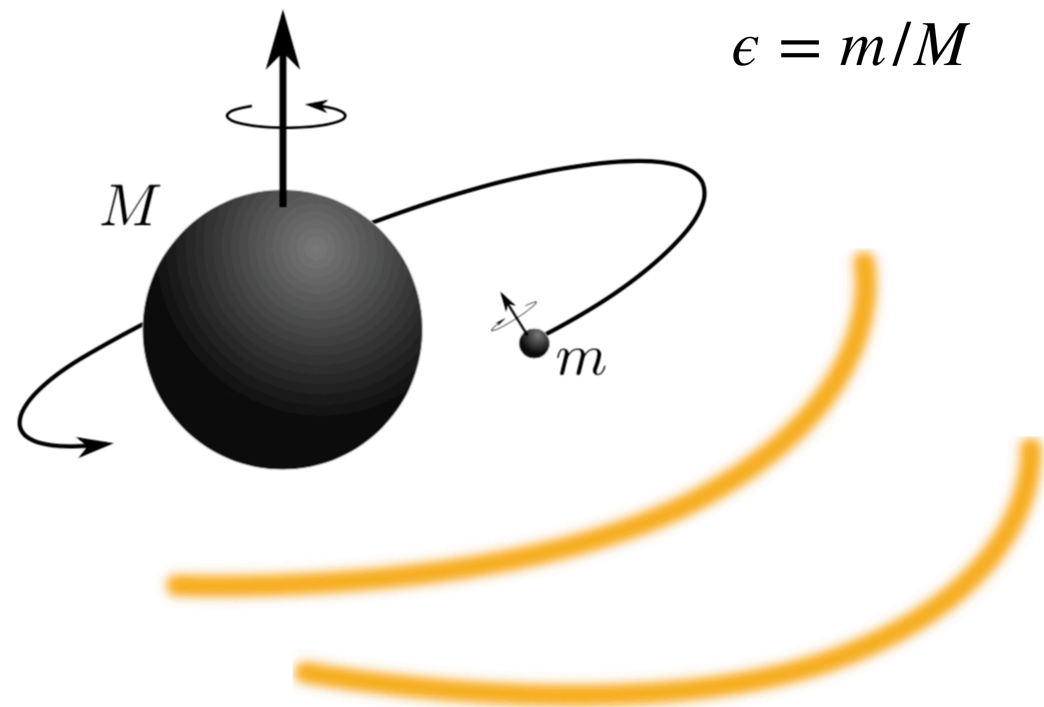

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Write metric as perturbative expansion about a background

$$g_{\mu\nu} = g^0_{\mu\nu} + \epsilon h^1_{\mu\nu} + \epsilon^2 h^2_{\mu\nu} + \mathcal{O}(\epsilon^3)$$

Substitute expansion into Einstein equation

$$G_{\mu\nu}[g] = 8\pi T_{\mu\nu}$$

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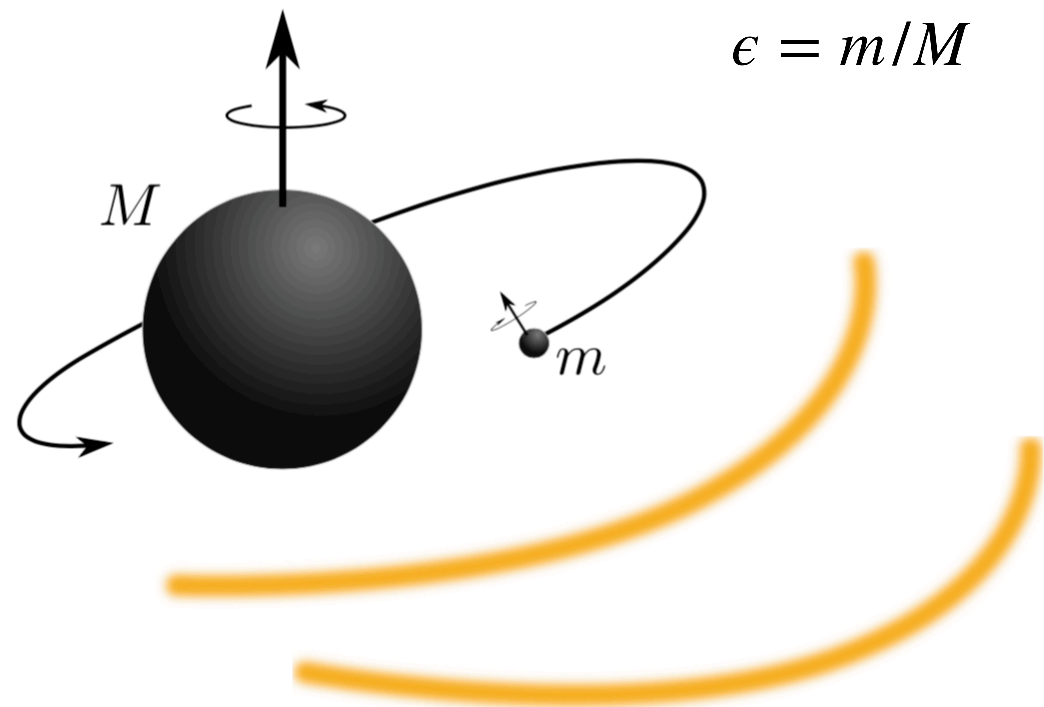

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Expanding out,

$$G_{\mu\nu}[\mathrm{g}] = G_{\mu\nu}[g^0] + \epsilon G^1_{\mu\nu}[h^1] + \epsilon^2 \left(G^1_{\mu\nu}[h^2] + G^2_{\mu\nu}[h^1, h^1] \right) + \mathcal{O}\left(\epsilon^3\right)$$

Obtain equations at each order in ϵ, which we can solve for $h^1_{\mu\nu}$, $h^2_{\mu\nu}$, …, along with equations of motion for a worldline. Goal: compute $h^1_{\mu\nu}$, $h^2_{\mu\nu}$, z^μ.

$$\frac{Dz^\alpha}{d\tau} = \epsilon F^\alpha_1 + \epsilon^2 F^\alpha_2$$

Why go to second-order?

On an inspiral timescale $t \sim M^2/m$, the phase of the gravitational wave has an expansion (excluding resonances):

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From the orbit averaged piece of first-order self-force $\langle F_1^\alpha \rangle$

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Post-Adiabatic order (ϕ_1)

Three contributions:

- First-order, oscillatory dissipative self-force $F_1^{\alpha(diss,osc)}$
- First-order conservative self-force $F_1^{\alpha(cons)}$
- Second-order orbit averaged self-force $\langle F_2^\alpha \rangle$

Needed for precision tests of GR

Potential application to IMRIs

Non-linear perturbation theory

Expand Einstein tensor and make regular/singular split in second-order perturbation

$$G_{\mu\nu}[\mathrm{g}] = G_{\mu\nu}[g^0] + \epsilon G^1_{\mu\nu}[h^1] + \epsilon^2 \left(G^1_{\mu\nu}[h^2] + G^2_{\mu\nu}[h^1, h^1] \right) + \mathcal{O}\left(\epsilon^3\right)$$

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Equations of motion take the form:

$$\frac{D^2 z^\mu}{d\tau} = \epsilon F^{1\mu}[h^{R1}] + \epsilon^2 F^{2\mu}[h^{R2}]$$

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$$\frac{d}{dt} = \frac{d\varphi_p}{dt}\frac{\partial}{\partial \varphi_p} + \frac{d\tilde{t}}{dt}\frac{\partial}{\partial \tilde{t}}$$

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General form of the two-timescale expanded field equations:

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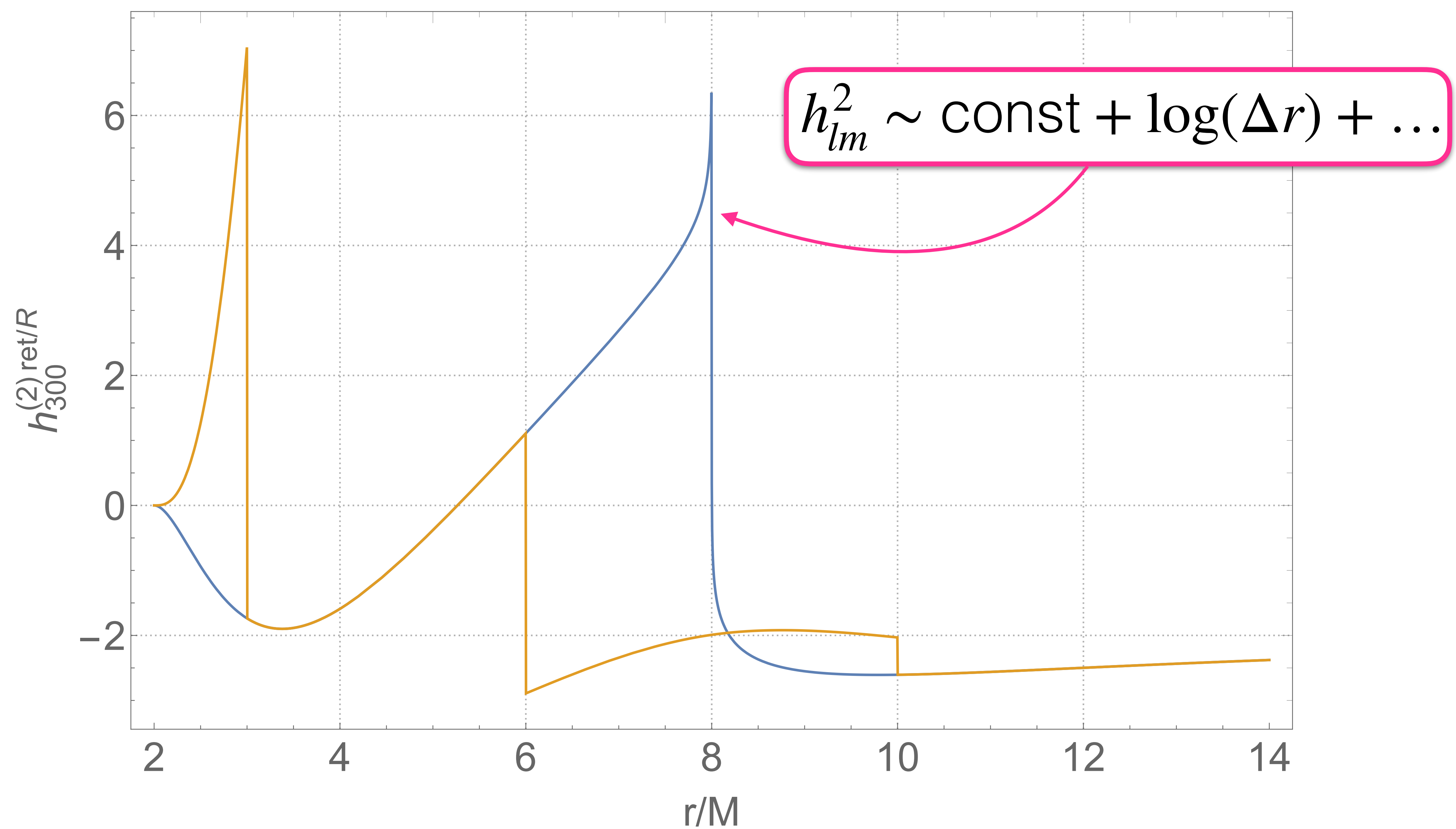
Adam will discuss the two-timescale expanded EoM

First results: monopole of 2nd-order metric perturbation

Particle on circular orbit of Schwarzschild black hole



First application: Gravitational binding energy

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Mass of smaller black hole

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Expanding out for small mass ratio

$$E_{\rm bind} = \hat{\mathscr{E}}_0(y) - 1 + \epsilon E_{SF} + O(\epsilon^2)$$

$$E_{SF} = \hat{M}_B^{(2)} - \hat{M}_{\rm BH}^{(2)} + \hat{\mathscr{E}}_0(y) - 1 + \frac{\delta\hat{M}(1-6y)y}{3(1-3y)^{3/2}}$$

We directly compute each of these quantities from the monopole piece of the first and second order metric perturbations on the horizon and at scri.

$$\hat{M}_B^{(2)} = \lim_{r\to\infty} \frac{1}{4}\left(\bar{h}^{2\,(1)}_{0,0} + \bar{h}^{2\,(6)}_{0,0}\right) Y_{0,0}$$

$$\delta\hat{M} = -\frac{y}{(1-3y)^{1/2}}$$

$$\hat{M}_{irr}^{(2)} \sim \lim_{r\to 2M}\left(\bar{h}^{2\,(i)}_{0,0} + \partial_{\tilde{w}}\bar{h}^{1\,(j)}_{0,0} + \bar{h}^{1\,(i)}_{0,0}\bar{h}^{1\,(j)}_{0,0}\right)$$

$$\hat{\mathscr{E}}_0(y) = \frac{1-2y}{(1-3y)^{1/2}}$$

First application: Gravitational binding energy

Binding energy can also be computed from the first-law of binary black hole mechanics using first-order self-force data*

$$\delta M + \Omega\,\delta J = z_1 \delta m_1 + z_2 \delta m_2$$

Rewriting this in terms of binding energy as a function of y, we get

$$E_{SF}^{\text{1st law}} = \frac{1}{2} z_{SF}(y) - \frac{y}{3}\frac{dz_{SF}}{dy} - 1 + \sqrt{1-3y} + \frac{y}{6}\frac{5-12y}{(1-3y)^{3/2}}$$

Just as the energy balance law relates second-order fluxes to the first-order local SF, the FLBM relates a binary's second-order energy, as defined at infinity, to the first-order, local Detweiler redshift z_{SF}.

Important to note that the first-law assumes a conservative, helically symmetric, asymptotically flat spacetime.

*Le Tiec, Barausse, Buonanno, Phys. Rev. Lett.108:131103 (2012)

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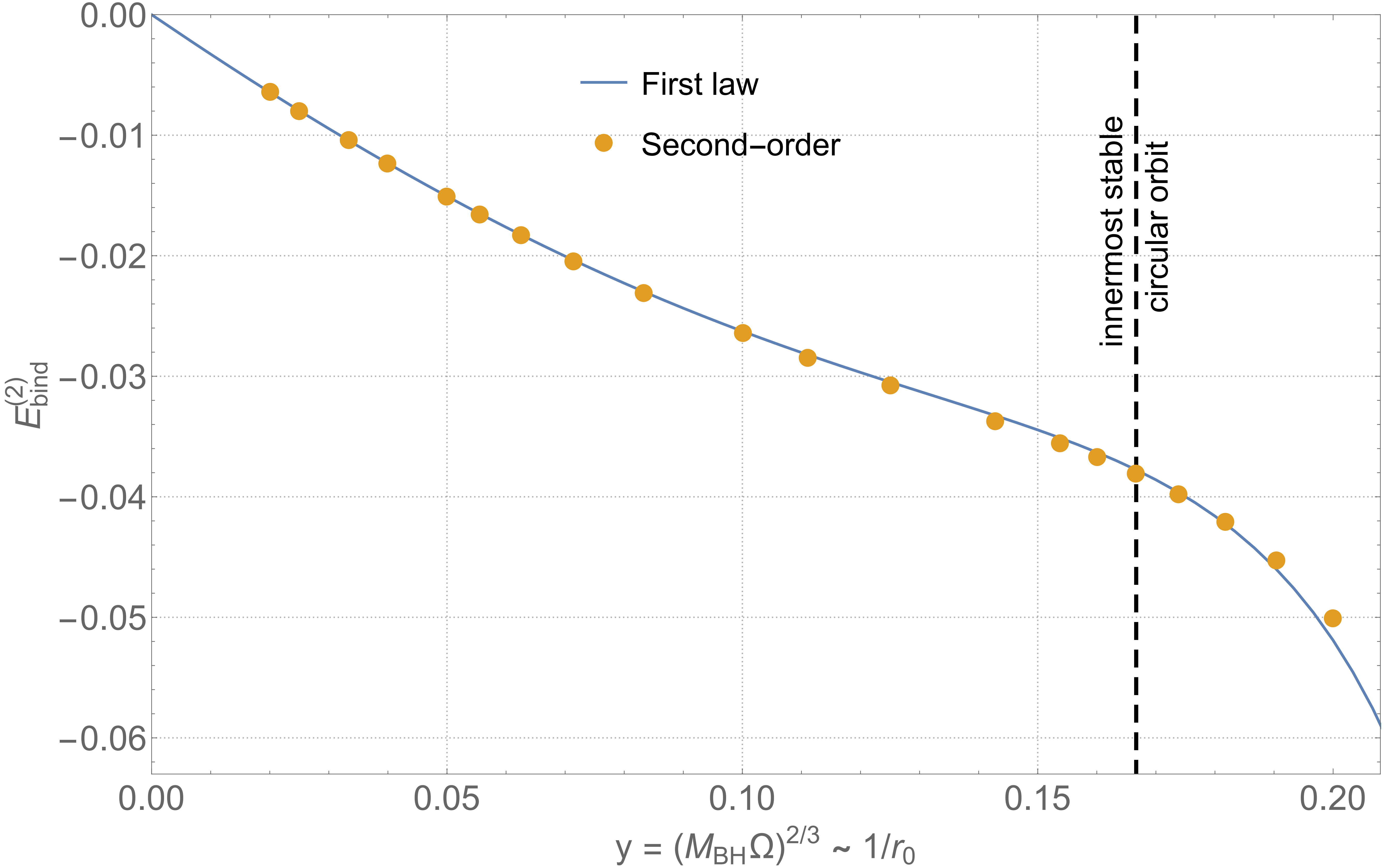

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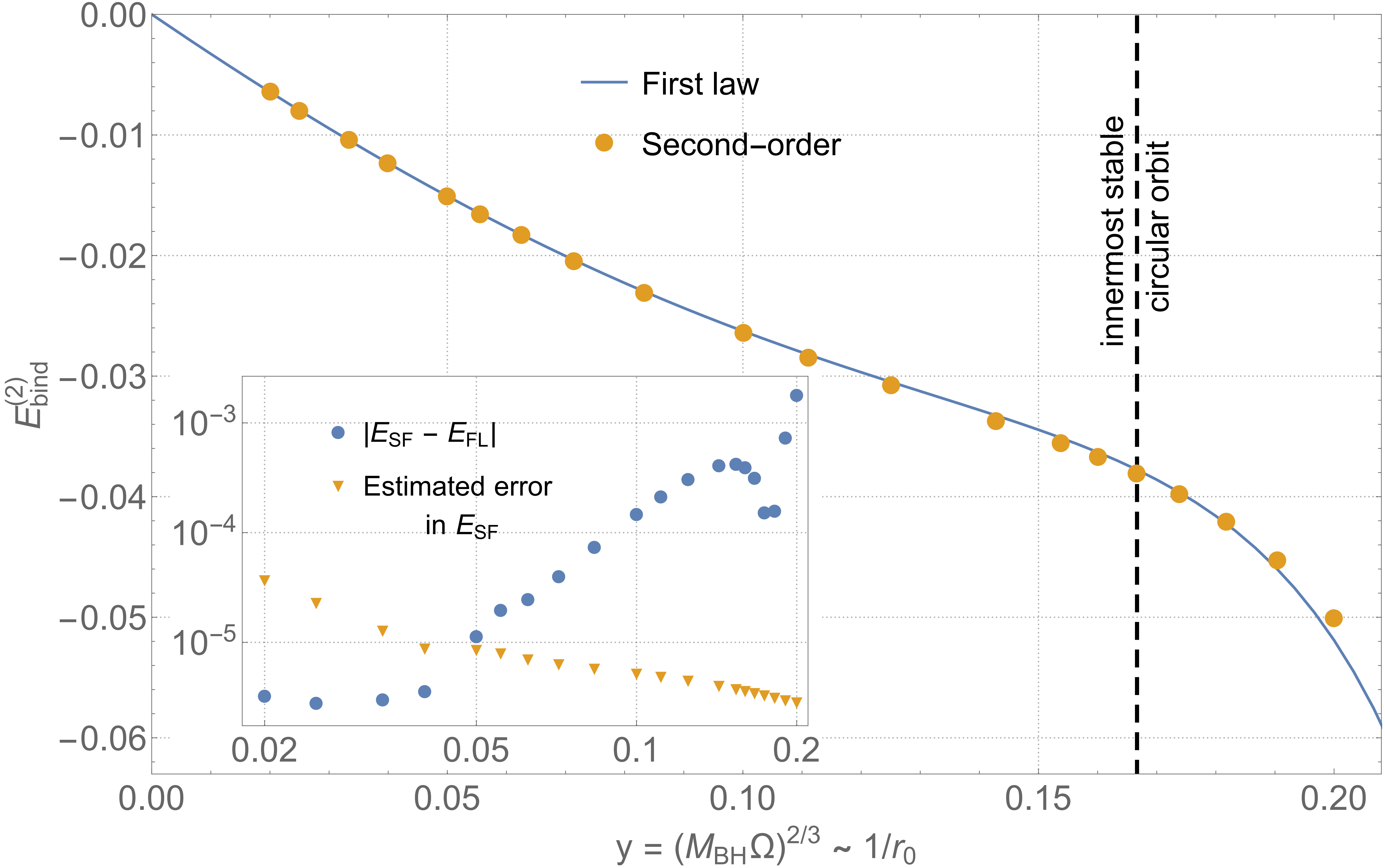

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- Derivatives w.r.t. slow time:

$$\frac{dh^1}{d\tilde{t}} \sim \frac{dr_0}{d\tilde{t}}\frac{dh^1}{dr_0} \sim \left(\frac{dE}{dr_0}\right)^{-1} \dot{\mathscr{E}}\frac{dh^1}{dr_0}$$

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Apply retarded boundary conditions

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Explicitly given by:

$$\begin{aligned} G^2_\omega[h,h] \equiv & -\tfrac{1}{2} h^{\mu\nu}\left(2h_{\mu(\alpha;\beta)\nu} - h_{\alpha\beta;\mu\nu} - h_{\mu\nu;\alpha\beta}\right) \\ & + \tfrac{1}{4} h^{\mu\nu}{}_{;\alpha} h_{\mu\nu;\beta} + \tfrac{1}{2} h^{\mu}{}_{\beta}{}^{;\nu}\left(h_{\mu\alpha;\nu} - h_{\nu\alpha;\mu}\right) \\ & - \tfrac{1}{2} \bar{h}^{\mu\nu}{}_{;\nu}\left(2h_{\mu(\alpha;\beta)} - h_{\alpha\beta;\mu}\right) \end{aligned}$$

Challenges constructing this term:

- divergent at the particle: as $(\Delta r)^{-2}$ in Lorenz gauge
- mode coupling with finite number of first-order modes
- numerical noise near the horizon
- decays too slowly at the horizon and infinity

Non-linear perturbation theory: behaviour at the particle

mode decomposition

$$h^1 \sim (\Delta r)^{-1}$$

$$\implies h^1_{lm} \sim \text{const} + \Delta r + \ldots$$

$$h^1_{lm}(r_0) = h^{ret1}_{lm}(r_0) - h^{P1}_{lm}(r_0)$$

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Must use effective-source regularization

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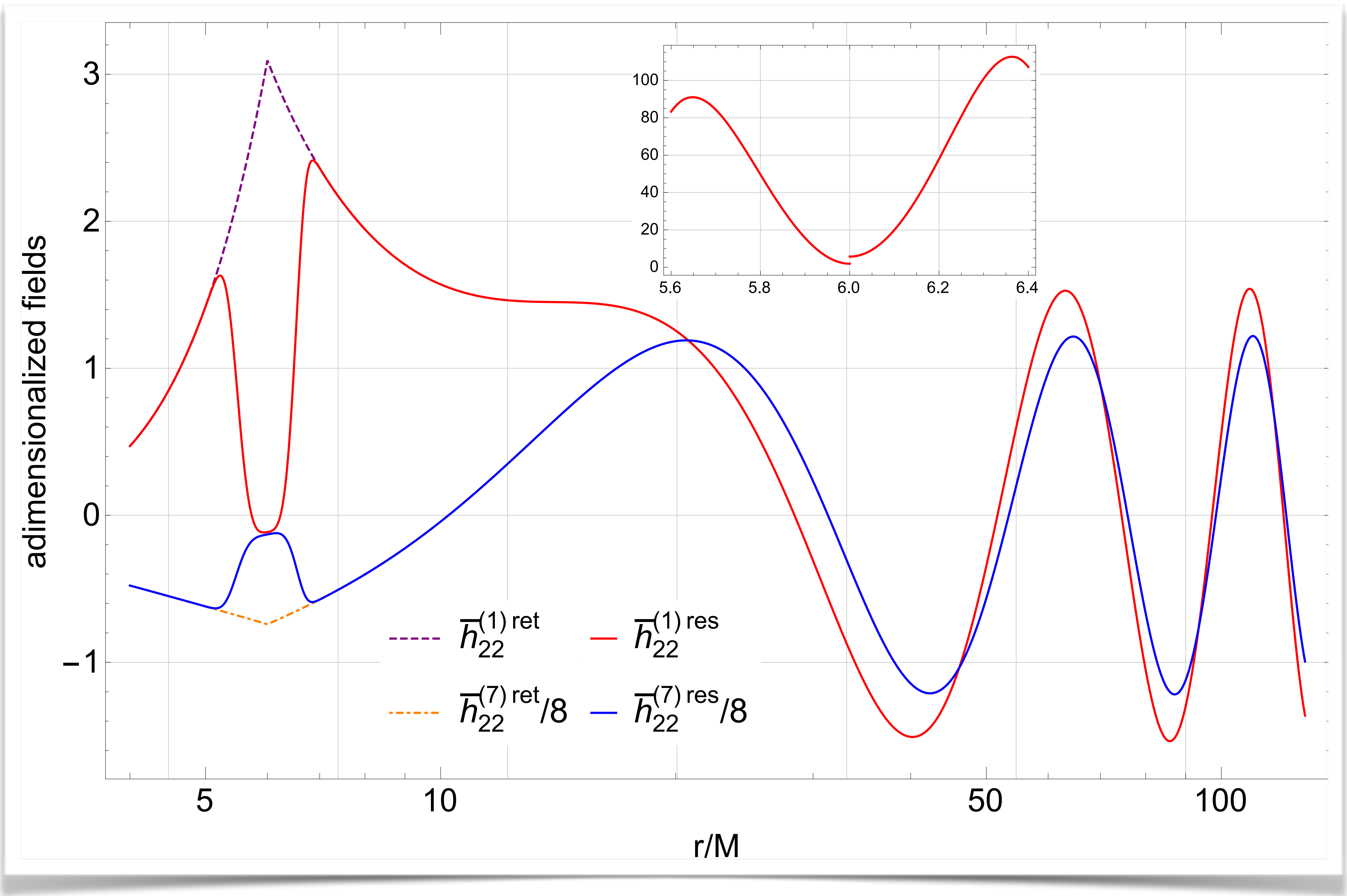

Wardell, NW, Phys. Rev. D 92, 084019 (2015)

Non-linear perturbation theory: mode coupling

$$G^2_{\mu\nu}[h^1, h^1] = \sum_{ilm} G_{ilm}(r; r_0) e^{-im\Omega t} Y^{ilm}_{\mu\nu}(r, \theta^A)$$

$$G^2_{ilm} = \sum_{\substack{i'l'm' \\ i''l''m''}} \mathscr{D}^{i'l'm'i''l''m''}_{ilm}[h^1_{i'l'm'}, h^1_{i''l''m''}]$$

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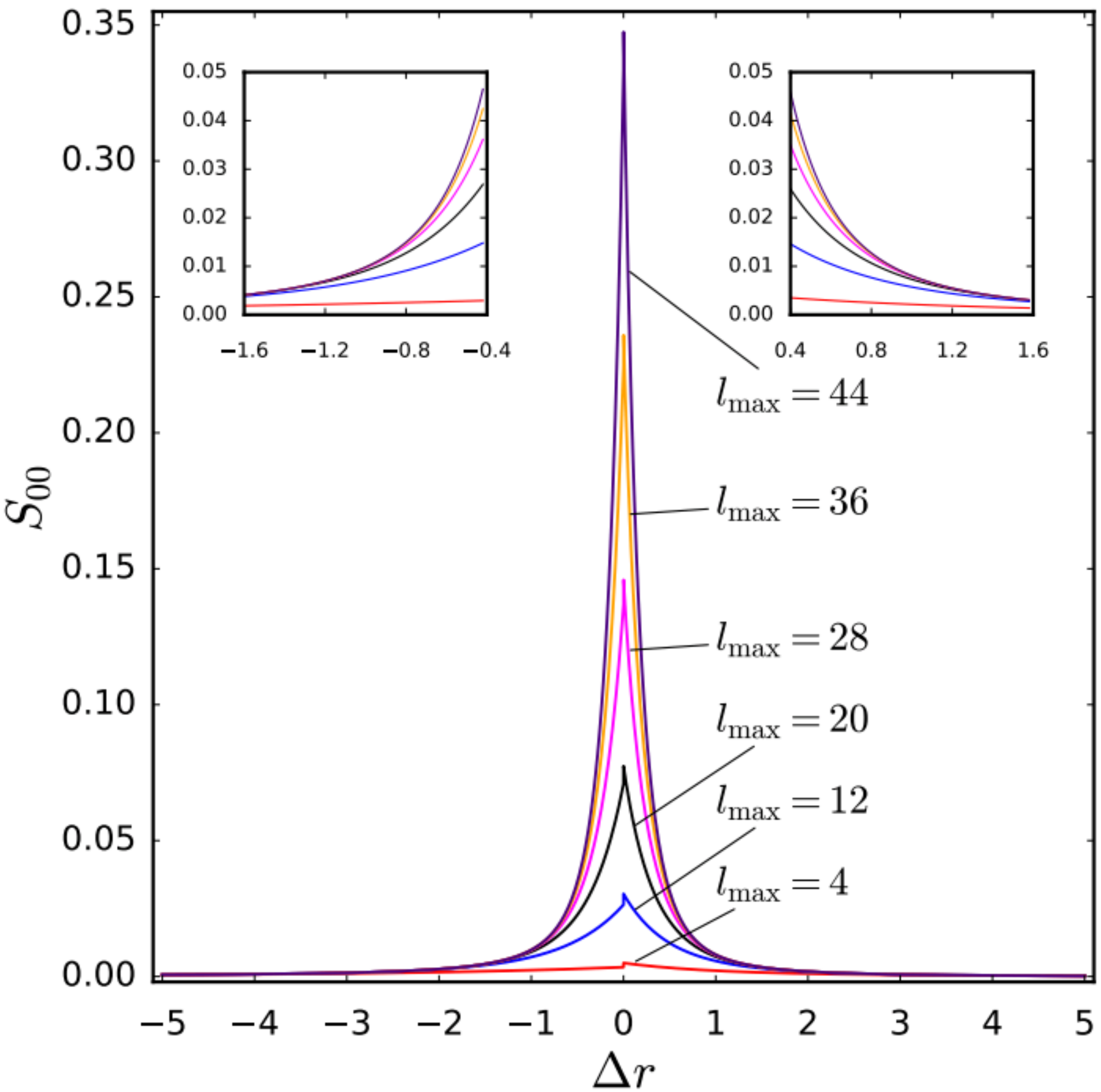

J. Miller, B. Wardell, A. Pound, Phys. Rev. D 94, 104018 (2016)

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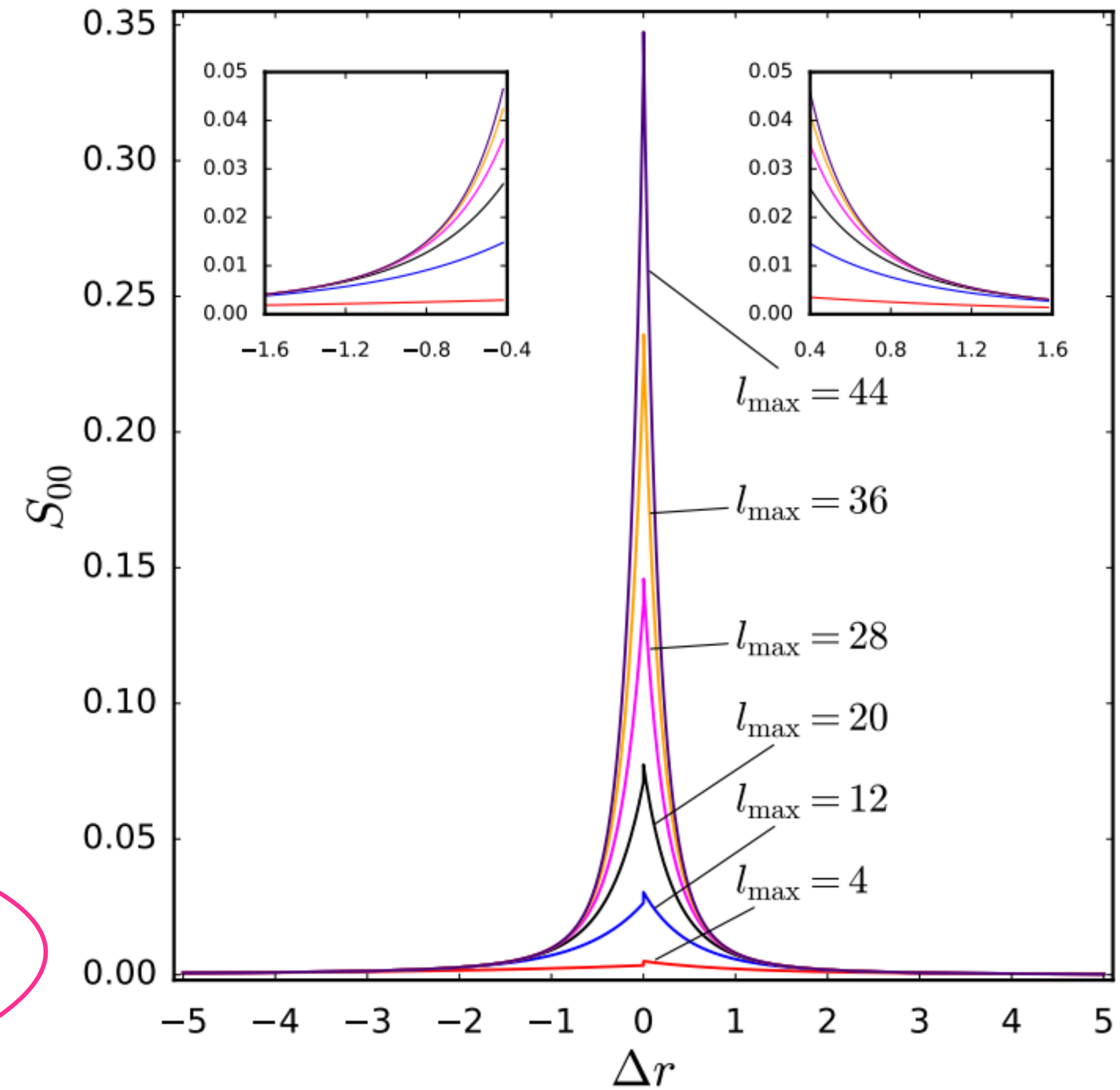
Solution:

$$h^1 = h^{R1} + h^{P1}$$

$$G^2[h^1,h^1] = G^2[h^{P1},h^{P1}] + 2G^2[h^{P1},h^{R1}] + G^2[h^{R1},h^{R1}]$$

Can tackle this term analytically

(but have to use **exact** mode decomposition of h^{P1} for other terms done using mode coupling - no shortcuts)



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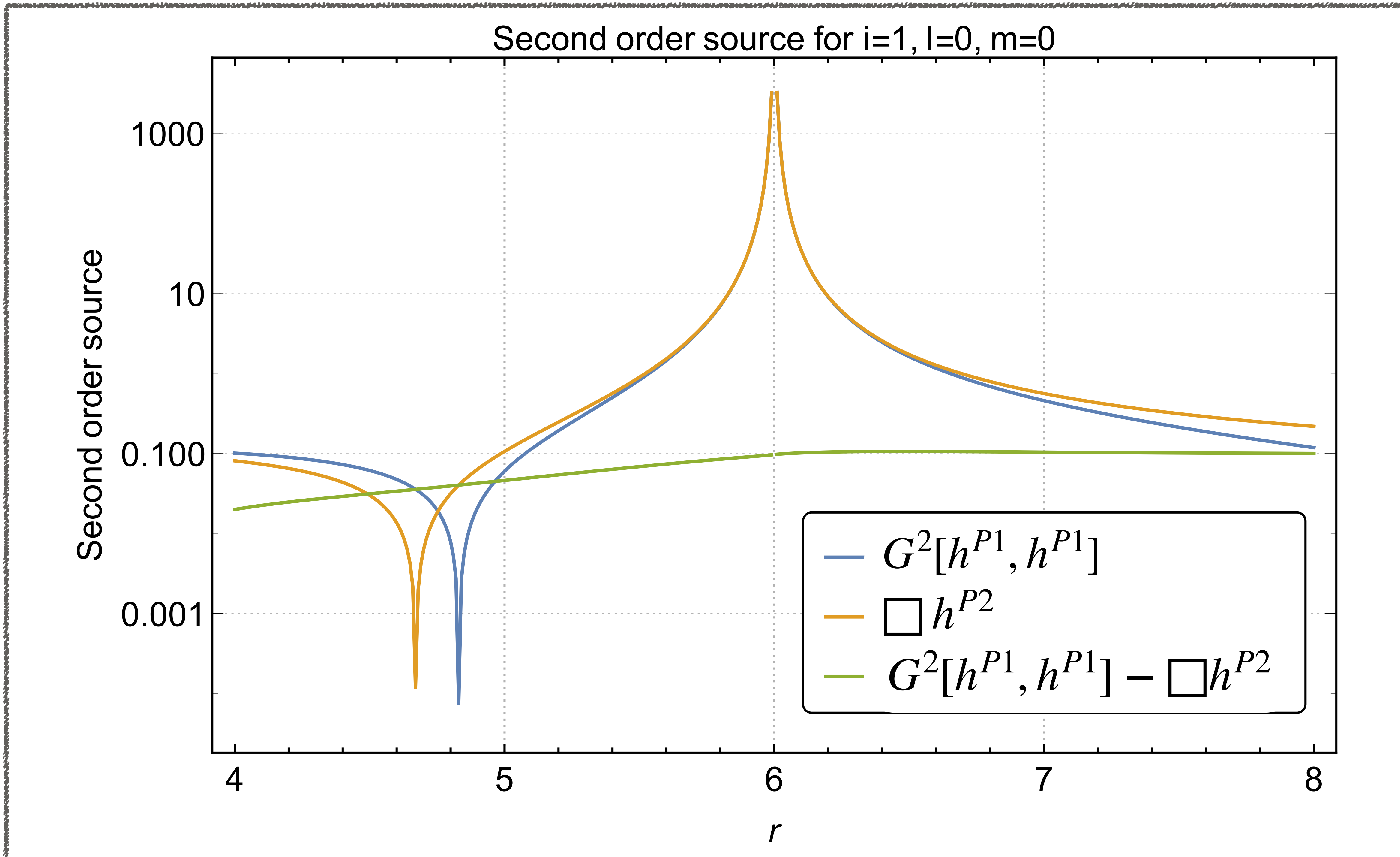
effective source near the worldline



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Nonlinear gravitational self-force. I. Field outside a small body

Adam Pound[1]

[1] *School of Mathematics, University of Southampton, Southampton, United Kingdom, SO17 1BJ*

(Dated: September 5, 2012)

A small extended body moving through an external spacetime $g_{\alpha\beta}$ creates a metric perturbation $h_{\alpha\beta}$, which forces the body away from geodesic motion in $g_{\alpha\beta}$. The foundations of this effect, called the gravitational self-force, are now well established, but concrete results have mostly been limited to linear order. Accurately modeling the dynamics of compact binaries requires proceeding to nonlinear orders. To that end, I show how to obtain the metric perturbation outside the body

Nonlinear gravitational self-force: second-order equation of motion

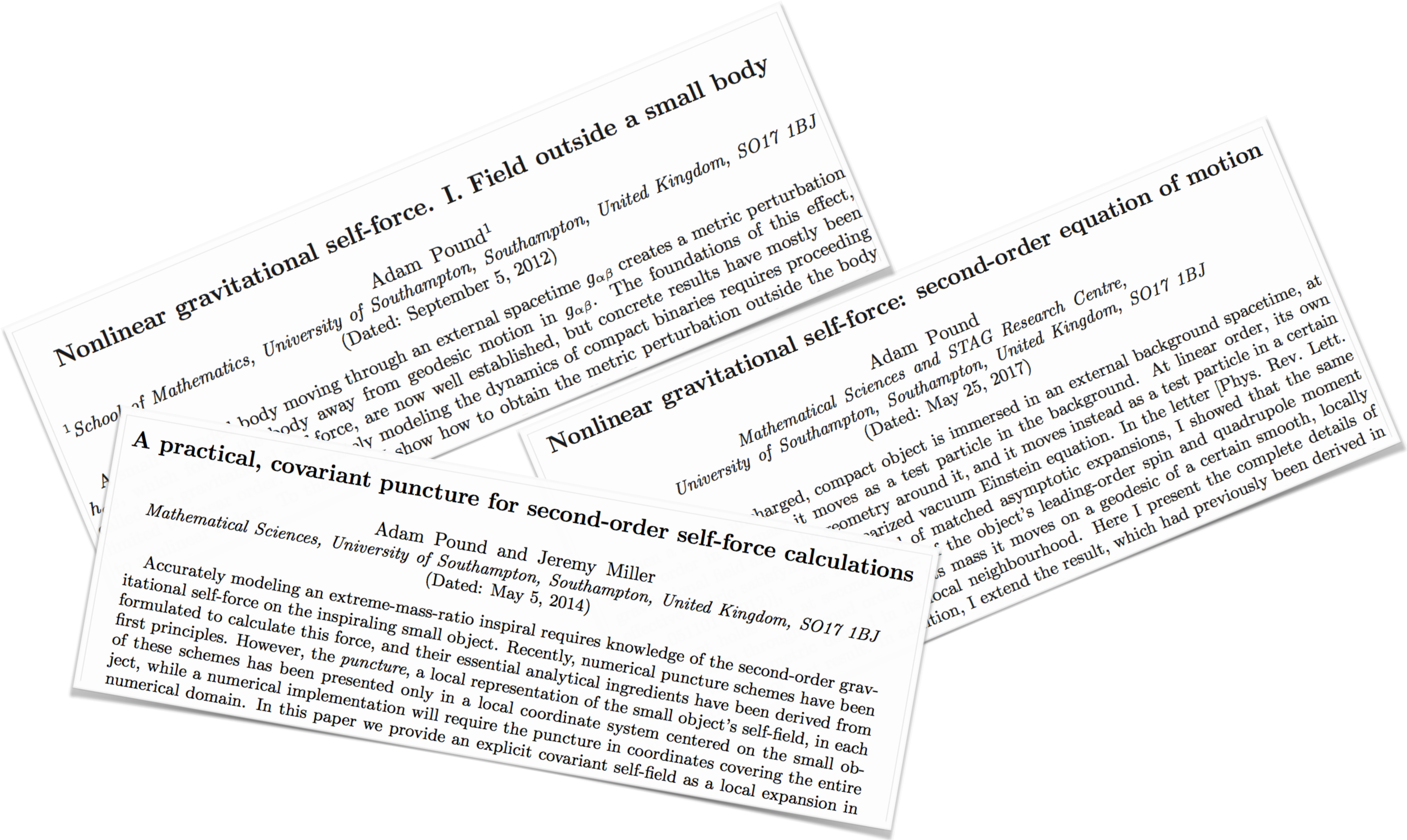
Adam Pound

Mathematical Sciences and STAG Research Centre,
University of Southampton, Southampton, United Kingdom, SO17 1BJ

(Dated: May 25, 2017)

When a small, uncharged, compact object is immersed in an external background spacetime, at zeroth order in its mass it moves as a test particle in the background. At linear order, its own gravitational field alters the geometry around it, and it moves instead as a test particle in a certain effective metric satisfying the linearized vacuum Einstein equation. In the letter [Phys. Rev. Lett. 109, 051101 (2012)], using a method of matched asymptotic expansions, I showed that the same statement holds true at second order: if the object's leading-order spin and quadrupole moment vanish, then through second order in its mass it moves on a geodesic of a certain smooth, locally causal vacuum metric defined in its local neighbourhood. Here I present the complete details of the derivation of that result. In addition, I extend the result, which had previously been derived in

Many details now set out in 10+ papers:



Nonlinear gravitational self-force. I. Field outside a small body

Adam Pound[1]

[1] School of Mathematics, University of Southampton, Southampton, United Kingdom, SO17 1BJ

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A practical, covariant puncture for second-order self-force calculations

Adam Pound and Jeremy Miller

Mathematical Sciences, University of Southampton, Southampton, United Kingdom, SO17 1BJ

(Dated: May 5, 2014)

Accurately modeling an extreme-mass-ratio inspiral requires knowledge of the second-order gravitational self-force on the inspiraling small object. Recently, numerical puncture schemes have been formulated to calculate this force, and their essential analytical ingredients have been derived from first principles. However, the *puncture*, a local representation of the small object's self-field, in each of these schemes has been presented only in a local coordinate system centered on the small object, while a numerical implementation will require the puncture in coordinates covering the entire numerical domain. In this paper we provide an explicit covariant self-field as a local expansion in

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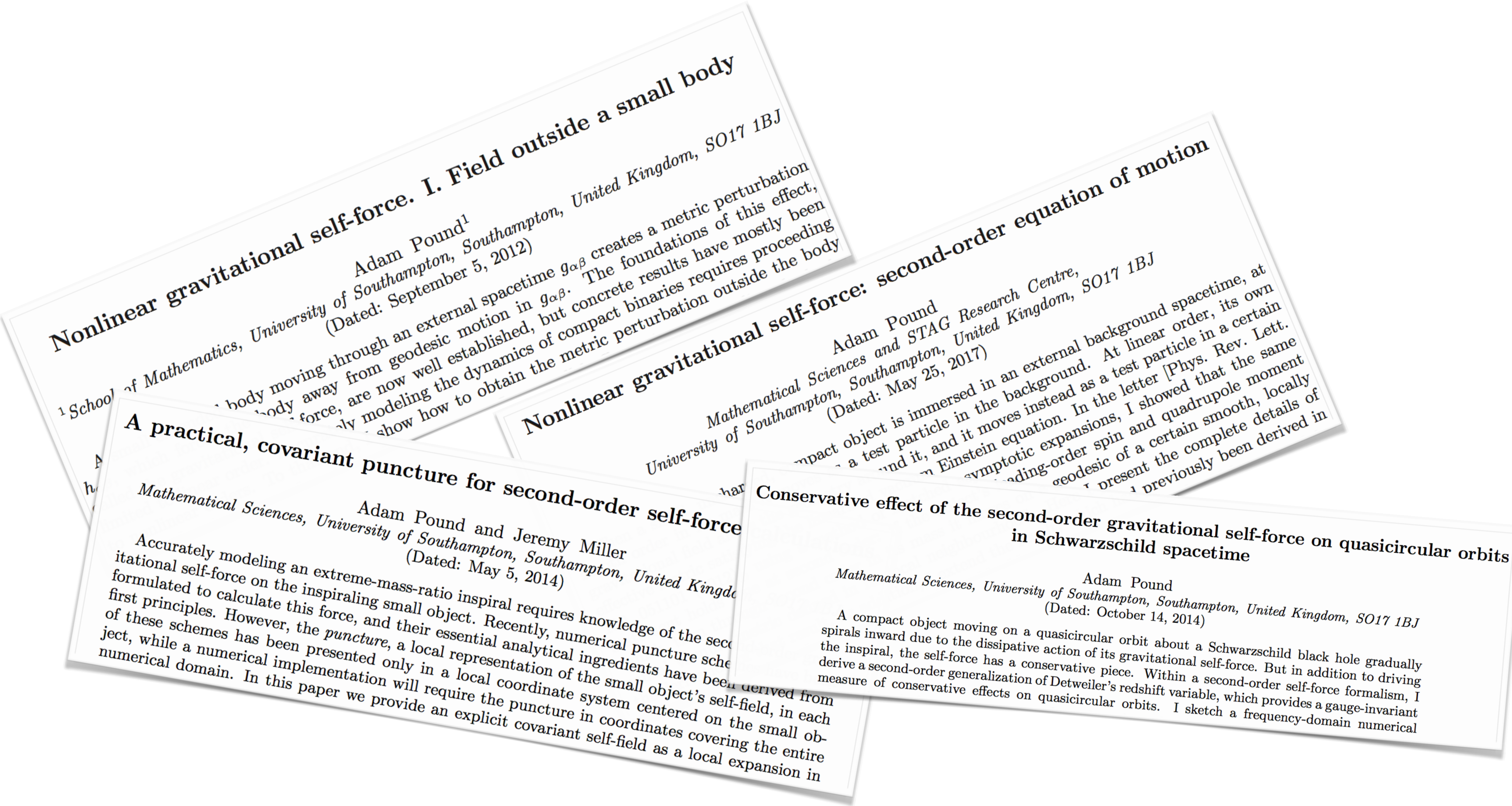

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Conservative effect of the second-order gravitational self-force on quasicircular orbits in Schwarzschild spacetime

Adam Pound

Mathematical Sciences, University of Southampton, Southampton, United Kingdom, SO17 1BJ

(Dated: October 14, 2014)

A compact object moving on a quasicircular orbit about a Schwarzschild black hole gradually spirals inward due to the dissipative action of its gravitational self-force. But in addition to driving the inspiral, the self-force has a conservative piece. Within a second-order self-force formalism, I derive a second-order generalization of Detweiler's redshift variable, which provides a gauge-invariant measure of conservative effects on quasicircular orbits. I sketch a frequency-domain numerical

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Applying the effective-source approach to frequency-domain self-force calculations

Niels Warburton[1] and Barry Wardell[2,1]

[1]School of Mathematical Sciences and Complex & Adaptive Systems Laboratory, University College Dublin, Belfield, Dublin 4, Ireland

[2]Department of Astronomy, Cornell University, Ithaca, NY 14853, USA

(Dated: 28th October 2014)

Applying the effective-source approach to frequency-domain self-force calculations: Lorenz-gauge gravitational perturbations

Barry Wardell[1,2] and Niels Warburton[3,2]

[1]Department of Astronomy, Cornell University, Ithaca, NY 14853, USA

[2]School of Mathematical Sciences and Complex & Adaptive Systems Laboratory, University College Dublin, Belfield, Dublin 4, Ireland

[3]MIT Kavli Institute for Astrophysics and Space Research, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

With a view to developing a formalism that will be applicable at second perturbative order, we devise a new practical scheme for computing the gravitational self-force experienced by a point mass moving in a curved background spacetime. Our method works in the frequency domain and employs the effective-source approach, in which a distributional source for the retarded metric perturbation is replaced with an effective source for a certain regularized self-field. A key ingredient of the calculation is the analytic determination of an appropriate puncture field from which the effective source and regularized residual field can be calculated. In addition to its application in our effective-

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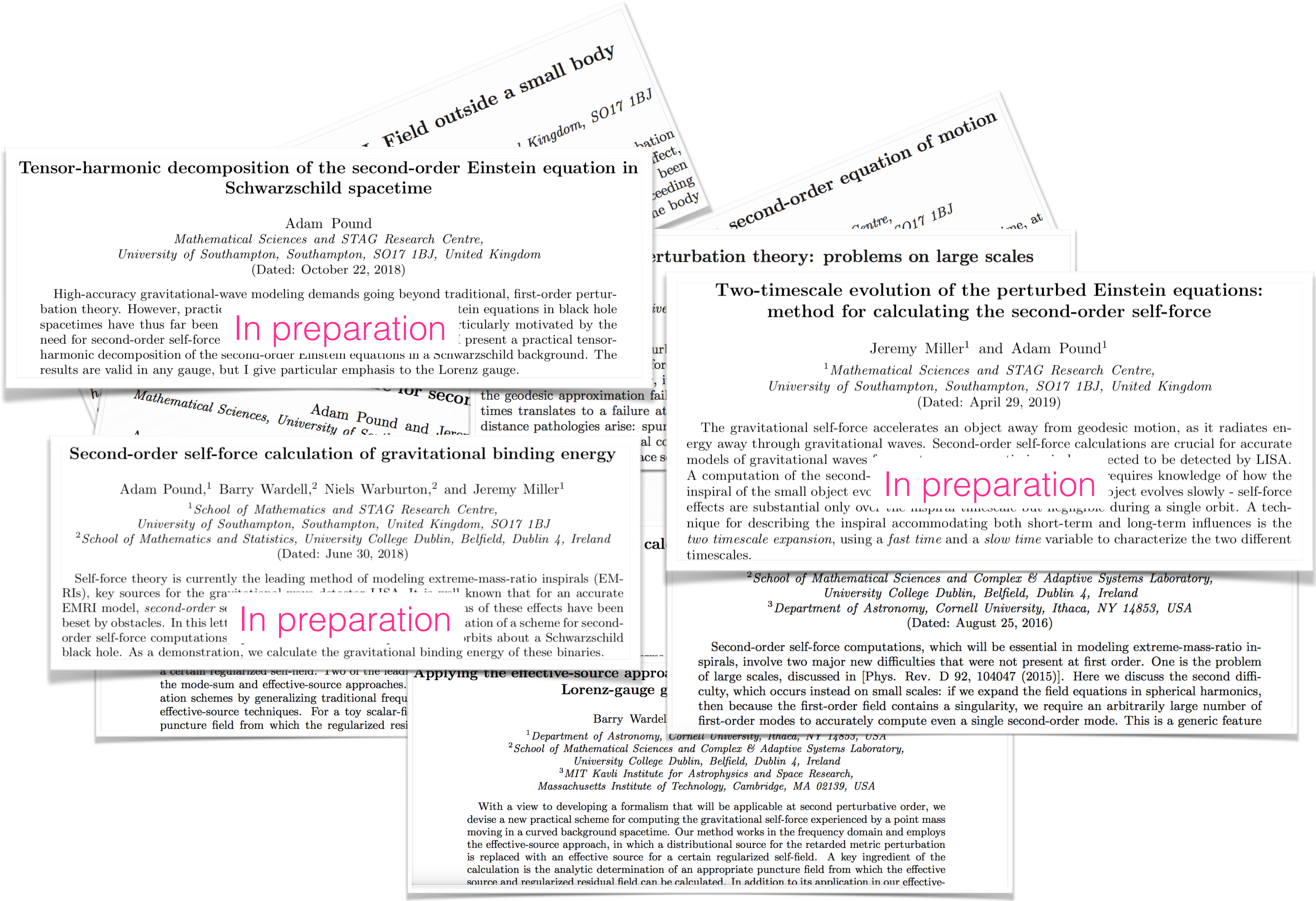

$$\Box^0_\omega h^{R2}_\omega = G^2_\omega[h^1, h^1] - \Box^0_\omega h^{P2}_\omega - \Box^1_\omega h^1_\omega$$

with retarded boundary conditions

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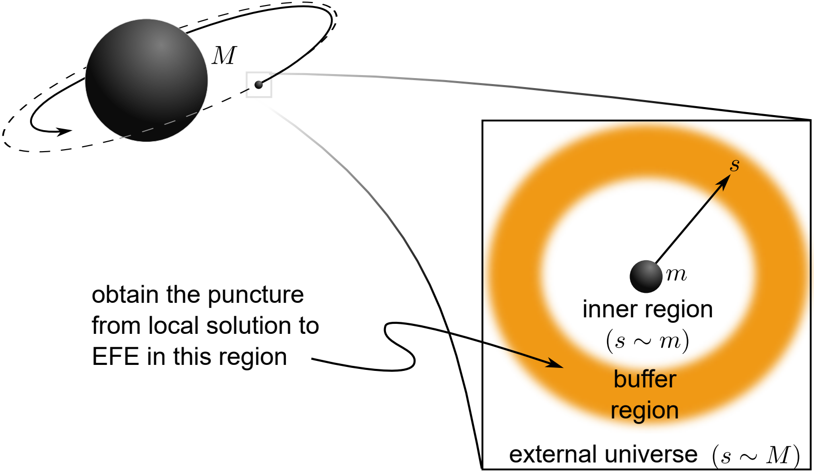
Basic idea



M
obtain the puncture from local solution to EFE in this region
s
m
inner region
$(s \sim m)$
buffer region
external universe $(s \sim M)$

Form of puncture

- In most gauges, the puncture has the form

$$h^{P1}_{\mu\nu} \sim \frac{m}{s} + O(s^0)$$
$$h^{P2}_{\mu\nu} \sim \frac{m^2}{s^2} + \frac{m h^{\rm R1}}{s} + O(s^0)$$

- h^{P2} is available in Lorenz gauge, in arbitrary spacetime, through order s^1, in covariant form [Pound and Miller 2014]
- Also available, in less ready-to-use form, in "P smooth" gauges [Gralla 2012] and "highly regular" gauges [Pound 2017]

Expansion of worldline

- The worldline z^μ is where the puncture diverges
- Quasicircular orbit: $z^\alpha = (t, r_p(\tilde{t}, \epsilon), \pi/2, \phi_p(\tilde{t}, \epsilon))$, where

$$r_p = r_0(\tilde{t}) + \epsilon r_1(\tilde{t}) + \dots$$
$$\frac{d\phi_p}{dt} := \Omega = \Omega_0(\tilde{t}) + \epsilon \Omega_1(\tilde{t}) + \dots$$

- Plug expansion into the puncture:

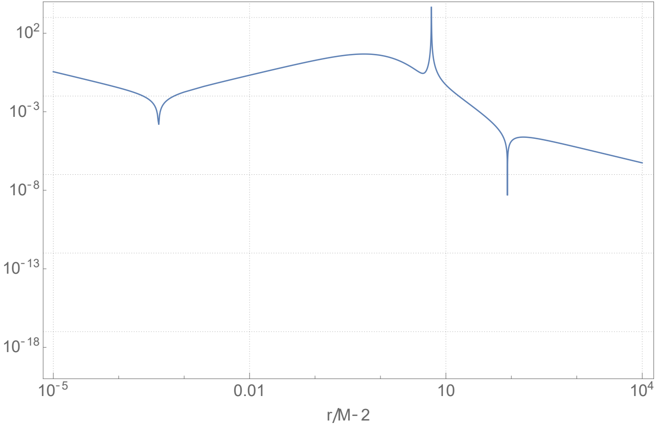
$$\Rightarrow h^{P2} \to h^{P2} + \frac{\delta h^{P1}}{\delta r_p} r_1 + \frac{\delta h^{P1}}{\delta \dot{r}_p} \dot{r}_0 + \frac{\delta h^{P1}}{\delta \Omega} \Omega_1$$

- The two-timescale puncture is also available in covariant form for generic bound orbits

Demonstration in Lorenz gauge

For quasicircular orbits in Schwarzschild,

$$h^{P2}_{\omega lm} \sim m^2 \log|r - r_0| + (m h^{\mathcal{R}1} + r_1 + \dot{r}_0)|r - r_0|$$



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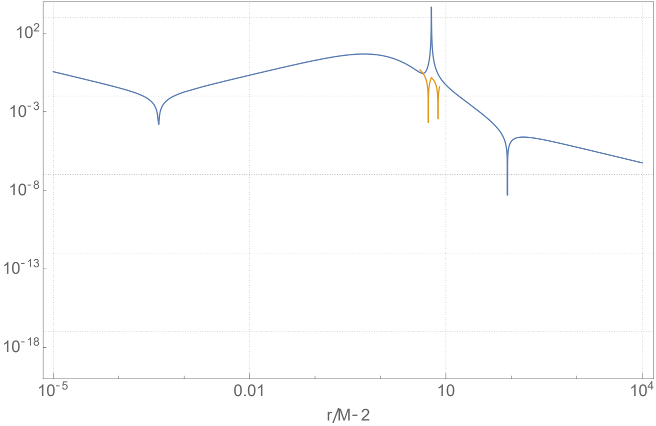

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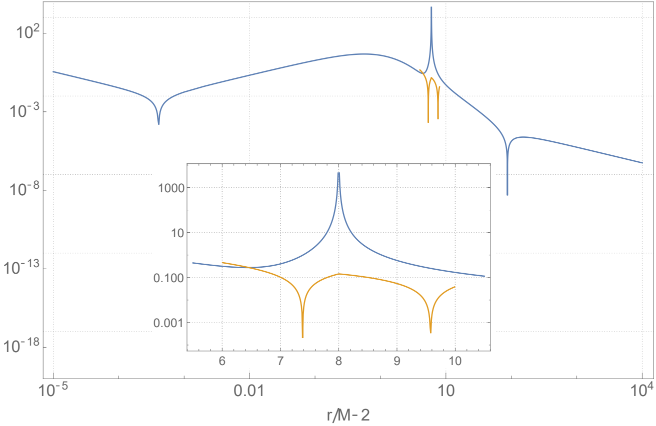

Extension to other gauges or variables

- Almost all calculations of first-order self-force have been built on Lorenz-gauge $h^{\rm S1}_{\mu\nu}$, even if they calculate the retarded field in a different gauge
- Can do the same at second order:
  - work in locally Lorenz gauge: $h^{P2}_{\mu\nu}$ in Lorenz, $h^{R2}_{\mu\nu}$ in whatever's convenient
  - transform from $h^{P2,Lor}_{\mu\nu}$ to some desired gauge
  - calculate punctures for Teukolsky (or Regge-Wheeler, Zerilli-Moncrief, etc.) variables: $\psi^{P2} = \psi[h^{P2,Lor}]$

$$\Box^0_\omega h^{R2}_\omega = G^2_\omega[h^1, h^1] - \Box^0_\omega h^{P2}_\omega - \Box^1_\omega h^1_\omega$$

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with retarded boundary conditions

Computing parametric derivative of h^1

- $\Box^1_\omega h^1_\omega \sim \partial_{\tilde{t}} h^1_\omega + \Omega_1 h^1_\omega$
- can convert to derivatives wrt parameters: e.g.,

$$h^1_\omega(\tilde{t}) = h^1_\omega\left(r_0(\tilde{t}), \delta M(\tilde{t}), \delta J(\tilde{t})\right)$$
$$\Rightarrow \partial_{\tilde{t}} h^1_\omega = \frac{dr_0}{d\tilde{t}}\partial_{r_0} h^1_\omega + \frac{d\delta M}{d\tilde{t}}\partial_{\delta M} h^1_\omega + \frac{d\delta J}{d\tilde{t}}\partial_{\delta J} h^1_\omega$$

- to find $\partial_{r_0} h^1_\omega$, differentiate $\Box^0_\omega h^1_\omega = T^1_\omega$ wrt r_0:

$$\partial_{r_0}(\Box^0_\omega h^1_\omega) = \partial_{r_0} T^1_\omega$$
$$\Rightarrow \Box^0_\omega(\partial_{r_0} h^1_\omega) = \partial_{r_0} T^1_\omega - (\partial_{r_0}\Box^0_\omega) h^1_\omega$$

- solve for $\partial_{r_0} h^1_\omega$ with your favorite frequency-domain method

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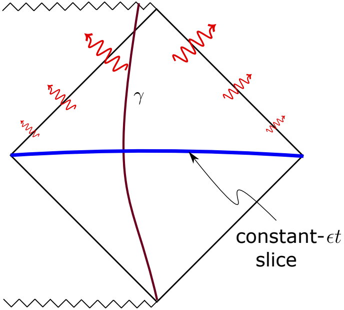
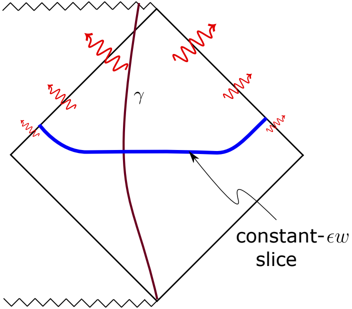
Dependence on choice of slow time





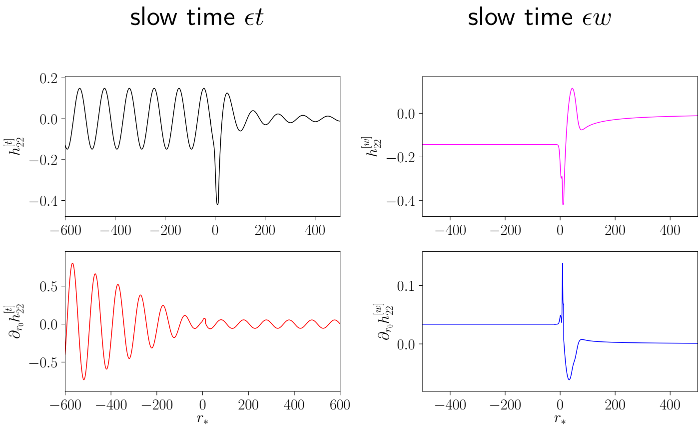
$$h^1_\omega \sim \frac{e^{\pm i\omega r^*}}{r} \text{ near bd'ries}$$

$$\Rightarrow \partial_{r_0} h^1_\omega \sim \frac{i\omega' r^* e^{\pm i\omega r^*}}{r}$$

$$h^1_\omega \sim \frac{A(r_0)}{r} \text{ near bd'ries}$$

$$\Rightarrow \partial_{r_0} h^1_\omega \sim \frac{A'(r_0)}{r}$$

Demonstration in Lorenz gauge



Extension to other gauges and variables

- Exactly the same idea applies
- e.g., two-timescale expansion of Teukolsky (or Zerilli-Moncrief, etc.) equation,

$$\hat{\Box}^0_\omega \psi^{R2}_\omega = S_\omega[h^1, h^1] - \hat{\Box}^0_\omega \psi^{P2}_\omega - \hat{\Box}^1_\omega \psi^1_\omega,$$

$$\Box^0_\omega h^{R2}_\omega = G^2_\omega[h^1, h^1] - \Box^0_\omega h^{P2}_\omega - \Box^1_\omega h^1_\omega$$

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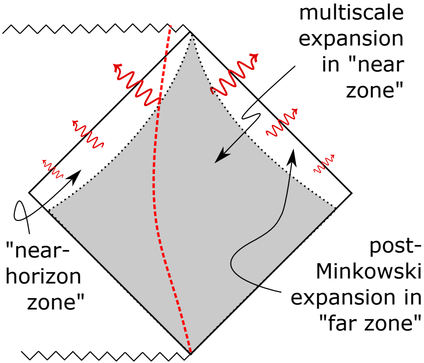
Matched expansions [Pound, Moxon, Flanagan, Hinderer, Yamada, Isoyama, Tanaka]



- We want to impose retarded BCs on full, physical metric perturbation
- This *does not* trivially imply simple BCs for two-timescale fields because the two-timescale expansion breaks down near horizon and infinity

- To find BCs for two-timescale fields, obtain other (time-domain) approximations near $\mathscr{H}^+$ and $\mathscr{I}^+$

Boundary conditions

- Use post-Minkowski expansion near $\mathscr{I}^+$ and a near-horizon expansion near $\mathscr{H}^+$
- Obtain solutions by using leading-order retarded (time domain) Green's functions
- post-Minkowski solution looks like

$$h^{PM} = h^{\mathcal{P},PM} + h^{\mathcal{R},PM}$$

$h^{\mathcal{P},PM} =$ particular solution, $h^{\mathcal{R},PM} =$ outgoing wave

- Analogous near horizon

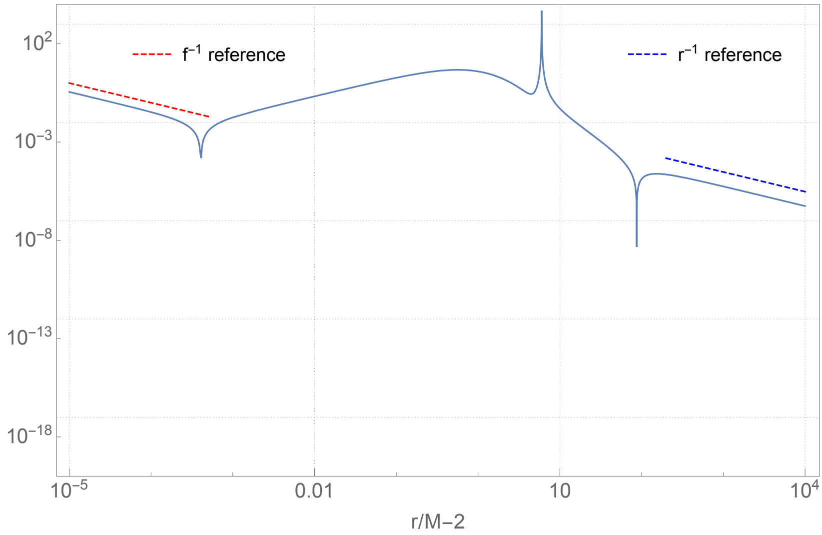
Boundary punctures

- BC for two-timescale solution h^{TT} is $\mathrm{FZ}[h^{TT}] = \mathrm{NZ}[h^{PM}]$
- To enforce BC, use $\mathrm{NZ}[h^{\mathcal{P},PM}]$ as a puncture at large r, impose standard outgoing, retarded BC on residual field

$$h^{\mathcal{R},TT} = h^{TT} - \mathrm{NZ}[h^{\mathcal{P},PM}]$$

- Numerically computed $h^{\mathcal{R},TT}$ determines $h^{\mathcal{R},PM}$
- Analogous near horizon

Demonstration in Lorenz gauge



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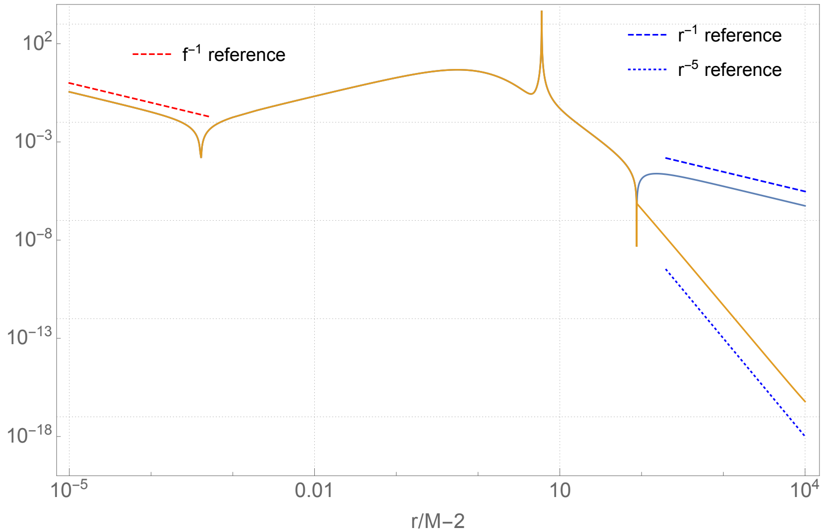

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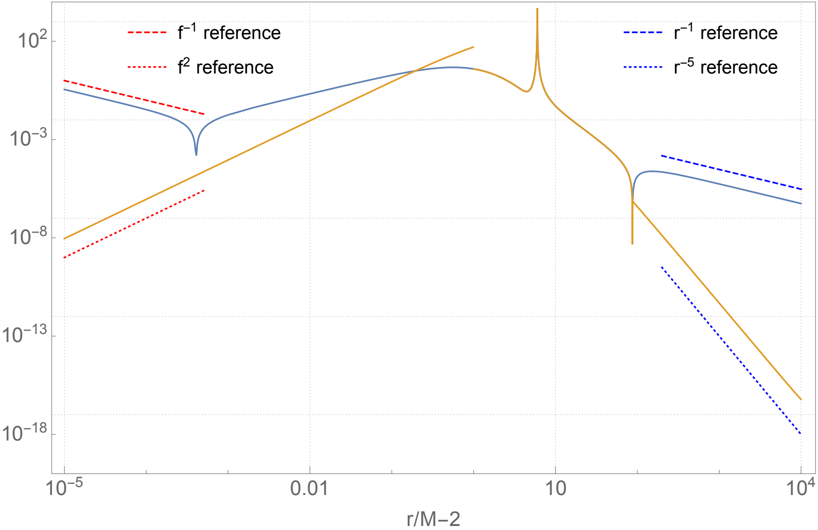

Extension to other gauges and variables

- The boundary punctures may not be strictly necessary in some cases
  - nonstationary modes with $\tilde{w}$ as slow time
  - in "nicer" gauges
- When needed, punctures can be obtained from Lorenz-gauge punctures
  — or directly, using same methods as in Lorenz gauge (obtaining approximate solutions in time domain near boundaries)

Reminder: quasicircular orbits

$$r_p = r_0(\tilde{t}) + \epsilon\, r_1(\tilde{t}) + O(\epsilon^2)$$

$$\Omega = \Omega_0(\tilde{t}) + \epsilon\, \Omega_1(\tilde{t}) + O(\epsilon^2)$$

$$\phi_p = \int \Omega\, dt = \frac{1}{\epsilon} \int (\Omega_0 + \epsilon\Omega_1 + \ldots) d\tilde{t}$$

$$h \sim \sum_{nlm} \epsilon^n h^n_{\omega_m lm}(\tilde{t}, r) e^{-im\phi_p(\tilde{t})} Y_{lm}$$

$$(\omega_m = m\Omega_0)$$

Reminder: governing equations

$$\Box h^1 = T^1$$

$$\Box h^{R2} = G^2[h^1, h^1] - \Box h^{P2}$$

$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon f_1^\mu[h^{R1}] + \epsilon^2 f_2^\mu[h^{R2}]$$

$$h^{P1} \sim \frac{m}{|x^\alpha - z^\alpha|}$$

$$h^{P2} \sim \frac{m^2}{|x^\alpha - z^\alpha|^2} + \frac{m h^{R1}}{|x^\alpha - z^\alpha|}$$

Reminder: governing equations

$$\Box^0_\omega h^1_\omega = T^1_\omega$$

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$$\Omega_0 = \sqrt{M/r_0^3}$$

$$\frac{dr_0}{d\tilde{t}} \propto f_1^{\text{diss}}[h^{R1}_\omega]$$

$$\Omega_1 \propto f_1^{\text{cons}}[h^{R1}_\omega] + r_1$$

$$\frac{dr_1}{d\tilde{t}} \propto f_2^{\text{diss}}[h^{R2}_\omega]$$

$$h^{P1} \sim \frac{m}{|x^\alpha - z^\alpha|}$$

$$h^{P2} \sim \frac{m^2}{|x^\alpha - z^\alpha|^2} + \frac{mh^{R1}}{|x^\alpha - z^\alpha|}$$

Reminder: governing equations

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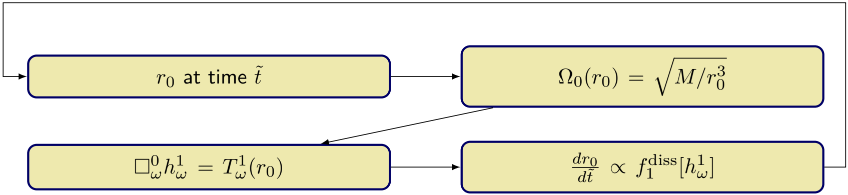
$$\Omega_1 \propto f_1^{\text{cons}}[h^{R1}_\omega] + r_1$$

$$\frac{dr_1}{d\tilde{t}} \propto f_2^{\text{diss}}[h^{R2}_\omega]$$

$$h^{P1}_\omega \sim const. + |r - r_0|$$

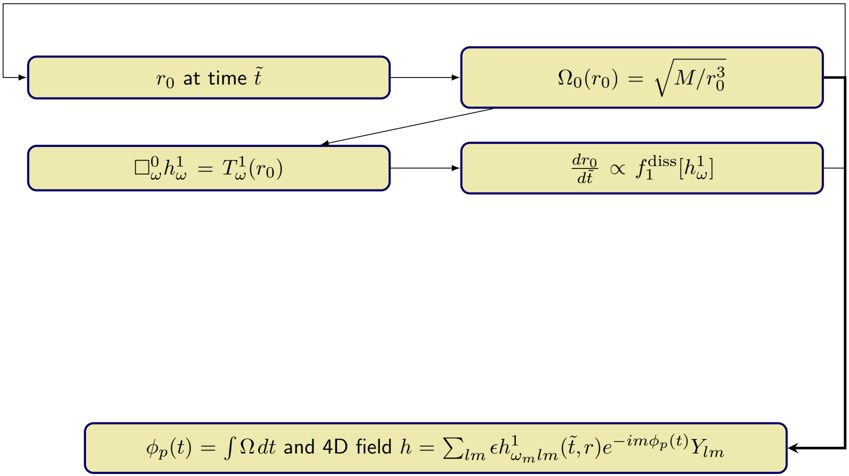
$$h^{P2}_\omega \sim m^2 \log|r - r_0| + const. + (mh^{R1} + r_1 + \dot{r}_0)|r - r_0|$$

Framework for computing a waveform



$\phi_p(t) = \int \Omega\, dt$ and 4D field $h = \sum_{lm} \epsilon h^1_{\omega_m lm}(\tilde{t}, r) e^{-im\phi_p(t)} Y_{lm}$

Framework for computing a waveform



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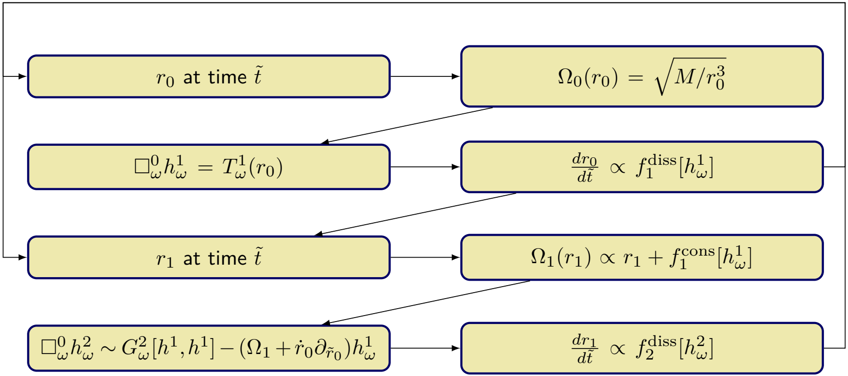
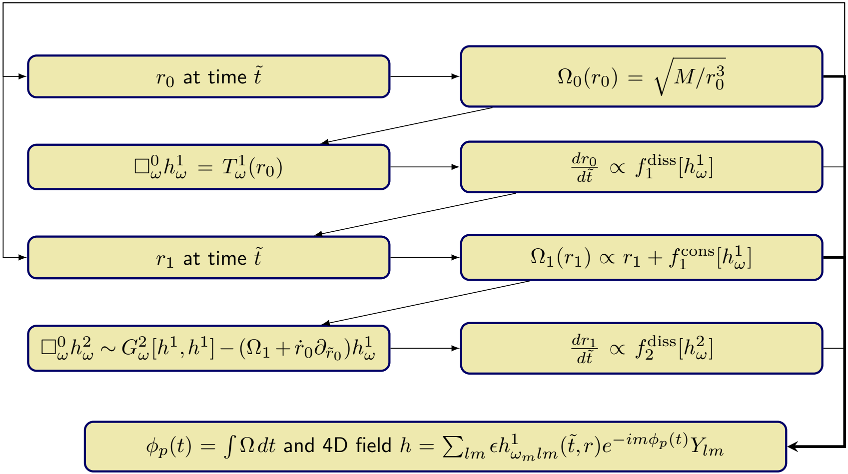


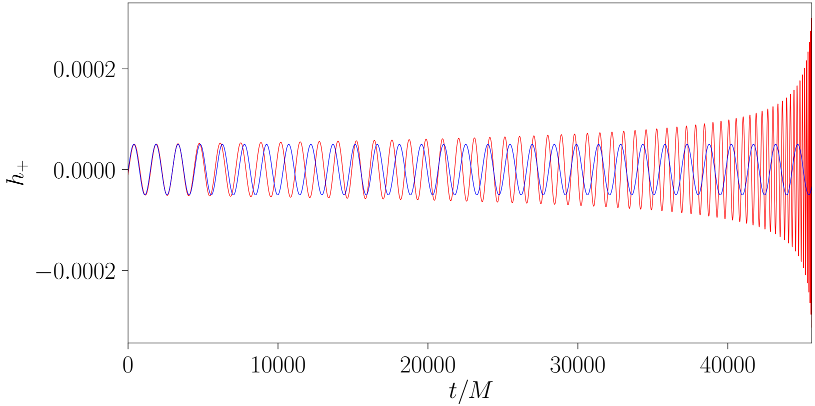
$$\phi_p(t) = \int \Omega\, dt \text{ and 4D field } h = \sum_{lm} \epsilon h^1_{\omega_m lm}(\tilde{t}, r) e^{-im\phi_p(t)} Y_{lm}$$

Framework for computing a waveform



Demonstration in Lorenz gauge

22 mode of adiabatic vs geodesic waveform



Notes on two-timescale framework

- solving the two-timescale equations gives us the post-adiabatic inspiral and waveform as an output
- the two-timescale expansion is currently the main framework under development for computing post-adiabatic waveforms — or for doing any second-order calculations at all

How you can contribute

- optimize numerical implementation of $G^2_\omega[h^1, h^1]$
- calculate explicit punctures h^{P2}_ω (in coordinates, decomposed into harmonics) for generic orbits in Schwarzschild and Kerr
- calculate derivatives of h^1_ω with respect to orbital parameters for generic orbits
- calculate explicit punctures at horizon and infinity for generic orbits
- incorporate particle's spin into two-timescale framework
- develop near-resonance expansions to transition across resonances
- develop method of computing h^n_ω in Lorenz gauge in Kerr
- develop tools for computing ψ^2_ω (e.g., coupling formula $h^1_{\omega lm} \to S_{\omega lm}[h^1, h^1]$)
- develop method of analytically calculating second-order quantities using PN expansions