I. The many faces of the first law in general relativity

Laws of Mechanics for Isolated Black Hole

Bardeen et. al 1973

Zeroth law :

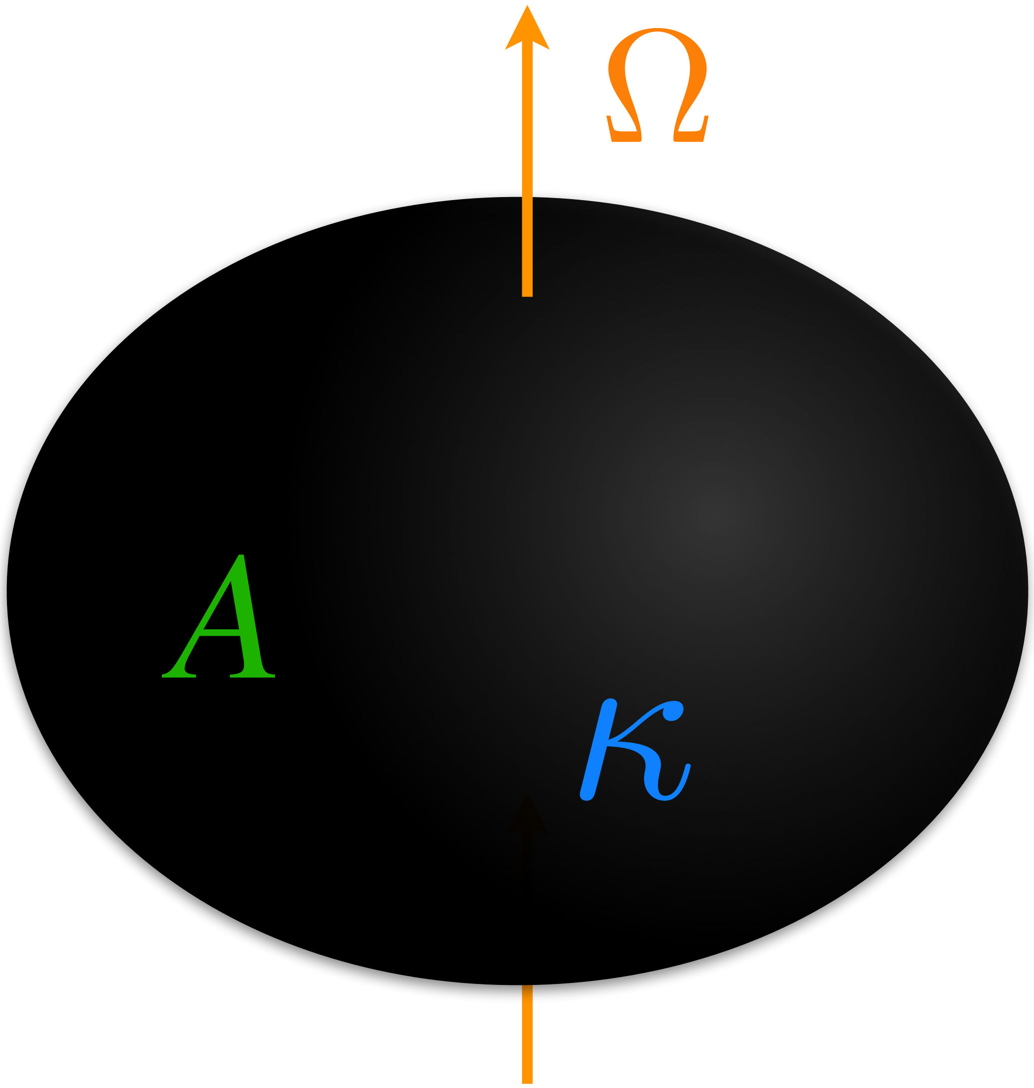
$$\kappa = \mathrm{cst}$$

First law :

$$\delta M = \Omega\,\delta J + \frac{\kappa}{8\pi}\delta A$$

Second law :

$$\delta A \geq 0$$



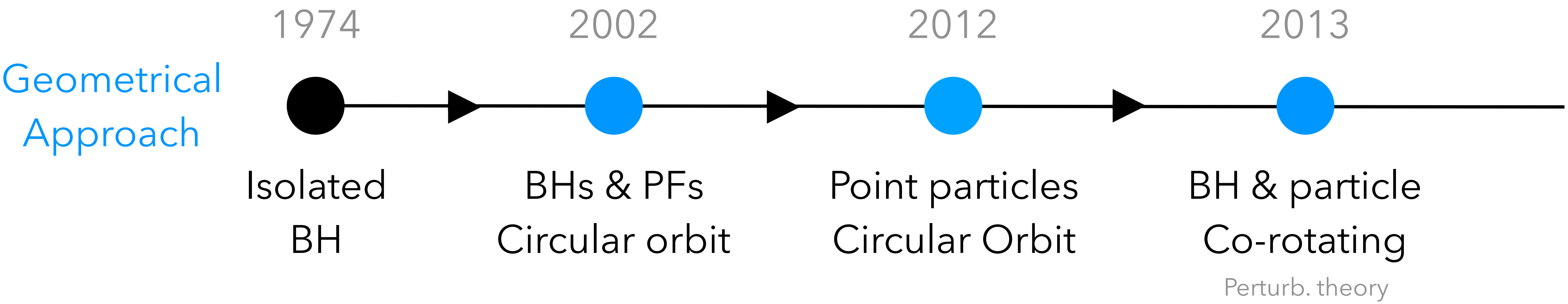
Add **semi-classical** treatment
to get *"Thermodynamics"*

Extensions of the First Law

1974

Isolated
BH

Extensions of the First Law



Geometrical Approach

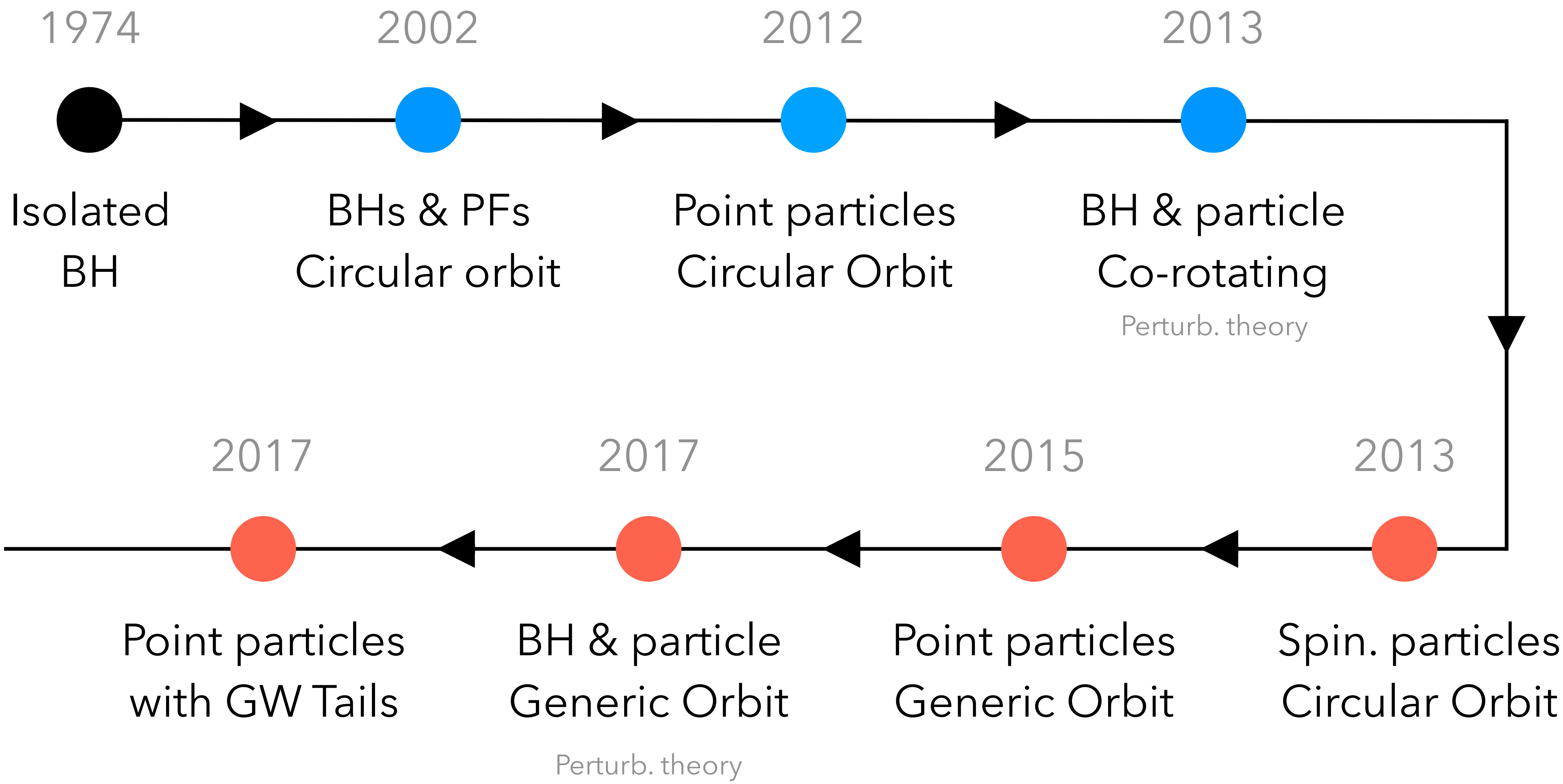
1974
Isolated
BH

2002
BHs & PFs
Circular orbit

2012
Point particles
Circular Orbit

2013
BH & particle
Co-rotating
Perturb. theory

Extensions of the First Law



Geometrical Approach

1974
Isolated BH

2002
BHs & PFs Circular orbit

2012
Point particles Circular Orbit

2013
BH & particle Co-rotating
Perturb. theory

Hamiltonian Approach

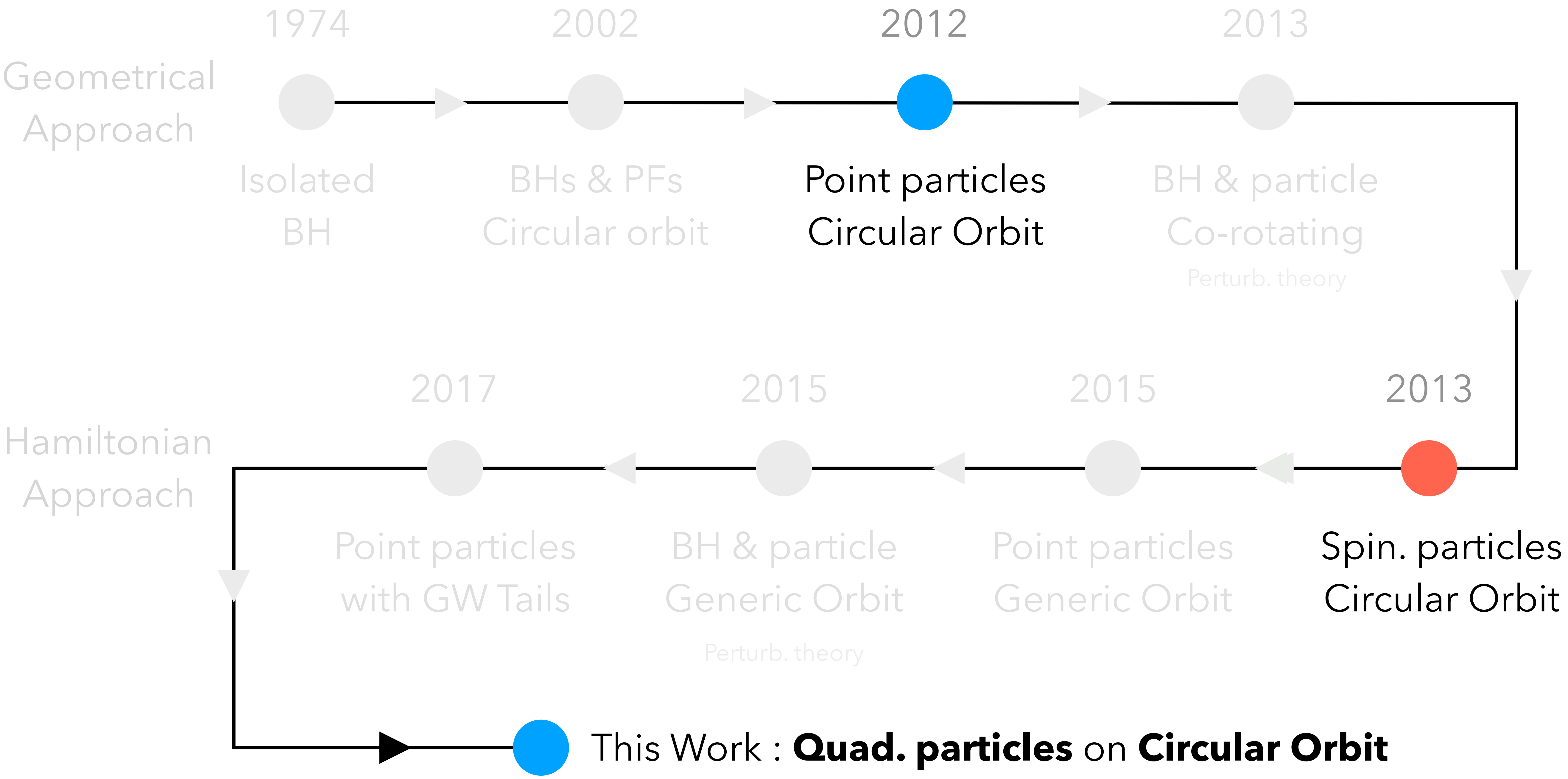
2013
Spin. particles Circular Orbit

2015
Point particles Generic Orbit

2017
BH & particle Generic Orbit
Perturb. theory

2017
Point particles with GW Tails

Extensions of the First Law



Geometrical Approach

1974
Isolated BH

2002
BHs & PFs
Circular orbit

2012
Point particles
Circular Orbit

2013
BH & particle
Co-rotating
Perturb. theory

Hamiltonian Approach

2013
Spin. particles
Circular Orbit

2015
Point particles
Generic Orbit

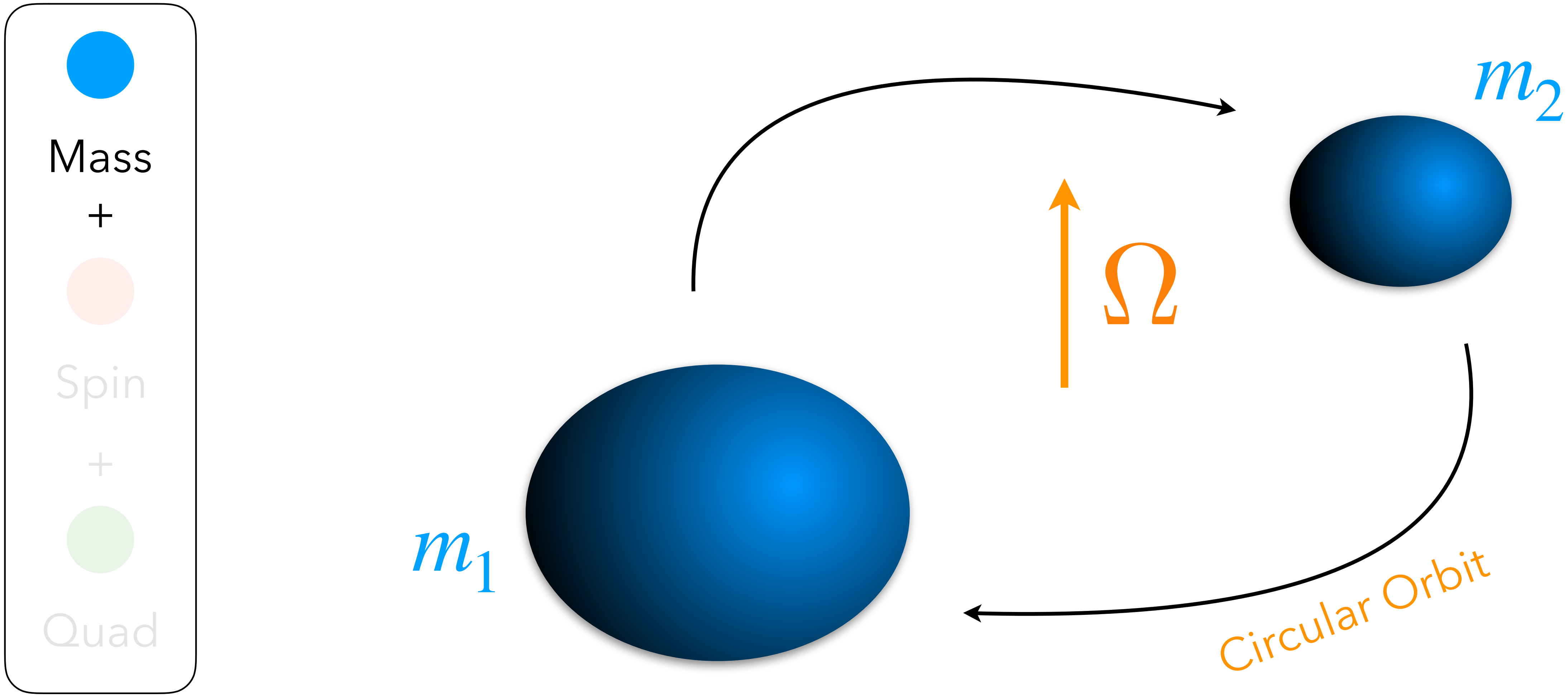
2015
BH & particle
Generic Orbit
Perturb. theory

2017
Point particles
with GW Tails

This Work : **Quad. particles** on **Circular Orbit**

First law for monopolar particles

$$\delta M - \Omega \delta J = \sum_i z_i \delta m_i - z_i \omega_i \delta S_i + F(Q_i)?$$



First law for dipolar particles

$$\delta M - \Omega \delta J = \sum_i z_i \delta m_i - z_i \omega_i \delta S_i + F(Q_i)?$$

●

Mass

+

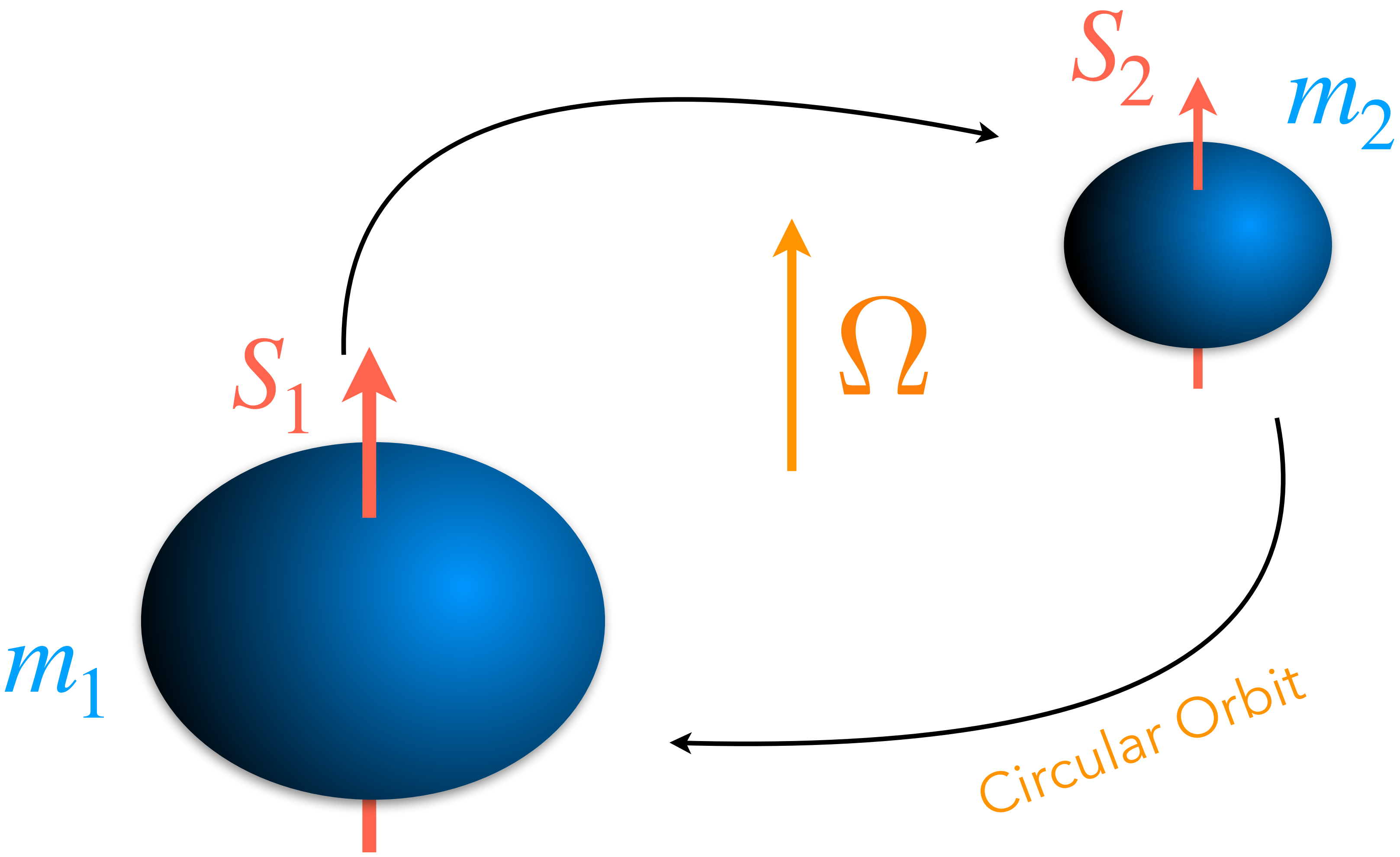
●

Spin

+

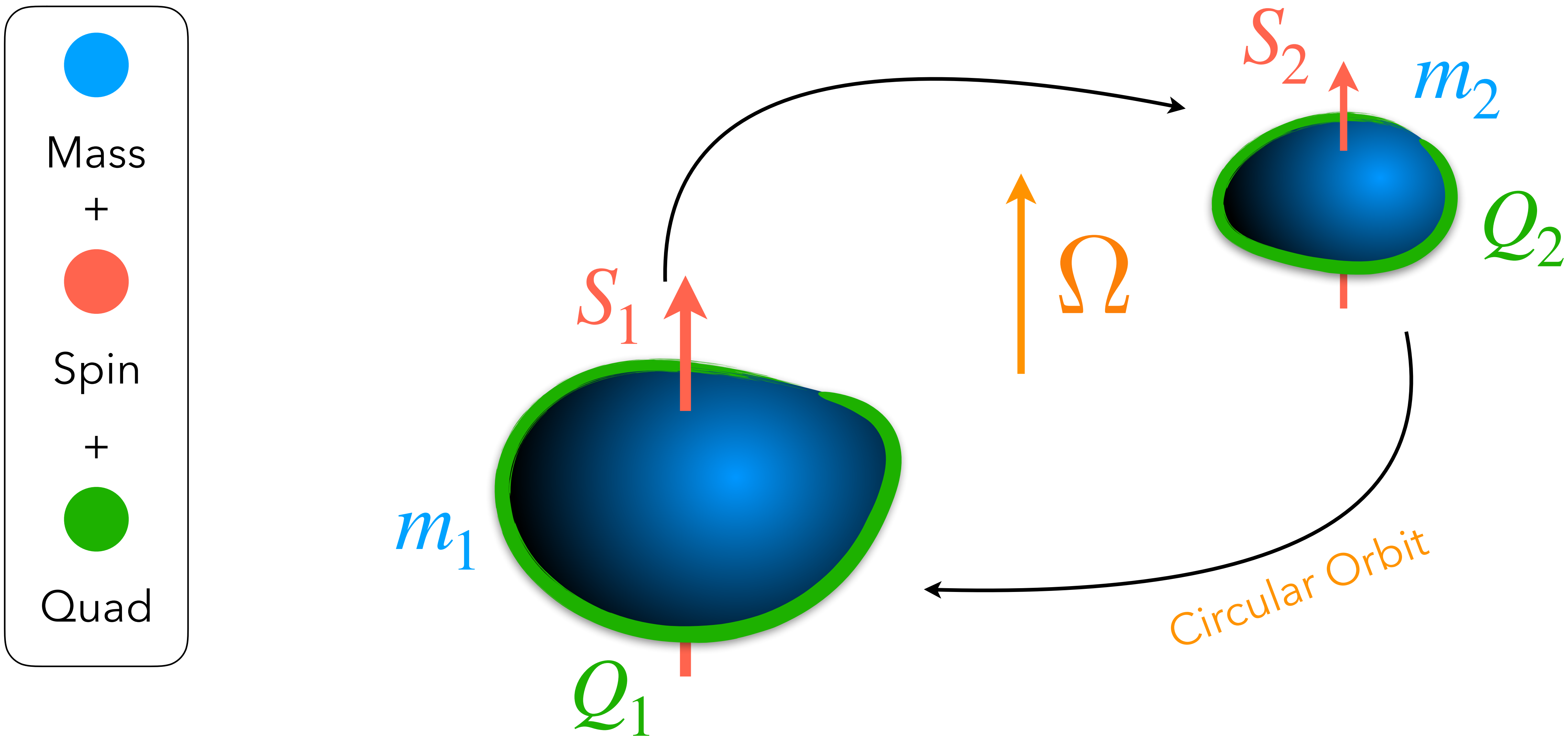
●

Quad



Goal : First law for quadrupolar particles

$$\delta M - \Omega \delta J = \sum_i z_i \delta m_i - z_i \omega_i \delta S_i + F(Q_i)?$$



Variant of Wald & Iyer identity

Variational identity btw Noether charge Q and SE tensor T^{ab}

$$\delta Q = \delta \int_{\mathcal{S}} \varepsilon_{abcd} T^{de} k_e - \frac{1}{2} \int_{\mathcal{S}} \varepsilon_{abcd} k^d T^{ef} \delta g_{ef}$$

for a spacetime g_{ab} with a Killing vector field k^a

Essentially comes from Stokes theorem applied to $G_{ab} = 8\pi T_{ab}$

Variant of Wald & Iyer identity

Variational identity btw Noether charge Q and SE tensor T^{ab}

$$\delta Q = \delta \int_{\mathcal{S}} \varepsilon_{abcd} T^{de} k_e - \frac{1}{2} \int_{\mathcal{S}} \varepsilon_{abcd} k^d T^{ef} \delta g_{ef}$$

for a spacetime g_{ab} with a Killing vector field k^a

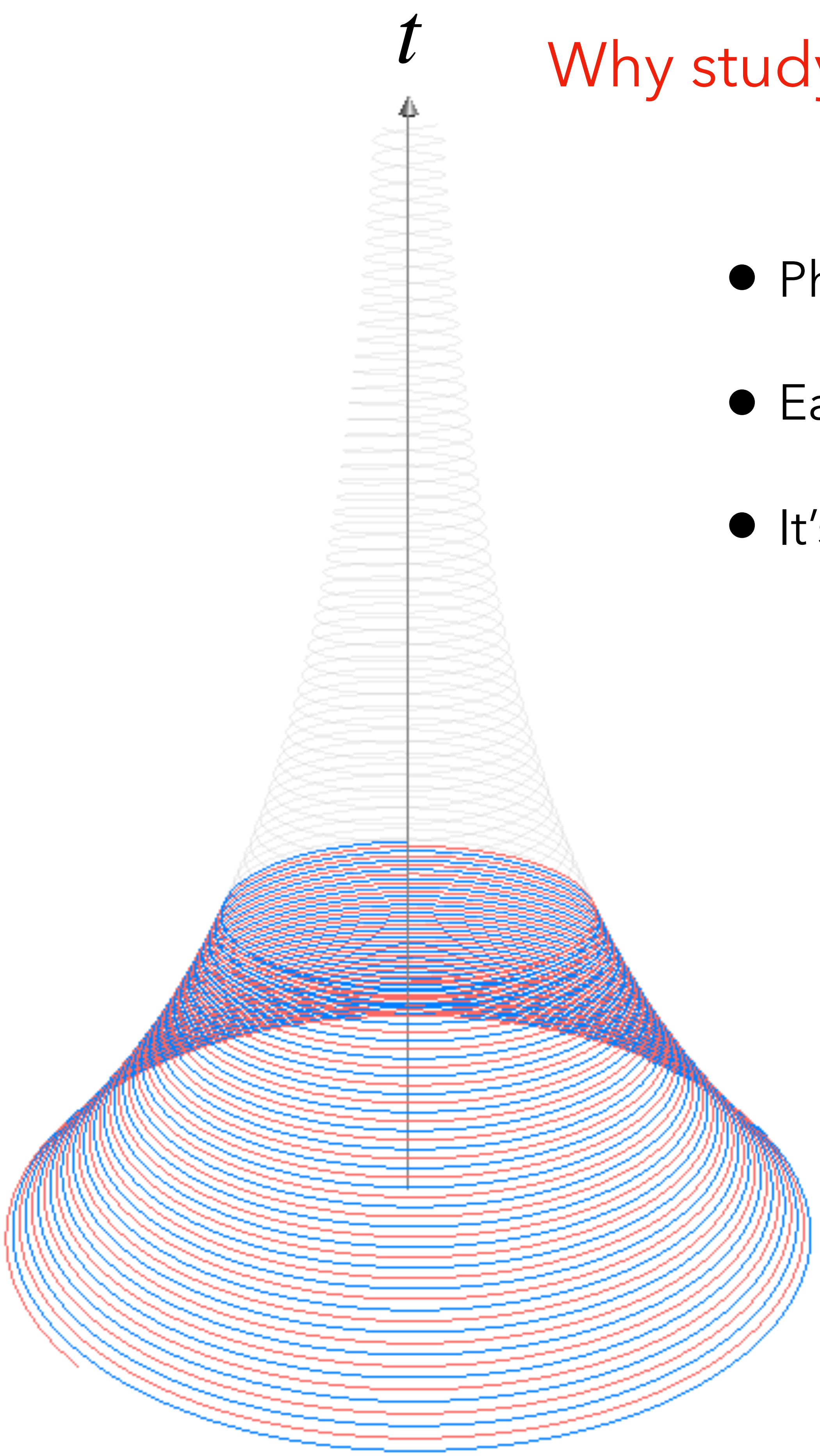
Essentially comes from Stokes theorem applied to $G_{ab} = 8\pi T_{ab}$

Application to binaries : circular orbit with **helical** symmetry

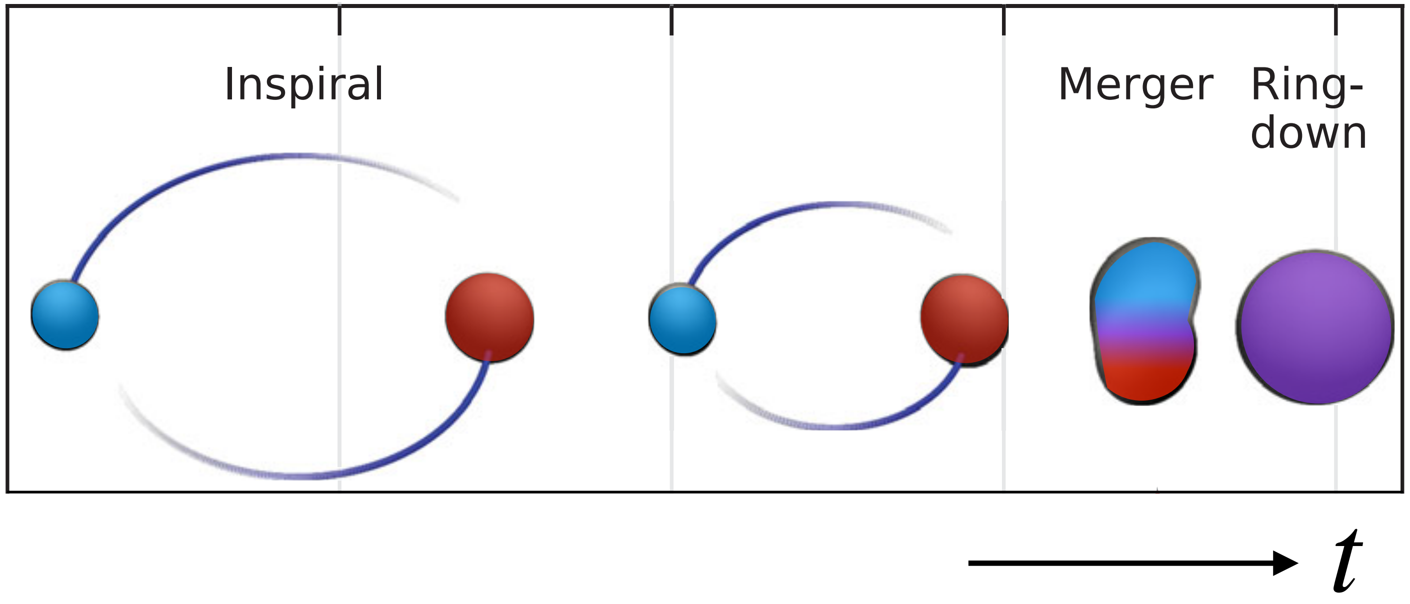
Therefore, we just need the T^{ab} of the compact objects…

II. Circular orbits & helical Killing vectors

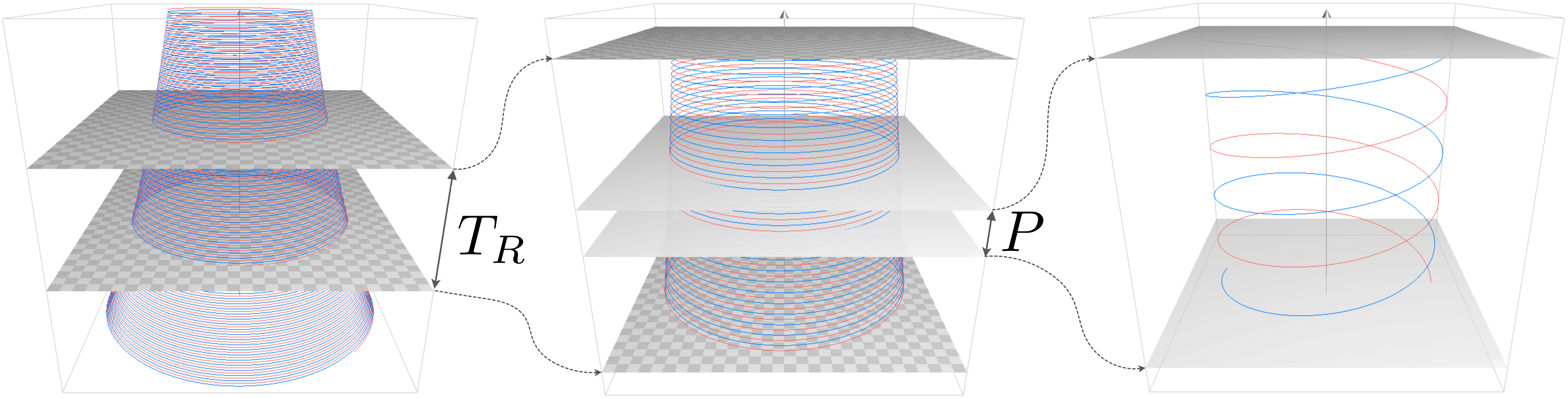
Why study (quasi)-circular orbits ?



- Physically relevant to LIGO/Virgo, SMBH binaries, ..
- Easy, starting case for EMRI’s
- It’s amenable to geometrical approach (Killing vector)



Relevance of circular orbits



T_R
Radiation reaction

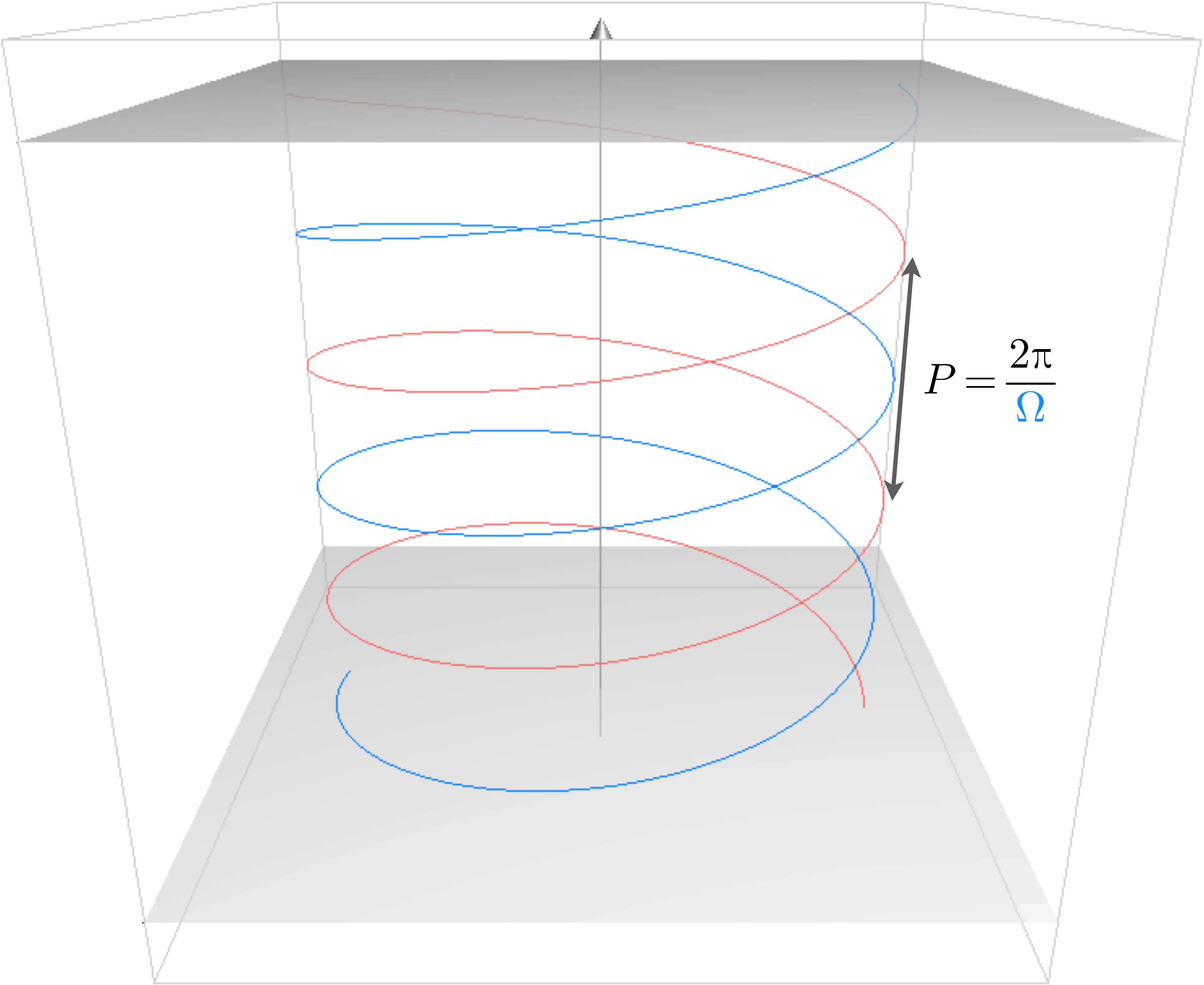
P
Orbital period

Adiabatic approx. :
(succession of circular orbits)

$$\frac{P}{T_R} \propto \frac{m_1 m_2}{(m_1 + m_2)^2} \left(\frac{v}{c}\right)^5 \ll 1$$

Helical Killing vector

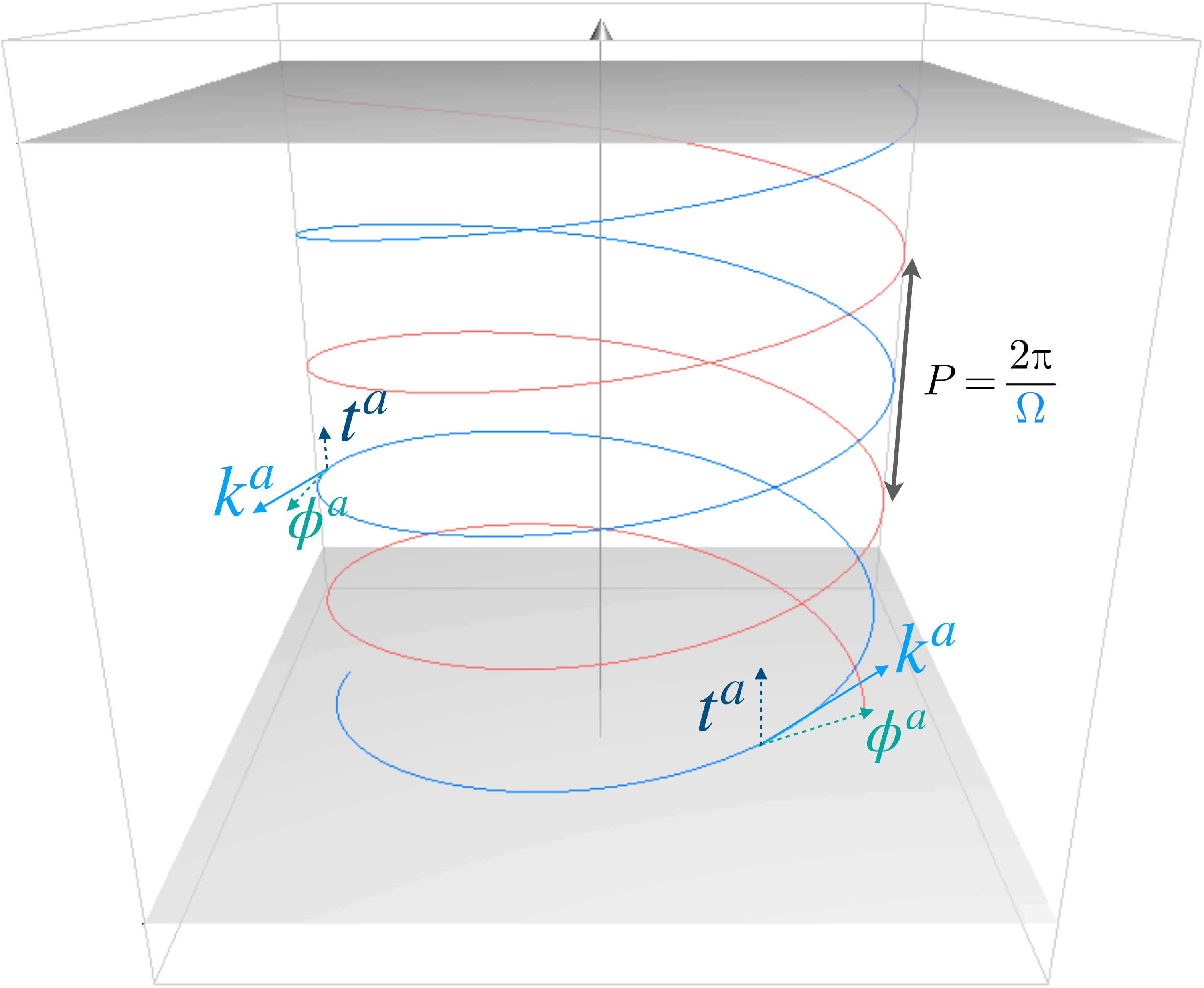
Circular Orbit



$$P = \frac{2\pi}{\Omega}$$

Helical Killing vector

Circular Orbit

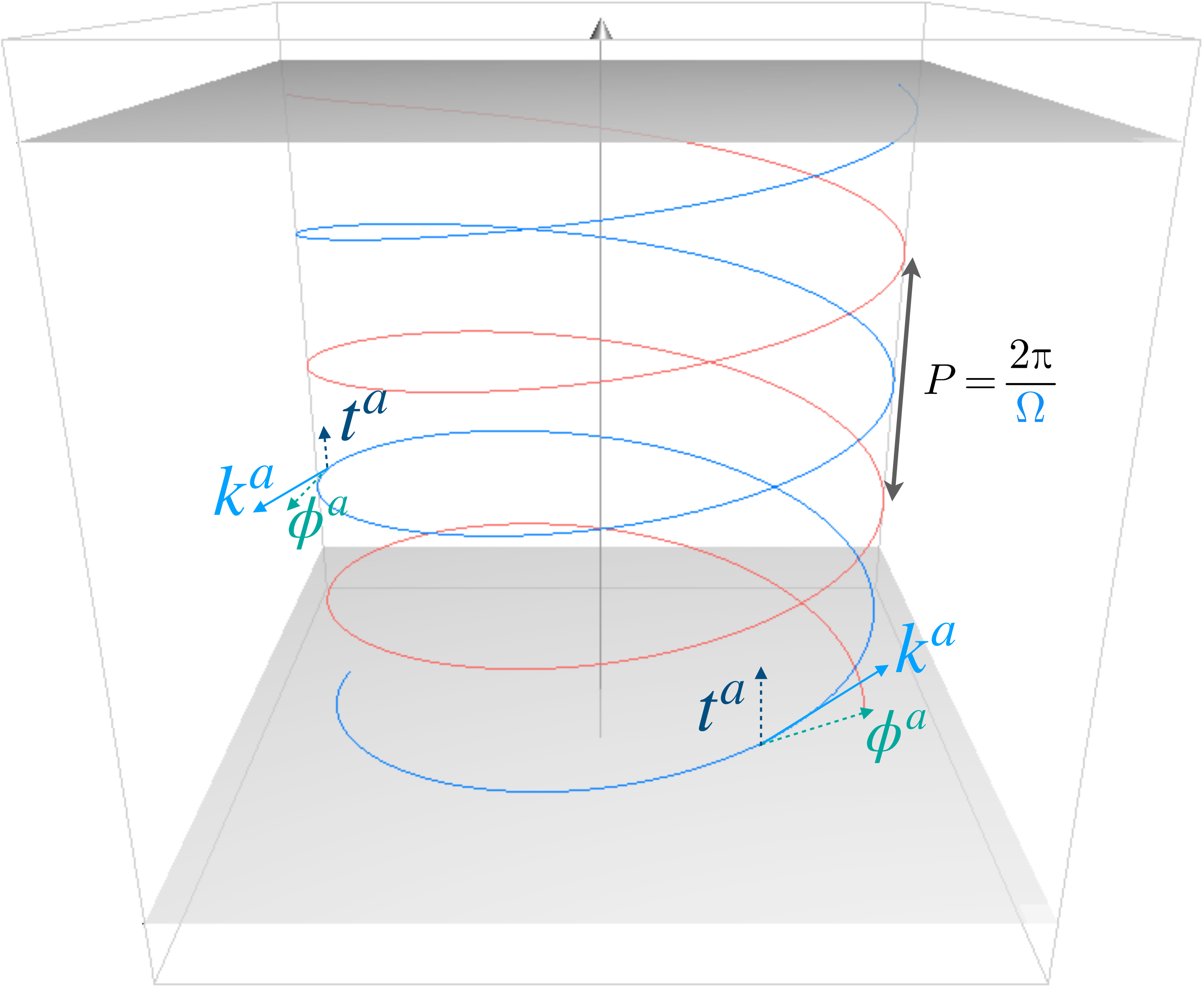


Killing vector

$$k^a = t^a + \Omega \phi^a$$

Helical Killing vector

Circular Orbit



Helical Killing vector

$$k^a = t^a + \Omega\phi^a$$

$$\Downarrow$$

Lie dragging

$$\mathcal{L}_k g_{ab} = 0$$

+

Conserved Noether charge

$$\delta Q \overset{\star}{=} \delta M - \Omega\delta J$$

III. Gravitational skeleton

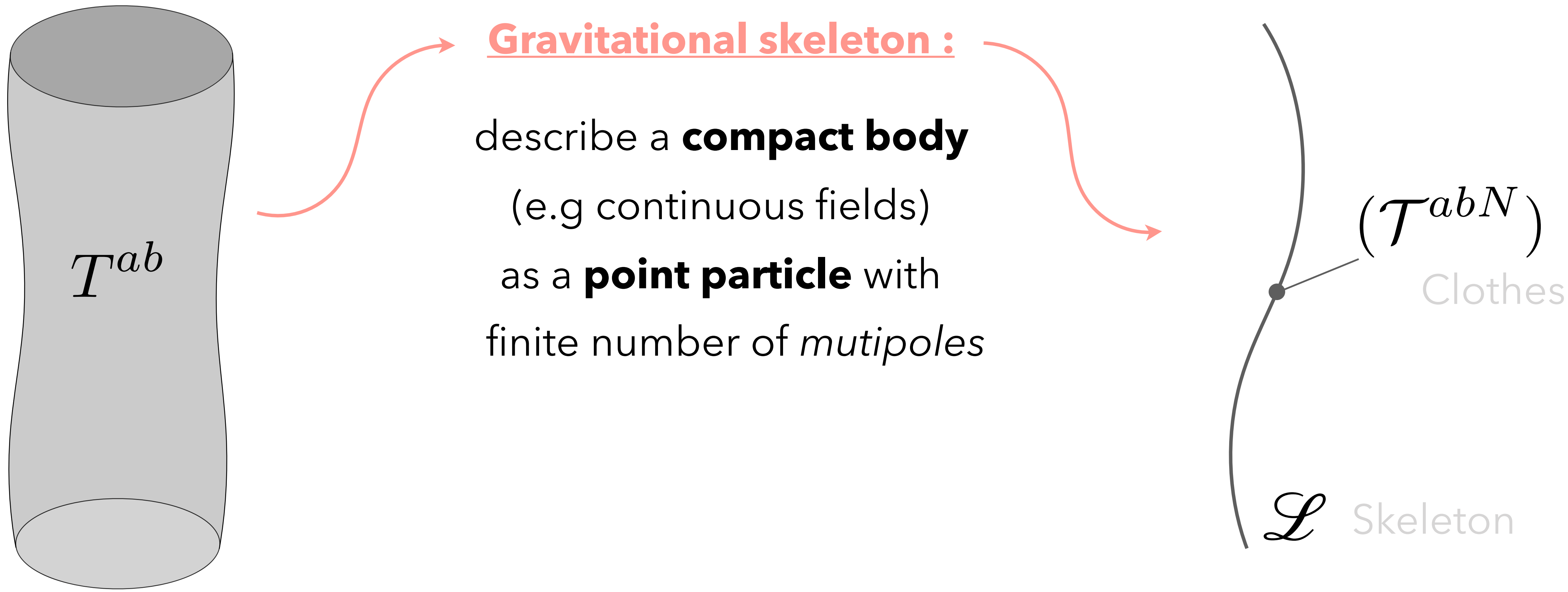
Multipole-particle formalisms

- MP, Tulczyjew : **gravitational skeleton**
- Harte : generalised Killing vectors

- Dixon : integral fluid-like moments
- Bailey, Israel : Lagrangian formalism

Multipolar particle formalisms

- MP, Tulczyjew : **gravitational skeleton**
- Harte : generalised Killing vectors
- Dixon : integral fluid-like moments
- Bailey, Israel : Lagrangian formalism



Gravitational skeleton :

describe a **compact body**

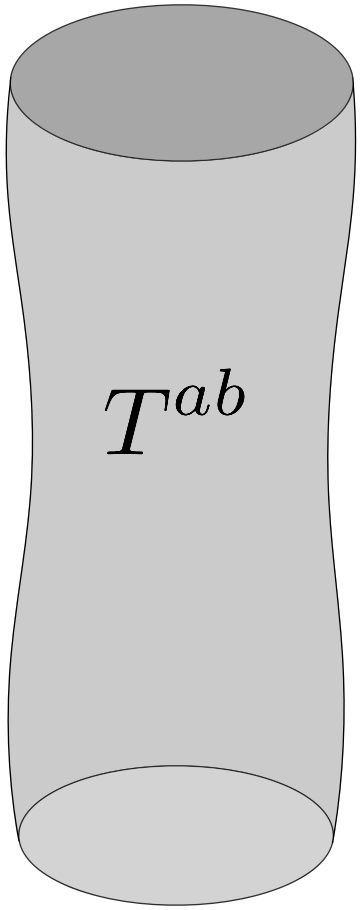
(e.g continuous fields)

as a **point particle** with

finite number of *mutipoles*

Multipolar particle formalisms

- MP, Tulczyjew : **gravitational skeleton**
- Harte : generalised Killing vectors
- Dixon : integral fluid-like moments
- Bailey, Israel : Lagrangian formalism



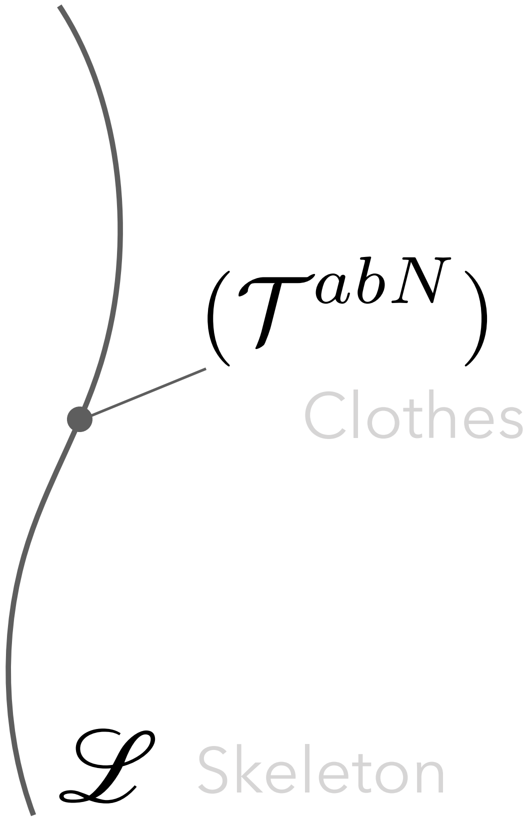
Gravitational skeleton :

describe a **compact body**

(e.g continuous fields)

as a **point particle** with

finite number of *mutipoles*



$$\boxed{T^{ab} = \sum_n \nabla_N \int_{\mathscr{L}} \mathcal{T}^{abN} \delta_4 \mathrm{d}\tau}$$

Exemple : dipolar particle

1. **Truncate** at $n = 1$ $$T^{ab} = \int_{\mathscr{L}} \mathcal{T}^{ab} \delta_4 \mathrm{d}\tau + \nabla_c \int_{\mathscr{L}} \mathcal{T}^{abc} \delta_4 \mathrm{d}\tau$$

Exemple : dipolar particle

1. **Truncate** at $n = 1$ $$T^{ab} = \int_{\mathscr{L}} \mathcal{T}^{ab} \delta_4 \mathrm{d}\tau + \nabla_c \int_{\mathscr{L}} \mathcal{T}^{abc} \delta_4 \mathrm{d}\tau$$

2. Use **SE conservation** $$\nabla_a T^{ab} = 0$$

Exemple : dipolar particle

1. Truncate at $n = 1$ $$T^{ab} = \int_{\mathscr{L}} \mathcal{T}^{ab} \delta_4 \mathrm{d}\tau + \nabla_c \int_{\mathscr{L}} \mathcal{T}^{abc} \delta_4 \mathrm{d}\tau$$

2. Use SE conservation $$\nabla_a T^{ab} = 0$$

then there exists p^a **and** S^{ab} **such that**

3. **Reduced** SE tensor $$T^{ab} = \int_{\mathscr{L}} u^{(a} p^{b)} \delta_4 \mathrm{d}\tau + \nabla_c \int_{\mathscr{L}} u^{(a} S^{b)c} \delta_4 \mathrm{d}\tau$$

4. **Evolution** equations $$\dot{p}^a = \frac{1}{2} R_{bcd}{}^a S^{bc} u^d$$ and $$\dot{S}^{ab} = 2 p^{[a} u^{b]}$$

SSC and helical isometry

(u^a, p^a, S^{ab}) = 3 + 4 + 6 degrees of freedom

Evolution Eqs give 4 + 6

$$\dot{p}^a = \frac{1}{2} R_{bcd}{}^a S^{bc} u^d$$

$$\dot{S}^{ab} = 2\, p^{[a} u^{b]}$$

need 3
add. Eqs.

Spin Supp. Condition

$$D^a := u_b S^{ab} = 0$$

- closes the diff. system
- eq. to choice of worldline

SSC and helical isometry

(u^a, p^a, S^{ab}) = 3 + 4 + 6 degrees of freedom

Evolution Eqs give 4 + 6

$$\dot{p}^a = \frac{1}{2} R_{bcd}{}^a S^{bc} u^d$$

$$\dot{S}^{ab} = 2\, p^{[a} u^{b]}$$

need 3
add. Eqs.

Spin Supp. Condition

$$D^a := u_b S^{ab} = 0$$

- closes the diff. system
- eq. to choice of worldline

$$\mathcal{L}_k g_{ab} = 0 \quad \Rightarrow \quad \mathcal{L}_k T^{ab} = 0 \quad \Rightarrow \quad \begin{cases} \mathcal{L}_k u^a = 0 \\ \mathcal{L}_k p^a = 0 \\ \mathcal{L}_k S^{ab} = 0 \end{cases}$$

IV. Derivation of the First Law

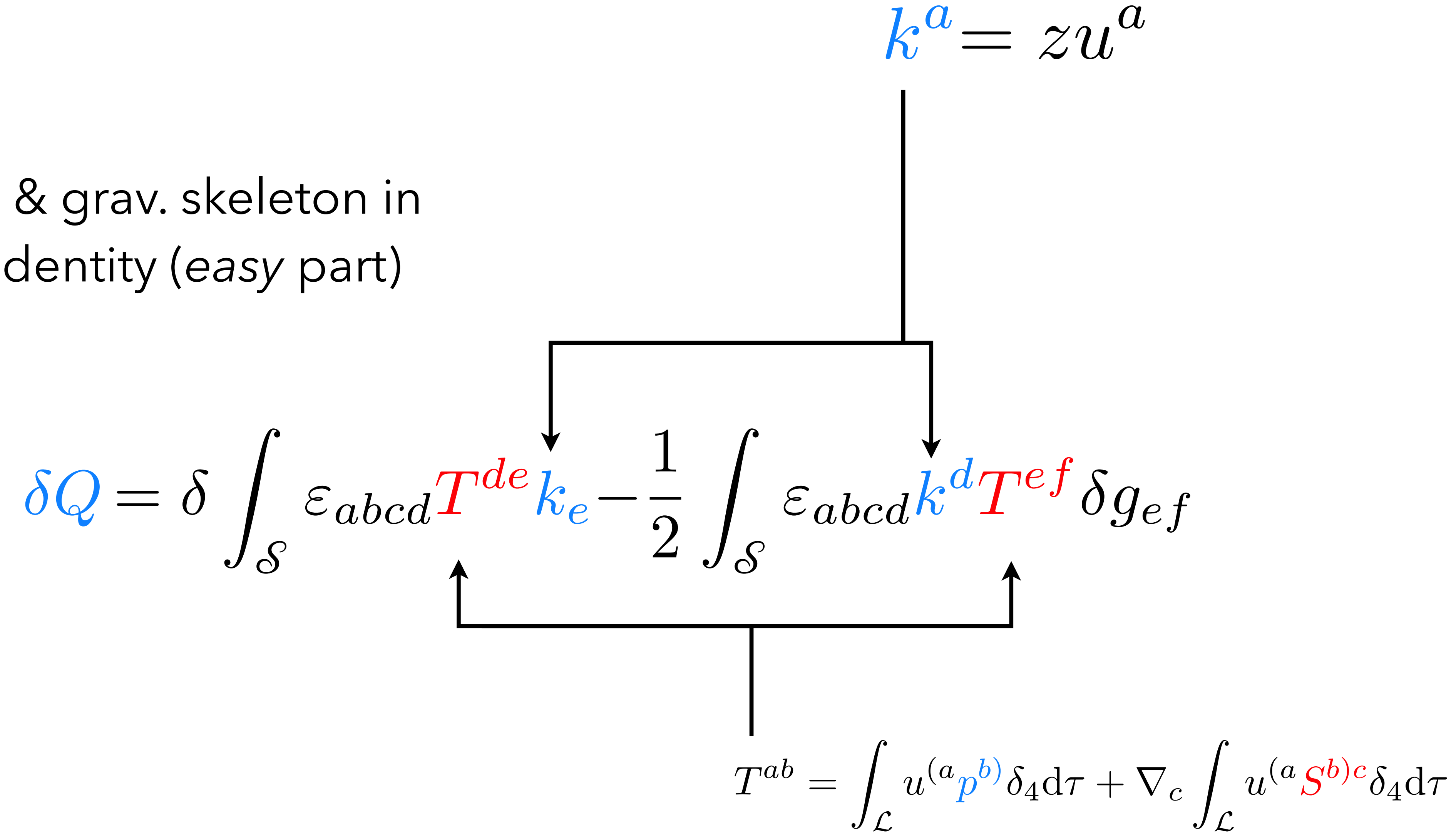
An assumption and the recipe

1. particle must follow helical trajectory (enforces circular orbit) $\Leftrightarrow$ $\mathscr{L}$ is an integral curve of k^a

$$k^a = z u^a$$

An assumption and the recipe

1. particle must follow helical trajectory (enforces circular orbit) ⇔ $\mathscr{L}$ is an integral curve of k^a



$$k^a = zu^a$$

2. Insert this & grav. skeleton in variational identity (*easy* part)

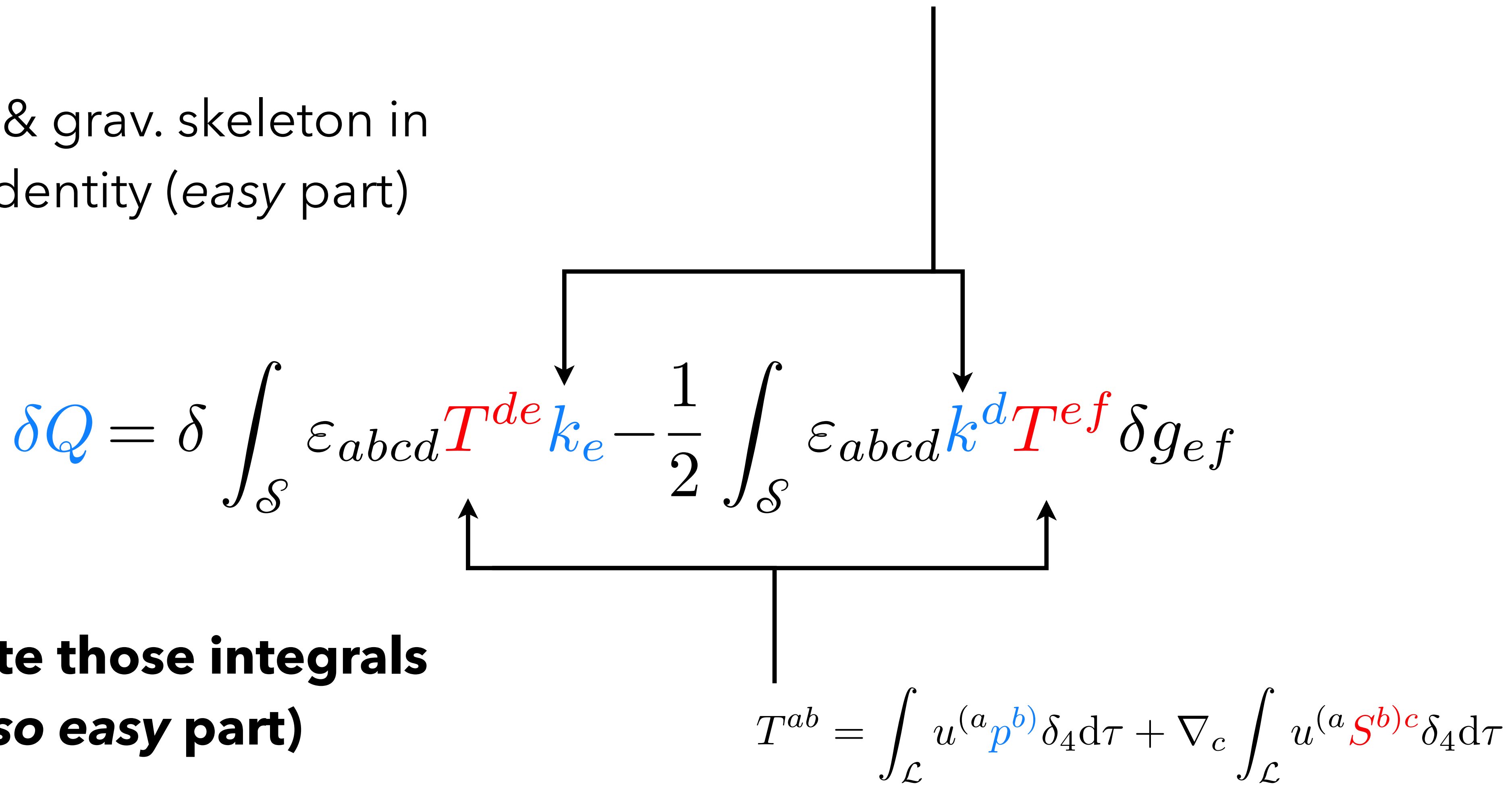
$$\delta Q = \delta \int_{\mathcal{S}} \varepsilon_{abcd} T^{de} k_e - \frac{1}{2} \int_{\mathcal{S}} \varepsilon_{abcd} k^d T^{ef} \delta g_{ef}$$

$$T^{ab} = \int_{\mathcal{L}} u^{(a} p^{b)} \delta_4 \mathrm{d}\tau + \nabla_c \int_{\mathcal{L}} u^{(a} S^{b)c} \delta_4 \mathrm{d}\tau$$

An assumption and the recipe

1. particle must follow helical trajectory (enforces circular orbit) ⇔ $\mathscr{L}$ is an integral curve of k^a

$$k^a = z u^a$$



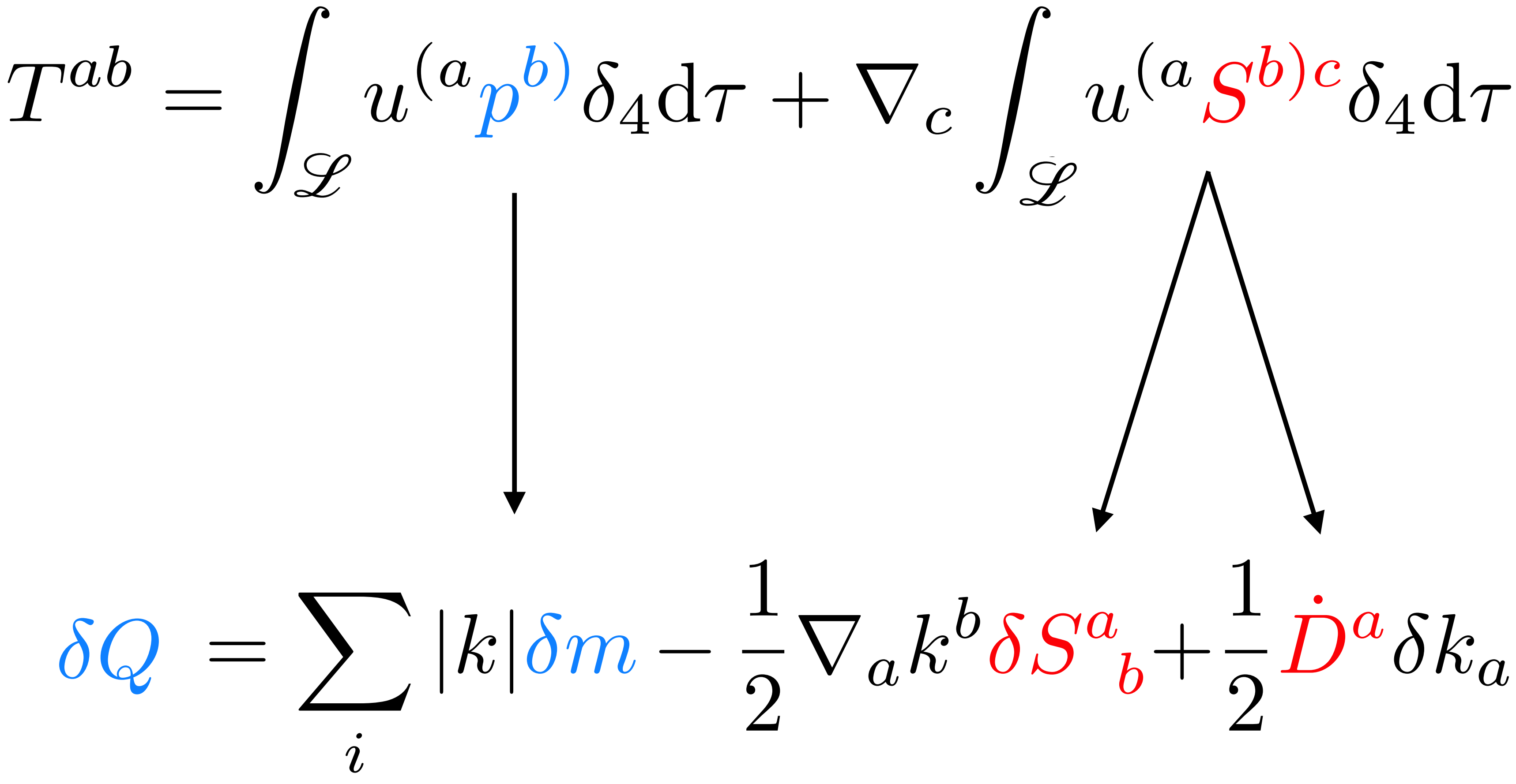
2. Insert this & grav. skeleton in variational identity (*easy* part)

$$\delta Q = \delta \int_{\mathcal{S}} \varepsilon_{abcd} T^{de} k_e - \frac{1}{2} \int_{\mathcal{S}} \varepsilon_{abcd} k^d T^{ef} \delta g_{ef}$$

3. Compute those integrals (*not so easy* part)

$$T^{ab} = \int_{\mathcal{L}} u^{(a} p^{b)} \delta_4 \mathrm{d}\tau + \nabla_c \int_{\mathcal{L}} u^{(a} S^{b)c} \delta_4 \mathrm{d}\tau$$

Result at dipolar order



$$T^{ab} = \int_{\mathscr{L}} u^{(a} p^{b)} \delta_4 \mathrm{d}\tau + \nabla_c \int_{\mathscr{L}} u^{(a} S^{b)c} \delta_4 \mathrm{d}\tau$$

First law : $$\delta Q = \sum_i |k| \delta m - \frac{1}{2} \nabla_a k^b \delta S^a{}_b + \frac{1}{2} \dot{D}^a \delta k_a$$

This result

- is **exact** and **SSC–independant**
- includes all **spin non-linearities**

Scalar first law

First law : $\delta Q = \sum_i |k| \delta m - \frac{1}{2} \nabla_a k^b \delta S^a{}_b + \frac{1}{2} \dot{D}^a \delta k_a$

Scalar first law

First law :

$$\delta Q = \sum_i |k|\delta m - \frac{1}{2}\nabla_a k^b \delta S^a{}_b + \frac{1}{2}\dot{D}^a \delta k_a$$

$\big\downarrow$ approx. scheme $\big\downarrow$ SSC

$$\delta M - \Omega\delta J = \sum_i |k|\delta m - \frac{1}{2}\nabla_a k^b \delta S^a{}_b + 0$$

Scalar first law

First law : $\delta Q = \sum_i |k| \delta m - \frac{1}{2} \nabla_a k^b \delta S^a{}_b + \frac{1}{2} \dot{D}^a \delta k_a$

approx. scheme

SSC

$$\delta M - \Omega \delta J = \sum_i |k| \delta m - \frac{1}{2} \nabla_a k^b \delta S^a{}_b + 0$$

spin expansion

$$\delta M - \Omega \delta J = \sum_i |k| \delta m - |\nabla k| \delta S + O(S^2)$$

V. Summary and conclusions

How is the first law used ?

Compare results between approx. schemes and NR and compute SF corrections

arXiv:1111.5609

Compute frequency shifts of ISCO from cons. part of SF

arXiv:1209.0964
arXiv:1404.6133

Calibrate EOB models

arXiv:1512.03392,
arXiv:1511.04533

Define redshift for NR, relate redshift variable to surface gravity

arXiv:1606.08056
arXiv: 1710.03673

At quad order : constrain NS equation of state

arXiv:1805.11581

What has been done

- Derive a variant of the Wald&Iyer variational formula
- Fully covariant and geometric derivation of the first law
- Recover all previous results on circular orbits
- Interesting results regarding EoM in *helical* context

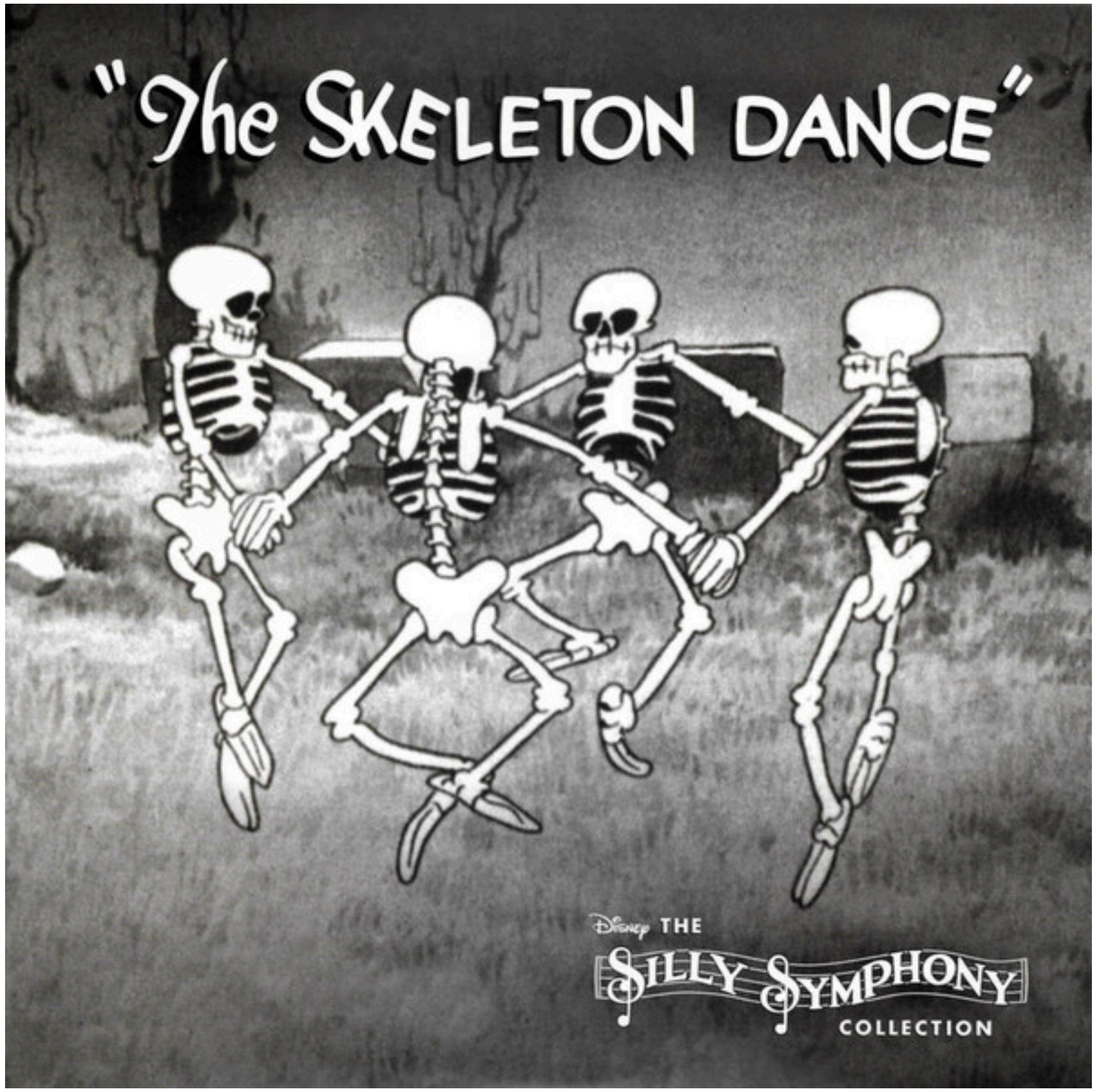
What remains to be done

- Geometrical setup allows to go easily to quad. order
- Can technically be applied to any kind of quad. model (spin, tidal, etc)
- What we expect to find (cf **Capra23**, hopefully…)

$$\delta M - \Omega\delta J = \sum_i |k|\delta m - |\nabla k|\delta S + |\nabla\nabla k|\delta Q \quad (?)$$

First law of mechanics
for quadrupolar compact objects

or



Paul Ramond   Capra 22 – 20/06/19   CBPF – Rio

Quadrupolar order

SE tensor $$T^{ab}_{\textit{quad}} = \int_{\mathscr{C}} {}^{1}\!/_{3} \left(R_{cde}{}^{(a} J^{b)cde} \delta_4 - 2\nabla_c \nabla_d [J^{c(ab)d} \delta_4] \right) \} \mathrm{d}\tau \,,$$

Evolution Eqs $$\begin{cases} \dot{p}^a = \dfrac{1}{6} R_{bcd}{}^{a} u^d S^{bc} - \dfrac{1}{6} \nabla^a [R_{bcde}] J^{bcde} \,, \\ \dot{S}^{ab} = 2 p^{[a} u^{b]} + \dfrac{4}{3} R_{edc}{}^{[a} J^{b]cde} \,. \end{cases}$$