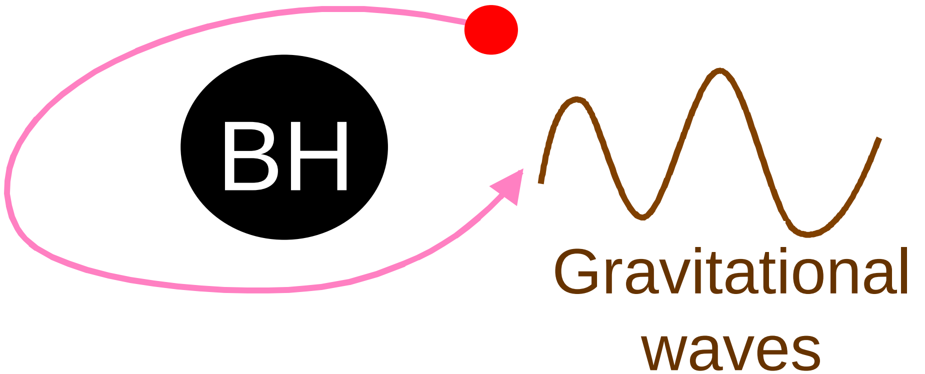
Hamiltonian formulation for extreme mass-ratio inspirals(EMRIs)



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arXiv:1809.11118
(arXiv:1612.02504)

Our new Innovative Area (grant) has just started in summer 2017.

Synergy between data analysis and theory researches

A: BH binaries

A-01
Testing gravity using gravitational waves

A-02
Gravity and Cosmology

A-03
BH binary formation

B: NS binaries

B-01
Internal structure of NS

B-02
Gamma-ray burst and BH

B-03
r –process elements

C: Supernovae

C-01
SN explosion mechanism

C-02
SN explosion mechanism via ν observation

Physics and astronomy motivated by GW observations

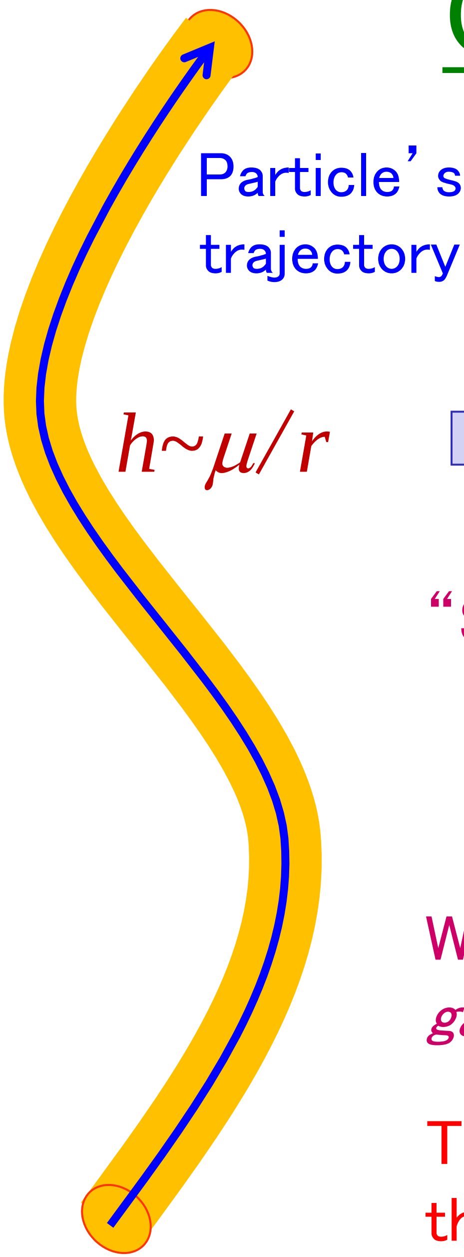
Out Target

Giving a prescription how to predict
the gravitational waveform from EMRIs,
using black hole perturbation.

Accurate and fast evaluation of
GW phase evolution is especially important to
compare the template with observations.

So, we want to establish
an economical way to compute
the orbital phase evolution
taking into account the self-force.

Gauge invariance



Perturbation is everywhere small outside the world tube

"tube radius" >> μ (mass of satellite)

Unavoidable ambiguity in the perturbed trajectory of $O(\mu)$

"Self-force is *gauge dependent*"

$$F^{\mu}_{\text{self}}(\tau, \gamma)$$ has unnecessary information.

Source trajectory

While, "long term orbital evolution is *gauge invariant*"

There must be a concise description that keeps only the gauge invariant information

Leading order wave form

Energy balance argument is sufficient.

$$\boxed{\frac{dE_{GW}}{dt} = \frac{dE_{orbit}}{dt}}$$

Waveform $\equiv \dfrac{df}{dt}$ for quasi-circular orbits, for example.

$$\frac{df}{dt} = \frac{dE_{GW}}{dt} \Big/ \frac{dE_{orbit}}{df}$$

leading order

$$\frac{dE_{GW}}{dt} = \qquad 0 \qquad + O(\mu) + O(\mu^2)$$

$$\frac{dE_{orbit}}{df} = (\text{geodesic}) + O(\mu) + O(\mu^2)$$

Radiation reaction for general orbits

Radiation reaction to the Carter constant

Schwarzschild "constants of motion" $E, L_i \Leftrightarrow$ Killing vector ξ

$$E = -\xi^{\mu}_{(t)} u_{\mu}$$

Conserved current for the field corresponding to Killing vector exists.

$$E_{GW} = \int d\Sigma^{\mu} t^{(\mathrm{GW})}_{\mu\nu} \xi^{\nu}$$

$$\dot{E} = -\dot{E}_{\mathrm{GW}}$$ In total, conservation law holds.

Kerr conserved quantities $E, L_z \Leftrightarrow$ Killing vector

Q $\cancel{\Leftrightarrow}$ Killing vector

$$Q \equiv K^{\mu\nu} u_{\mu} u_{\nu} \qquad K_{(\mu\nu;\rho)} = 0$$

Killing tensor

We need to directly evaluate the self-force acting on the particle.

To obtain the self-forced motion, we just need to solve the geodesic equation on an appropriately regularized perturbed spacetime.

$$S=\frac{1}{2}\int g^{\mu\nu}u_\mu u_\nu d\tau=\underbrace{\frac{1}{2}\int g_{(0)}^{\mu\nu}u_\mu u_\nu d\tau}_{H_0}-\underbrace{\frac{1}{2}\int h_{(ret-S)}^{\mu\nu}u_\mu u_\nu d\tau}_{\text{interaction Hamiltonian } H_{\rm int}}$$

Canonical transformation to the action angle variables with the aid of constants of motion in the background:

$$P_\alpha\equiv\{H_0,-E,L_z,Q\}$$

Generating function:

$$W(x,J)=J_t t+J_\phi\phi+\int^r \underbrace{\tilde{u}_r(r',J)}_{}dr'+\int^\theta \underbrace{\tilde{u}_\theta(\theta',J)}_{}d\theta'$$

well-known for Kerr geodesic motion

$$J_r=\oint\tilde{u}_r(r',P)dr'\quad J_\theta=\oint\tilde{u}_\theta(\theta',P)d\theta'$$

$$u_\mu=\frac{\partial W}{\partial x^\mu}\qquad w^\alpha=\frac{\partial W}{\partial J_\alpha}$$

Gauge invariance of the angular velocity

Angle variables $w^a = O(\eta^{-1})$ ($\propto$ radiation reaction time) are gauge invariant in the context of long term evolution.

$\eta := \mu/M$

allowing $O(\eta)$ gauge ambiguity, if not $O(\eta\,\Delta T)$.

$\Rightarrow$ $\omega^a = \langle \dot{w}^a \rangle = O(\eta^0)$ should be invariant up to $O(\eta)$

$\because$

$$W(x,J) = J_t t + J_\phi \phi + \int^r \tilde{u}_r(r',J)\,dr' + \int^\theta \tilde{u}_\theta(\theta',J)\,d\theta'$$

$$J_r = \oint \tilde{u}_r(r',J)\,dr', \quad J_\theta = \oint \tilde{u}_\theta(\theta',J)\,d\theta'$$

$$W(x,J) = \tilde{W}(x,J) + n_r J_r + n_\theta J_\theta$$

where we introduce a single valued function with respect to x:

$$\tilde{W}(x,J) = J_t t + J_\phi \phi + \int_{r_0}^r \tilde{u}_r(r',J)\,dr' + \int_{\theta_0}^\theta \tilde{u}_\theta(\theta',J)\,d\theta'$$

$$w^I = \frac{\partial W(x,J)}{\partial J_I} = \frac{\partial \tilde{W}(x,J)}{\partial J_I} + n_I \quad (I = r,\theta) \qquad w^i = \frac{\partial W(x,J)}{\partial J_i} = \frac{\partial \tilde{W}(x,J)}{\partial J_i} \quad (i = t,\phi)$$

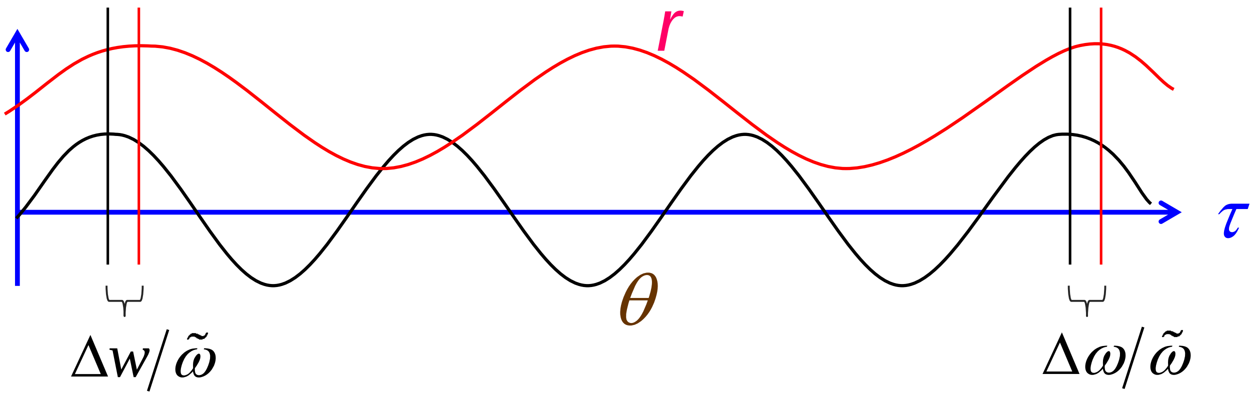
Small variations of x and J are not amplified in w.

Resonant orbits

For resonant case $\tilde{\omega} \equiv \omega_r / \beta_r = \omega_\theta / \beta_\theta$

holds for some integers β_r & β_q

$\Delta w \equiv w^\theta / \beta_\theta - w^r / \beta_r = w_0^\theta / \beta_\theta - w_0^r / \beta_r$ has physical meaning.



Under the time reversal, a resonant geodesic with Δw transforms into a resonant geodesics with $-\Delta w$.

Impact of the resonance on the phase evolution

(gravitational radiation reaction)

$$\frac{d\Delta w}{dt} \approx \Delta\omega \qquad \frac{d\Delta\omega}{dt} \approx \eta\omega^2$$

$\boxed{\eta := \mu/M}$

$\Rightarrow \Delta t_{res} = O\left(\sqrt{\eta^{-1}}\,\omega^{-1}\right)$: duration staying around resonance

$\Rightarrow \Delta\omega_{res} = O\left(\sqrt{\eta}\,\omega\right)$: extra frequency shift caused by passing through resonance

$\Rightarrow \delta w_{res} = \underline{O\left(\sqrt{\eta^{-1}}\right)}$: overall phase error due to resonance

$\neq O(\eta^0)$

If $\dfrac{d\Delta\omega}{dt} = 0$ for $\Delta w = \Delta w_c$,

Oscillation period is much shorter than the radiation reaction time

$$\begin{cases} \dfrac{d\left(\Delta w - \Delta w_c\right)}{dt} = O\left(\Delta\omega\right) \\ \dfrac{d\Delta\omega}{dt} = O\left(\eta\omega^2\left(\Delta w - \Delta w_c\right)\right) \end{cases} \Rightarrow \frac{d^2\left(\Delta w - \Delta w_c\right)}{dt^2} \approx \beta\eta\omega^2\left(\Delta w - \Delta w_c\right)$$

If β stays negative,
resonance may persist for a long time.

Radiation reaction to the action variables (constants of motion)

"retarded" field= "radiative" + "symmetric"

$$\underbrace{\text{"radiative"}}_{\frac{\text{"ret"}-\text{"adv"}}{2}} \qquad \underbrace{\text{"symmetric"}}_{\frac{\text{"ret"}+\text{"adv"}}{2}}$$

regularization is unnecessary

$$\left\langle \frac{dJ_\alpha}{d\tau} \right\rangle = \left\langle \frac{\partial H_{\mathrm{int}}}{\partial w^\alpha} \right\rangle = \left[\int d\tau \int d\tau' \frac{\partial}{\partial w^\alpha} G^{(ret-S)}(w, J; \gamma') \right]_{(w,J)=\gamma'=\gamma}$$

$\partial/\partial w^a$ can be replaced with $\underline{\partial/\partial w^\alpha_{\mathrm{ini}}}$.

initial value

$$\left\langle \frac{dJ_\alpha}{d\tau} \right\rangle = \left[\int d\tau \int d\tau' \frac{\partial}{\partial w^\alpha_{\mathrm{ini}}} G^{(ret-S)}(\gamma, \gamma') \right]_{\gamma'=\gamma}$$

$$= \left[\int d\tau \int d\tau' \frac{\partial}{\partial w^\alpha_{\mathrm{ini}}} G^{(rad)}(\gamma, \gamma') \right]_{\gamma'=\gamma} + \left[\int d\tau \int d\tau' \frac{\partial}{\partial w^\alpha_{\mathrm{ini}}} G^{(sym-S)}(\gamma, \gamma') \right]_{\gamma'=\gamma}$$

Simplification

Symmetric part

$$\left[\int d\tau \int d\tau' \frac{\partial}{\partial w_{\text{ini}}^{\alpha}} G^{(sym-S)}(\gamma,\gamma')\right]_{\gamma'=\gamma} = \frac{1}{2}\frac{\partial}{\partial w_{\text{ini}}^{\alpha}}\left[\int d\tau \int d\tau' G^{(sym-S)}(\gamma,\gamma')\right]_{\gamma'=\gamma} = \frac{1}{2}\frac{\partial}{\partial w_{\text{ini}}^{\alpha}}\left\langle H_{sym}\right\rangle$$

H_{sym} after substitution $\gamma'=\gamma$ is independent of w_{ini}^{a} for non-resonant case.

At the leading order in $\eta=\mu/M$, only the radiative part determines the change of “constants of motion”, except for resonance orbits.

(Mino (2003))

Even for resonant case, H_{sym} depends on w^{α} only through Δw.

$$\frac{\partial\left\langle H_{sym}\right\rangle}{\partial w_0^r} = \frac{\partial \Delta w}{\partial w_0^r}\frac{\partial\left\langle H_{sym}\right\rangle}{\partial \Delta w} = -\frac{1}{\beta^r}\frac{\partial\left\langle H_{sym}\right\rangle}{\partial \Delta w}$$

$$\frac{\partial\left\langle H_{sym}\right\rangle}{\partial w_0^\theta} = \frac{\partial \Delta w}{\partial w_0^\theta}\frac{\left\langle H_{sym}\right\rangle}{\partial \Delta w} = \frac{1}{\beta^\theta}\frac{\left\langle H_{sym}\right\rangle}{\partial \Delta w}$$

$$\beta^r\left\langle\frac{\partial H_{sym}}{\partial w^r}\right\rangle + \beta^\theta\left\langle\frac{\partial H_{sym}}{\partial w^\theta}\right\rangle = 0$$

relation between $\dot{J}_r$ and $\dot{J}_\theta$

radiative part

$$H_{rad}(x,u) = -\int d\omega \sum_{l,m} \frac{\mu}{4i\omega^4} \left\{ Z^{out}_{\omega lm} \Phi^{out}_{\omega lm}(x,u) + \frac{\omega}{p_{\omega lm}} Z^{down}_{\omega lm} \Phi^{down}_{\omega lm}(x,u) \right\} + (c.c.)$$

$$\Phi^{out/down}_{\omega lm}(x,u) \equiv {}_{-2}\tau^{\dagger}_{\mu\nu}\; {}_{-2}\Psi^{out/down}_{\omega lm}(x) u^{\mu} u^{\nu} \qquad Z^{out/down}_{\omega lm} \equiv \int d\tau' \, \overline{\Phi^{out/down}_{\omega lm}}(\tau')$$

$${}_{-2}\Psi^{out/down}_{\omega lm}(x) \propto {}_{2}R^{out/down}_{\omega lm}(r)\; {}_{-2}S_{\omega lm}(\theta,\varphi) e^{-i\omega t}$$

$$Z^{out/down}_{\omega lm} \equiv \int d\tau' \, \overline{\Phi^{out/down}_{\omega lm}}(\tau')$$

$$\langle \dot{j}_{\alpha} \rangle = -\int d\omega \sum_{l,m} \frac{\mu}{4i\omega^4} \left\{ Z^{out}_{\omega lm} \left\langle \frac{\partial \Phi^{out}_{\omega lm}(x,u)}{\partial w^{\alpha}} \right\rangle + \frac{\omega}{p_{\omega lm}} Z^{down}_{\omega lm} \left\langle \frac{\partial \Phi^{down}_{\omega lm}(x,u)}{\partial w^{\alpha}} \right\rangle \right\} + (c.c.)$$

Fourier exp. w.r.t. w $\quad \Phi^{out/down}_{\omega lm}(w,J) \equiv \sum_{k,n} \phi^{out/down}_{\omega lmkn}(J) e^{-i\left(\omega w^t - m w^{\varphi} - k w^{\theta} - n w^r\right)}$

$$\Rightarrow \quad Z^{out/down}_{\omega lm} \equiv \sum_{k,n} \int d\tau \sum_{k,n} \overline{\phi^{out/down}_{\omega lmkn}}(J) e^{i\left(\omega w^t - m w^{\varphi} - k w^{\theta} - n w^r\right)} \Big|_{\gamma}$$

$$= \sum_{k,n} \tilde{Z}^{out/down}_{lmkn} e^{i\chi_{mkn}} \delta(\omega - \omega_{mkn})$$

where $\quad \omega_{mkn} \equiv m\Omega^{\varphi} + k\Omega^{\theta} + n\Omega^{r}, \qquad \chi_{mkn} \equiv \omega_{mkn} w^t_0 - \left(m w^{\varphi}_0 + k w^{\theta}_0 + n w^{r}_0\right),$

$$\tilde{Z}^{out}_{lmkn} \equiv 2\pi z \, \overline{\phi^{out}_{\omega_{mkn} lmkn}}(J) \Big|_{\gamma}$$

$$\left\langle \dot{J}_\alpha \right\rangle = -\int d\omega \sum_{l,m} \frac{\mu}{4i\omega^4} \left\{ Z^{out}_{\omega lm} \left\langle \frac{\partial \Phi^{out}_{\omega lm}(x,u)}{\partial w^\alpha} \right\rangle + \frac{\omega}{p_{\omega lm}} Z^{down}_{\omega lm} \left\langle \frac{\partial \Phi^{down}_{\omega lm}(x,u)}{\partial w^\alpha} \right\rangle \right\} + (c.c.)$$

$$\Phi^{out/down}_{\omega lm}(w,J) \equiv \sum_{k,n} \phi^{out/down}_{\omega lmkn}(J) e^{-i\left(\omega w^t - m w^\varphi - k w^\theta - n w^r\right)}$$

$$Z^{out/down}_{\omega lm} = \sum_{k,n} \tilde{Z}^{out/down}_{lmkn} e^{i\chi_{mkn}} \delta(\omega - \omega_{mkn})$$

with

$$\omega_{mkn} \equiv m\Omega^\varphi + k\Omega^\theta + n\Omega^r,$$

$$\chi_{mkn} \equiv \omega_{mkn} w^t_0 - \left(m w^\varphi_0 + k w^\theta_0 + n w^r_0 \right),$$

$$\Rightarrow \quad \int d\omega Z^{out}_{\omega lm} \frac{\partial \Phi^{out}_{\omega lm}(x,u)}{\partial w^\alpha}$$

$$= \frac{\partial}{\partial w^\alpha} \sum_{k,n} \sum_{k',n'} \tilde{Z}^{out}_{lmkn} e^{i\chi_{mkn}} \phi^{out}_{\omega_{mkn} lmk'n'}(J) e^{-i\left(\omega_{mkn} w^t - m w^\varphi - k' w^\theta - n' w^r\right)} \Big|_\gamma$$

$$= i \sum_{k,n} \sum_{k',n'} \tilde{Z}^{out}_{lmkn} \left(\varepsilon_\alpha \phi^{out}_{\omega_{mkn} lmk'n'}(J) \right) e^{i\left((k'-k)w^\theta + (n'-n)w^r\right)} \Big|_\gamma$$

$$\varepsilon_\alpha \equiv (-\omega_{mkn}, n', k', m),$$

$$\int d\omega Z^{out}_{\omega lm}\frac{\partial\Phi^{out}_{\omega lm}(x,u)}{\partial w^\alpha}=i\sum_{k,n}\sum_{k',n'}\tilde{Z}^{out}_{lmkn}\left(\varepsilon_\alpha\phi^{out}_{\omega_{mkn}lmk'n'}(J)\right)e^{i\left((k'-k)w^\theta+(n'-n)w^r\right)}\Big|_\gamma$$

Non-resonant case

Orbital average implies that only contributions with $k=k'$ and $n=n'$ remain.

$$\boxed{\tilde{Z}^{out}_{lmkn}\equiv 2\pi z\,\overline{\phi^{out}_{\omega_{mkn}lmkn}}(J)\Big|_\gamma}$$

$$\Rightarrow\int d\omega Z^{out}_{\omega lm}\left\langle\frac{\partial\Phi^{out}_{\omega lm}(x,u)}{\partial w^\alpha}\right\rangle=\frac{i}{2\pi z}\sum_{k',n'}\varepsilon_\alpha\left|\tilde{Z}^{out}_{lmk'n'}\right|^2$$

The formulas for $\langle\dot{J}_\theta\rangle$ and $\langle\dot{J}_r\rangle$ are analogous to those for $\langle\dot{J}_t\rangle$ and $\langle\dot{J}_\varphi\rangle$.

Resonant case

same ω for different k and n.

$$\boxed{\tilde{\omega}=\omega^\theta/\beta_\theta=\omega^r/\beta_r}$$

$$N\equiv k\beta^\theta+n\beta^r\quad N'\equiv k'\beta^\theta+n'\beta^r$$

$$\boxed{(k'-k)w^\theta+(n'-n)w^r=(N'-N)\tilde{\omega}(\tau-\tau_0)+(k'-k)w_0^\theta+(n'-n)w_0^r}$$

Orbital average implies $N=N'$.

$$\boxed{\Delta w=w_0^\theta/\beta_\theta-w_0^r/\beta_r}$$

$$\Rightarrow\quad(n'-n)\beta^r=-(k'-k)\beta^\theta$$

$$(k'-k)w^\theta+(n'-n)w^r\to(k'-k)\beta^\theta\Delta w$$

Summation over k and n that satisfy $N=k\beta^\theta+n\beta^r$.

$$\int d\omega Z^{out}_{\omega lm}\left\langle\frac{\partial\Phi^{out}_{\omega lm}(x,u)}{\partial w^\alpha}\right\rangle=\frac{i}{2\pi z}\sum_{N=-\infty}^{\infty}\left(\sum_{(k,n)_N}\tilde{Z}^{out}_{lm(kn)}e^{-ik\beta^\theta\Delta w}\right)\left(\sum_{(k',n')_N}\varepsilon_\alpha\overline{\tilde{Z}^{out}_{lm(k'n')}}e^{ik'\beta^\theta\Delta w}\right)$$

$$\boxed{\tilde{Z}^{out}_{lm(kn)}\equiv 2\pi z\,\overline{\phi^{out}_{\omega_{mN}lmkn}}(J)\Big|_\gamma}\qquad\omega_{mN}\equiv m\omega^\varphi+N\tilde{\omega},$$

First-law relation

- Full Hamiltonian $H = H_0 + H_{\mathrm{int}} = \frac{1}{2} g_{\mu\nu} u^\mu u^\nu = -\frac{1}{2}$

H_0 is a function of J_α only.

$$\left\langle \dot{H}_0 \right\rangle = -\left\langle \dot{H}_{\mathrm{int}} \right\rangle = \lim_{T\to\infty} \frac{1}{2T} \left[H_{\mathrm{int}}(T) - H_{\mathrm{int}}(-T) \right] = 0$$

$$0 = \left\langle \dot{H}_0 \right\rangle = \frac{\partial H_0}{\partial J_\alpha} \left\langle \dot{J}_\alpha \right\rangle = \omega^\alpha \left\langle \dot{J}_\alpha \right\rangle$$

$\left\langle \dot{J}_\alpha \right\rangle$ are not mutually independent.

Next Leading Order of η in waveform

Orbital frequencies:

$$\omega^\alpha \equiv \frac{dw^\alpha}{d\tau}$$

Wave form is specified by $\dfrac{d\omega^\alpha}{d\tau}(\omega)$

$$\frac{d\omega^\alpha}{dt} = \frac{\partial \omega^\alpha}{\partial J_\beta}\frac{dJ_\beta}{d\tau}$$

linear perturbation

leading order ($O(\eta^{-1})$ phase)

$$\frac{dJ_\beta}{dt} = 0 + O(\eta) + O(\eta^2)$$

next leading order ($O(1)$ phase)

$$\frac{\partial \omega^\alpha}{\partial J_\beta} = (\text{geodesic}) + O(\eta) + O(\eta^2)$$

Long term evolution

EOM)

$$\frac{dJ_\alpha}{d\tau} = -\frac{\partial H_{\text{int}}}{\partial w^\alpha}$$

We need to solve this equation
to the second order

$$\frac{dw^\alpha}{d\tau} = \Omega^\alpha_{(0)}(J) + \frac{\partial H_{\text{int}}}{\partial J^\alpha}$$

We separate the variables

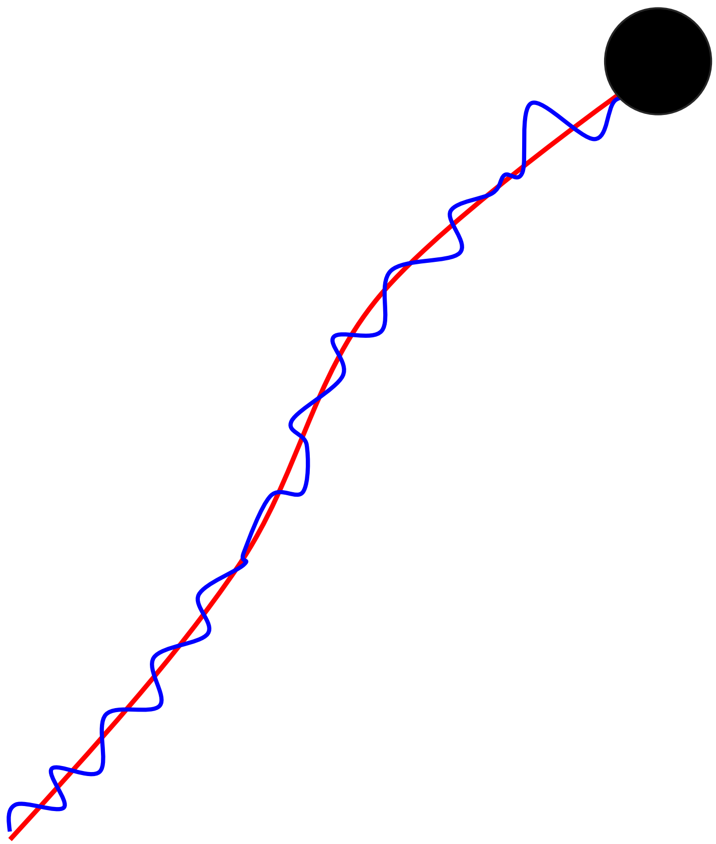
$$J_\alpha = \bar{J}_\alpha(\tilde{\tau}) + \delta J_\alpha(\tilde{\tau}, w^I)$$

$$w^\alpha = \bar{w}^\alpha(\tilde{\tau}) + \delta w^\alpha(\tilde{\tau}, w^I) \quad (I = r, \theta)$$

so that

$(\bar{J}, \bar{w})$:depends only on slow time

$(\delta J, \delta w)$:can rapidly oscillate
but always remains small



Linear perturbation)

$$H_{\text{int}} = H_{\text{int}}^{(1)} + H_{\text{int}}^{(2)} + \cdots$$

Source orbit for $H_{\text{int}}^{(1)} = -h_{(1)}^{\mu\nu} u_\mu u_\nu$

is approximated by the osculating geodesic orbit:

$$\bar{J}_{osc}(\tau_0;\tau) = \bar{J}(\tilde{\tau}_0) \qquad \bar{w}_{osc}^{\alpha}(\tau_0;\tau) = \bar{w}^{\alpha}(\tilde{\tau}_0) + \omega_{(0)}^{\alpha}\left(\bar{J}\left(\tilde{\tau}_0\right)\right)\left(\tau - \tau_0\right)$$

We decompose EOM as

$$\frac{d\bar{J}_\alpha}{d\tau} = -\left\langle \frac{\partial H_{\text{int}}^{(1)}}{\partial w^\alpha} \right\rangle \qquad \frac{d\bar{w}^\alpha}{d\tau} = \omega_{(0)}^{\alpha}\left(\bar{J}\right) + \left\langle \frac{\partial H_{\text{int}}^{(1)}}{\partial J_\alpha} \right\rangle$$

$$\frac{d\delta J_\alpha^{(1)}}{d\tau} = -\frac{\partial H_{\text{int}}^{(1)}}{\partial w^\alpha} + \left\langle \frac{\partial H_{\text{int}}^{(1)}}{\partial w^\alpha} \right\rangle \qquad \frac{d\delta w_{(1)}^\alpha}{d\tau} = \frac{\partial H_{\text{int}}^{(1)}}{\partial J_\alpha} - \left\langle \frac{\partial H_{\text{int}}^{(1)}}{\partial J_\alpha} \right\rangle$$

$\left\{ \delta J_\alpha^{(1)}, \delta w_{(1)}^\alpha, \left\langle \partial H_{\text{int}}^{(1)} / \partial J^\alpha \right\rangle_w \right\}$ can be erased by the choice of gauge for generic orbits, but such a treatment is too native especially when we discuss some limiting cases, such as the circular limit, equatorial limit, and ISCO limit.

Summary of the results obtained in arXiv:1612.02504)

The conservative self-force can be described by the effective Hamiltonian

$$\mathcal{H} = H^{(0)}(J) + \frac{1}{2}\mathcal{H}_{\rm int}(J)$$

$$\mathcal{H}_{\rm int} = -\frac{1}{2}\left\langle h^{\mu\nu}_{(1)sym} u_\mu u_\nu \right\rangle = -\frac{1}{2}\left\langle G_{sym}\left(u^{(0)}(\tau), z^{(0)}(\tau); u^{(0)}(\tau'), z^{(0)}(\tau')\right)\right\rangle$$

We can show the presence of a gauge in which

$$\Delta\left\langle \omega^a \right\rangle := \frac{1}{2}\frac{\partial \mathcal{H}_{\rm int}}{\partial J_a} - \left\langle \left(\frac{\partial H^{(1)}_{\rm int}}{\partial J_a}\right)_w \right\rangle = 0$$

The gauge transformation is constrained.

$$\delta_g \Delta\left\langle \omega^\alpha \right\rangle := -\delta_g \left\langle \left(\frac{\partial H^{(1)}_{\rm int}}{\partial J_\alpha}\right)_w \right\rangle = \frac{\partial^2 H^{(0)}}{\partial J_\alpha \partial J_\beta}\delta_g J_\beta = \frac{\partial \omega^\beta}{\partial J_\alpha}\delta_g J_\beta =: M^{\alpha\beta}\delta_g J_\beta$$

gauge invariance of ω

1) $\omega^\alpha J_\alpha = -1 \Rightarrow \omega^\alpha \delta_g J_\alpha = 0 \Rightarrow J_\alpha \delta_g \Delta\left\langle \omega^\alpha \right\rangle = J_\alpha \frac{\partial \omega^\beta}{\partial J_\alpha}\delta_g J_\beta = \omega^\beta \delta_g J_\beta = 0$

But one can show from the beginning

$$J_\alpha \Delta\left\langle \omega^\alpha \right\rangle = J_\alpha \frac{\partial}{\partial J_\alpha}\left(H^{(0)}(J) + \frac{1}{2}\mathcal{H}_{\rm int}(J)\right) - J_\alpha \omega^\alpha = 2\left(H^{(0)}(J) + \mathcal{H}_{\rm int}(J)\right) + (-1) = 0$$

2) When $\det M^{\alpha\beta} = 0 \Rightarrow {}^{\exists}\kappa_\alpha M^{\alpha\beta} = 0 \Rightarrow \kappa_\alpha \delta_g \Delta\left\langle \omega^\alpha \right\rangle = \kappa_\alpha M^{\alpha\beta}\delta_g J_\beta = 0$

But $\kappa_\alpha \Delta\left\langle \omega^\alpha \right\rangle = 0$ can be proven, although the proof is more involved.

Uniqueness of the effective Hamiltonian

Effective Hamiltonian that depends only on J is uniquely given by

$$\mathcal{H} = H^{(0)}(J) + \frac{1}{2}\mathcal{H}_{\text{int}}(J)$$

if we require the scaling: $\mathcal{H}_{\text{int}}(\lambda J) = \lambda^4 \mathcal{H}_{\text{int}}(J)$.

Here, the gauge dependence of J is allowed. .

Special care is required for some limiting cases:

$\begin{cases} J_r \to 0 \\ J_\theta \to 0 \end{cases}$ ⇨ new gauge invariant relation $\begin{cases} J_r = 0 \\ J_\theta = 0 \end{cases}$

$\omega_r \to 0$ marginally stable orbits

$$\delta r_{\min} = \frac{\partial r_{\min}}{\partial J_\alpha}\delta J_\alpha < \infty \quad \Rightarrow \quad \left(\frac{\partial r_{\min}}{\partial J_\alpha} \Big/ \left| \frac{\partial r_{\min}}{\partial J_\alpha} \right| \right) \delta J_\alpha = 0$$

All the components which cannot be changed by gauge transformations can be shown to vanish from the beginning.

(Proof for $\omega_r \to 0$ is given in our paper only for circular cases.)

Linear perturbation)

$$H_{\rm int} = H^{(1)}_{\rm int} + H^{(2)}_{\rm int} + \cdots$$

Source orbit for $H^{(1)}_{\rm int} = -h^{\mu\nu}_{(1)} u_\mu u_\nu$

is approximated by the osculating geodesic orbit:

$$\bar{J}_{osc}(\tau_0;\tau) = \bar{J}(\tilde{\tau}_0) \qquad \bar{w}^\alpha_{osc}(\tau_0;\tau) = \bar{w}^\alpha(\tilde{\tau}_0) + \Omega^\alpha_{(0)}\left(\bar{J}\left(\tilde{\tau}_0\right)\right)(\tau - \tau_0)$$

We decompose EOM as

$$\frac{d\bar{J}_\alpha}{d\tau} = -\left\langle \frac{\partial H^{(1)}_{\rm int}}{\partial w^\alpha} \right\rangle \qquad \frac{d\bar{w}^\alpha}{d\tau} = \Omega^\alpha_{(0)}\left(\bar{J}\right) + \left\langle \frac{\partial H^{(1)}_{\rm int}}{\partial J_\alpha} \right\rangle$$

$$\frac{d\delta J^{(1)}_\alpha}{d\tau} = -\frac{\partial H^{(1)}_{\rm int}}{\partial w^\alpha} + \left\langle \frac{\partial H^{(1)}_{\rm int}}{\partial w^\alpha} \right\rangle \qquad \frac{d\delta w^\alpha_{(1)}}{d\tau} = \frac{\partial H^{(1)}_{\rm int}}{\partial J_\alpha} - \left\langle \frac{\partial H^{(1)}_{\rm int}}{\partial J_\alpha} \right\rangle$$

We choose the gauge such that

$$\left\{ \delta J^{(1)}_\alpha = 0,\ \delta\omega^\alpha_{(1)} = 0,\ \left\langle \partial H^{(1)}_{\rm int} / \partial J^\alpha \right\rangle_w = \partial \mathcal{H}_{\rm int} / \partial J^\alpha \right\}$$

$$\Rightarrow \qquad \frac{d\bar{J}_\alpha}{d\tau} = -\left\langle \frac{\partial H^{(1)}_{\rm int}}{\partial w^\alpha} \right\rangle_{w^I} \qquad \frac{d\bar{w}^\alpha}{d\tau} = \frac{\partial \mathcal{H}_{\it eff}\left(\bar{J}\right)}{\partial J_\alpha}$$

Second order perturbation)

$$H_{\rm int} = H_{\rm int}^{(1)} + H_{\rm int}^{(2)} + \cdots$$

Source for $H_{\rm int}^{(2)} = -h_{(2)}^{\mu\nu} u_\mu u_\nu$ has two contributions:

1) Quadratic term of the first order metric perturbation "$\partial h\, \partial h$"

2) First order deviation of the source orbit from the osculating orbit

$$\Delta J_\alpha(\tau_0;\tau) := \bar{J}_\alpha(\tilde{\tau}) - \bar{J}_\alpha^{osc}(\tau_0;\tau) = \delta J_\alpha^{(1)} + \left\langle \frac{dJ_\alpha}{d\tau} \right\rangle (\tau - \tau_0) + \cdots$$

$$\Delta w^\alpha(\tau_0;\tau) := \bar{w}^\alpha(\tau) - \bar{w}_{osc}^\alpha(\tau_0;\tau) = \frac{1}{2} \frac{\partial \Omega_{(0)}^\alpha}{\partial J_\beta} \left\langle \frac{dJ_\beta}{d\tau} \right\rangle_{osc} (\tau - \tau_0)^2 + \cdots$$

$$\frac{dJ_a}{d\tau} = -\frac{\partial H_{\rm int}^{(1)}}{\partial w^a} - \underline{\frac{\partial H_{\rm int}^{(2,hh)}}{\partial w^a}} - \underline{\left(\Delta J_\beta \frac{\partial}{\partial J_\beta^{(s)}} + \Delta w^\beta \frac{\partial}{\partial w_{(s)}^\beta} \right)} \frac{\partial H_{\rm int}^{(1)}}{\partial w^\alpha}$$

differentiation with respect to the source orbit

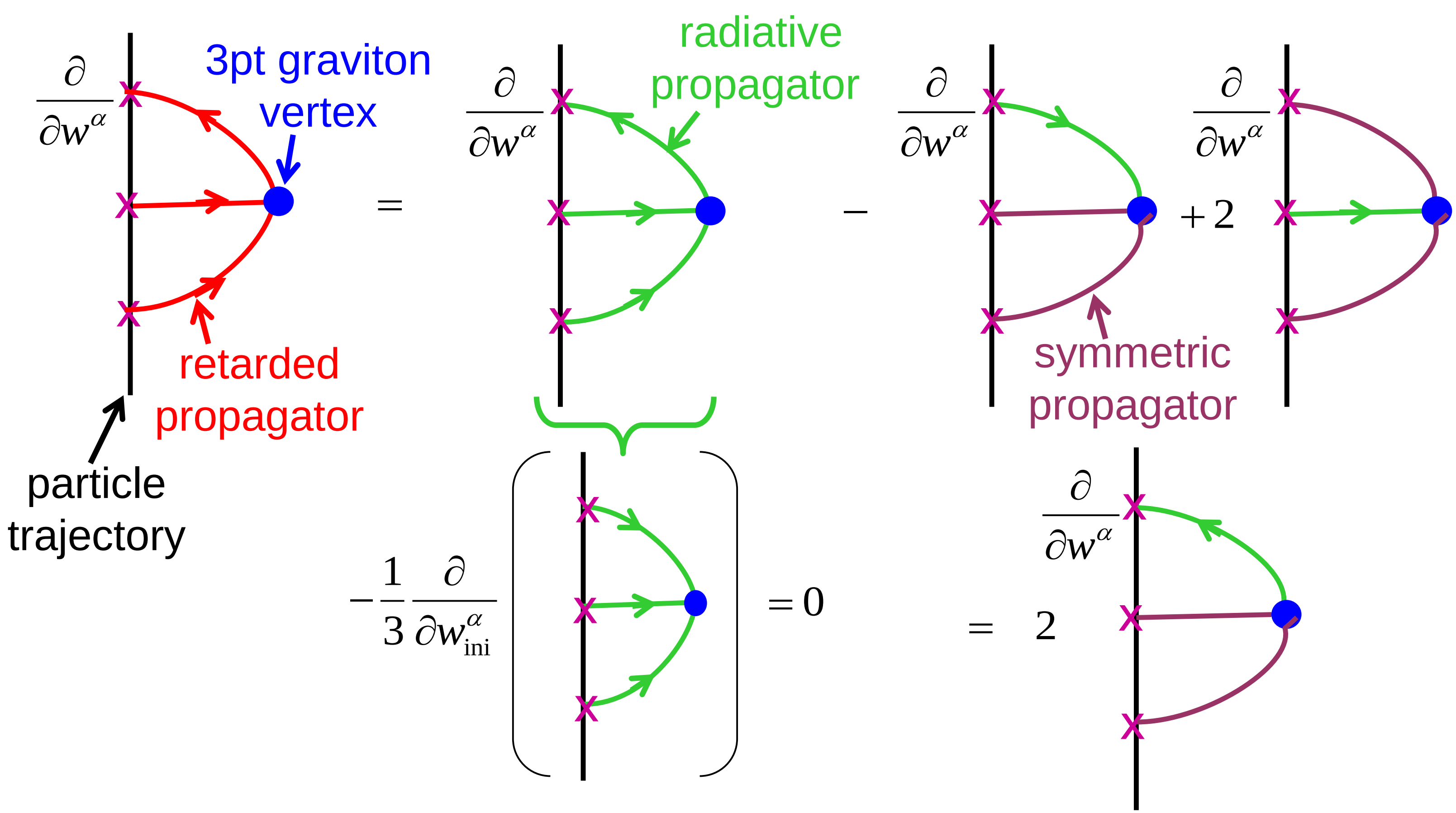
$$\frac{d\bar{J}_\alpha}{d\tau} = -\left\langle \frac{\partial H_{\rm int}^{(1)}}{\partial w^\alpha} \right\rangle_{osc} - \left\langle \frac{\partial H_{\rm int}^{(2,hh)}}{\partial w^\alpha} \right\rangle_{osc} - \left\langle \left(\Delta J_\beta \frac{\partial}{\partial J_\beta^{(s)}} + \Delta w^\beta \frac{\partial}{\partial w_{(s)}^\beta} \right) \frac{\partial H_{\rm int}^{(1)}}{\partial w^\alpha} \right\rangle_{osc}$$

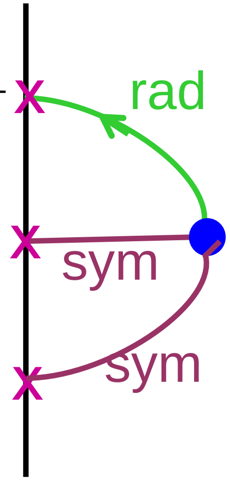
Drastic simplification may occur

From here on I just give a highly speculative argument.

$$\frac{d\bar{J}_\alpha}{d\tau} \supset -\frac{\partial H_{\text{int}}^{(2)}}{\partial w^\alpha}$$

Second order dissipative part = Graviton 2-loop



$\frac{\partial}{\partial w^\alpha}$ 

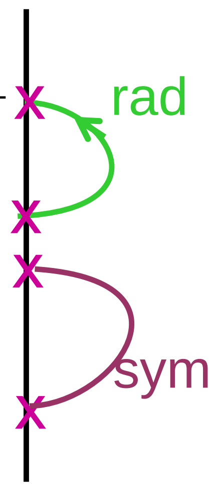
$$= \int d\tau \int d\tau' \int d\tau'' \int d^4 x'$$

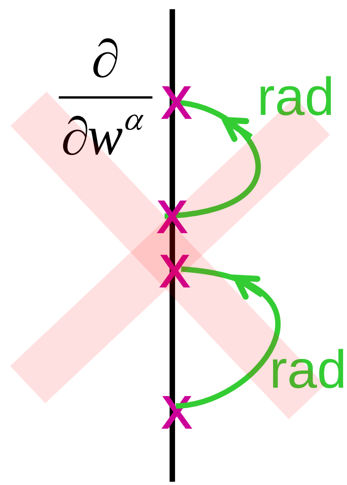
$$\times \frac{\partial}{\partial w^a} \hat{V}_{x_1', x_2', x_3'} G_{(rad)}\left(x, x_1'\right)\Big|_{x=\gamma(\tau)} G_{(sym)}\left(x_2', \gamma\left(\tau'\right)\right) G_{(sym)}\left(x_3', \gamma\left(\tau''\right)\right)\Big|_{x_1'=x_2'=x_3'=x'}$$

Second order source must satisfy conservation $T_{\mu\nu}{}^{;\nu}=0$ as a whole

$$\underline{\nabla^\nu T^{(\text{particle})}_{\mu\nu}} = \nabla^\nu G^{[2]}_{\mu\nu}\left(h_{(sym)}, h_{(sym)}\right)$$

necessary deviation from geodesic comes from symmetric contribution only.

$\frac{\partial}{\partial w^\alpha}$ 

$\frac{\partial}{\partial w^\alpha}$ 

$$= \int d\tau \frac{\partial}{\partial w^\alpha} \int d\tau' \delta J_\beta^{(1)}\left(\tau'\right) \frac{\partial}{\partial J_\beta\left(\tau'\right)} G_{(rad)}\left(x, \gamma\left(\tau'\right)\right)$$

Summary

The "flux-balance formulae" that determine the averaged evolution of energy, angular momentum and Carter constant in terms of the averaged asymptotic partial wave fluxes for EMRIs in Kerr spacetime were first derived ~15 years ago.

(Sago et al. 2005)

We here gave a new derivation of the flux formulae based on Hamiltonian dynamics of a self-forced particle motion using action-angle variables, which is much simpler than the previous one, and applies to the resonant inspirals without any complication.

The conservative effect of the first order perturbation can be encapsulated in the effective Hamiltonian.

Formal discussion may provide some insight into the simplification of the next leading order calculation.