Perturbed Schwarzschild horizons in second-order perturbation theory

23rd Capra meeting

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June 22-26, 2020

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Outline

- **Motivations**
- **Intrinsic metric and surface area**
- ***Event horizon* and *apparent horizon***
- **The two-time scale expansion**
- **Conclusions**

Motivations

- **EMRIs** $\rightarrow$
  - Gravitational two-body problem
  - Perturbative parameter $\epsilon = \frac{m}{M}$
- **High accuracy predictions**$\rightarrow$
  - GW detections (LISA, LIGO/VIRGO)
  - NR simulations
  - Theoretical models
- **Self-force theory** $\rightarrow$
  - Beyond test particle approximation
  - Consistent description up to second order in ϵ

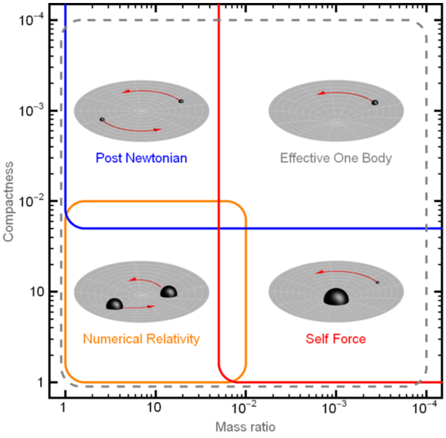


Figure: Credits, Maarten van de Meent

Motivations

Studying the central black hole

- **Complete knowledge of the behavior of the system**
- **High precision calculations** *A. Pound et al. "Second-Order Self-Force Calculation of Gravitational Binding Energy in Compact Binaries." Physical Review Letters 124.2 (2020)*
- **Comparison between different parameters definitions** e.g. mass, angular momentum

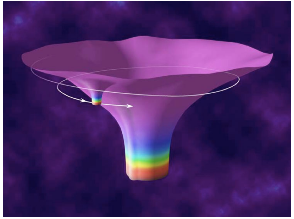


Figure: Credits, NASA

Intrinsic metric and surface area

The horizon surface is parametrized by

$$x_H^\alpha = (v, r_H, \theta^A) \tag{1}$$

The second order expansion for the horizon radius is:

$$r_H = 2M + \epsilon r^{(1)}(v, \theta^A) + \epsilon^2 r^{(2)}(v, \theta^A) \tag{2}$$

The intrinsic metric on the horizon is given by

$$\gamma_{AB} = (g_{\alpha\beta} + h_{\alpha\beta}) e^\alpha{}_A e^\beta{}_B. \tag{3}$$

- $g_{\alpha\beta} \rightarrow$ The *Schwarzschild metric* in *Eddington-Finkelstein coordinates*
- $h_{\alpha\beta} = \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} \rightarrow$ The perturbation metric
- $e^\alpha{}_A = \dfrac{\partial x_H^\alpha}{\partial \theta^A} \rightarrow$ The tangent vectors to the horizon

We use capital Latin letters $A, B, \ldots$ for the (θ, ϕ) components

Intrinsic metric and surface area

The horizon area is given by the integral

$$A = \int_{\mathscr{H}} d\theta^2 \sqrt{\gamma} \tag{4}$$

Remark

$$M_{\text{irr}} = \sqrt{\frac{A}{16\pi}} \tag{5}$$

The second order expansion for the area is:

$$A = 4M^2 \Big\{ 4\pi + \epsilon \frac{\sqrt{4\pi}}{8M^2} \left[2h^{(1)\ 00}_{\text{trace}} + 8Mr^{(1)\ 00} \right] + \frac{\epsilon^2}{8M^2} \Big[\sqrt{4\pi} (2h^{(2)\ 00}_{\text{trace}} + 8Mr^{(2)\ 00}) \\ + \sum_{lm} (2r^{(1)\ lm} \partial_r h^{(1)\ lm\ *}_{\text{trace}} + 2\lambda_1^2 r^{(1)\ lm} h^{(1)\ lm\ *}_{r+} + 2|r^{(1)\ lm}|^2 - \frac{\lambda_2^2}{16M^2} |h^{(1)\ lm}_{+}|^2 - \frac{\lambda_2^2}{16M^2} |h^{(1)\ lm}_{-}|^2) \Big] \Big\} \tag{6}$$

- The fields $(h^{(1)\ lm}, r^{(1)\ lm}, \ldots)$ in eq.(6) are the coefficients of the *tensorial spherical harmonics* expansion
- $\lambda_1^2 = l(l+1)$ and $\lambda_2^2 = (l-1)l(l+1)(l+2)$

Event horizon and apparent horizon

The *event horizon* is a congruence of null geodesics with k^α as tangent vector

$$k^\alpha k_\alpha = 0 \tag{7}$$

- In the *late future* the spacetime will be *stationary* $\rightarrow$ Trace back in time the location of the event horizon

On the *apparent horizon* the *expansion* is zero

$$\vartheta = k^\alpha{}_{;\alpha} = 0 \tag{8}$$

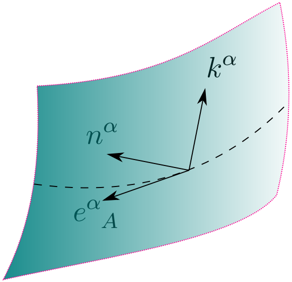
with k^α a null vector field normal to the surface and pointing outward

Remark

In the exact Schwarzschild manifold the event horizon and the apparent horizon are the same surface: a two-sphere of radius $2M$

How to determine the horizon location?

- Construct a basis $\{k^\alpha, n^\alpha, {e^\alpha}_A\}$ on the perturbed spacetime
- Use the specific relations for k^α to find the horizon equations
  - $k^\alpha k_\alpha = 0$ for the event horizon
  - $\vartheta = 0$ for the apparent horizon
- Evaluate every expression up to second order in ϵ
  (every expression is evaluated at $r = 2M$)
- Expand all the quantities in tensorial spherical harmonics
- Solve the equations for the perturbed horizons locations



Event horizon

The first order equation for the event horizon location is:

$$h_{vv}^{(1)\ lm} - \frac{r_{\mathcal{EH}}^{(1)\ lm}}{2M} + 2\,\partial_v r_{\mathcal{EH}}^{(1)\ lm} = 0 \tag{9}$$

For the second order ($l = m = 0$) we get:

$$4\sqrt{\pi}\,\partial_v r_{\mathcal{EH}}^{(2)\ 00} - \frac{\sqrt{\pi} r_{\mathcal{EH}}^{(2)\ 00}}{M} + 2\sqrt{\pi} h_{vv}^{(2)\ 00} + \sum_{lm}\left(\frac{h_{vr}^{(1)\ lm} r^{(1)\ lm\ *}}{2M} - \frac{\lambda_1^2\, h_{+\ v}^{(1) lm} r^{(1)\ lm\ *}}{2M^2} - h_{vv}^{(1)\ lm} h_{vr}^{(1)\ lm\ *}\right.$$
$$\left. + \frac{|r^{(1)\ lm}|^2}{4\ M^2} - \frac{\lambda_1^2\ |r^{(1)\ lm}|^2}{4M^2} - \frac{\lambda_1^2\ |h_{-\ v}^{(1)\ lm}|^2}{4M^2} - \frac{\lambda_1^2\ |h_{+\ v}^{(1)\ lm}|^2}{4M^2} + r^{(1)\ lm\ *}\partial_r h_{vv}^{(1)\ lm}\right) = 0 \tag{10}$$

These are differential equations of the form:

$$r_{\mathcal{EH}}^{(n)\ lm}(v) - 4M\ \partial_v r_{\mathcal{EH}}^{(n)\ lm}(v) = F^{lm}(v) \tag{11}$$

with *teleological* solutions

$$r_{\mathcal{EH}}^{(n)\ lm}(v) = \frac{1}{4M}\int_v^{+\infty} e^{-\frac{1}{4M}(v'-v)} F^{lm}(v') dv' + r_{\mathcal{EH}\ \text{Kerr}}^{(n)\ lm} \tag{12}$$

Apparent horizon

For the apparent horizon we can explicitly solve the equations.
The first order gives:

$$r^{(1)\ lm}_{\mathcal{AH}} = \frac{2M\ h^{(1)\ lm}_{vv} - \lambda_1^2 h^{(1)\ lm}_{+\ v} - \partial_v h^{(1)\ lm}_{\text{trace}}}{1+\lambda_1^2} \tag{13}$$

The second order leads to a very long expression. We show only the general structure

$$r^{(2)\ 00}_{\mathcal{AH}} = 2M\ h^{(2)\ 00}_{vv} - \partial_v h^{(2)\ 00}_{\text{trace}} + \sum_{lm} \left[h^{(1)\ lm} h^{(1)\ lm}\right] \tag{14}$$

The linear terms are given by the second order perturbations, the quadratic terms are products of first order perturbations

Two-timescale expansion

We introduce a map between the manifold with coordinates (v, r, θ^A) and a manifold with coordinates $(\epsilon v, \varphi(v, \epsilon), r, \theta^A)$ such that:

$$r^{(n)\ lm}(v) = \tilde{r}^{(n)\ lm}(\epsilon v, \varphi(v, \epsilon)) \tag{15}$$

- $\tilde{v} = \epsilon v$ slow-time $\rightarrow$ radiation-reaction timescale
- $\varphi(v, \epsilon)$ fast-time $\rightarrow$ orbital timescale
  - Assumption: the field is periodic in φ with frequency $\dfrac{d\varphi}{dv} = \omega$ (and bounded in $\tilde{v}$)

We expand $\tilde{r}^{(n)\ lm}$ in Fourier series

$$\tilde{r}^{(n)\ lm}(\epsilon v, \varphi(v, \epsilon)) = \sum_k \tilde{r}_k^{(n)\ lm}(\epsilon v) e^{-ik\varphi} \tag{16}$$

These relationships hold for every field in the event horizon (and apparent horizon) equations

Two-timescale expansion

- Specialize to systems that evolve to have zero angular momentum and total mass M at late times
- Apply the two-timescale expansion to the event horizon equations

At first order we get

$$\tilde{r}^{(1)\ lm}_{\mathcal{EH}\ k} = \frac{2M\ (\tilde{h}^{(1)\ lm}_{vv})_k}{1 + 4ikM\omega_0} \tag{17}$$

To make a comparison, the same expansion could be written for the apparent horizon solutions. The first order gives

$$\tilde{r}^{(1)\ lm}_{\mathcal{AH}\ k} = \frac{2M\ (\tilde{h}^{(1)\ lm}_{vv})_k - \lambda_1^2\ (\tilde{h}^{(1)lm}_{+\ v})_k + ik\omega_0(\tilde{h}^{(1)\ lm}_{\text{trace}})_k}{1 + \lambda_1^2} \tag{18}$$

Remark

If we set $l = m = k = 0$ at first order we obtain the same result for both the horizons

Two-timescale expansion

Let us apply the two-timescale expansion to the second order equation for the event horizon and to the apparent horizon solution. The results differ by many terms. However, if we keep only the $k_2 = l = m = 0$ terms of the sum, we get the following results:

$$\tilde{r}^{(2)\ 00}_{\mathcal{EH}\ k=0} = 2M(\tilde{h}^{(2)\ 00}_{vv})_{k=0} + 8M^2\ \partial_{\tilde{v}}(\tilde{h}^{(1)\ 00}_{vv})_{k=0} + \frac{M(\tilde{h}^{(1)\ 00}_{vv})^2_{(k_2=0)}}{\sqrt{\pi}} + \frac{2M^2(\tilde{h}^{(1)\ 00}_{vv})_{(k_2=0)}\ \partial_r(\tilde{h}^{(1)\ 00}_{vv})_{(k_2=0)}}{\sqrt{\pi}} + \sum_{k_2lm>0}[...]_{\mathcal{EH}} \tag{19}$$

$$\tilde{r}^{(2)\ 00}_{\mathcal{AH}\ k=0} = 2M(\tilde{h}^{(2)\ 00}_{vv})_{k=0} - \partial_{\tilde{v}}(\tilde{h}^{(1)\ 00}_{\text{trace}})_{(k=0)} + \frac{M(\tilde{h}^{(1)\ 00}_{vv})^2_{(k_2=0)}}{\sqrt{\pi}} + \frac{2M^2(\tilde{h}^{(1)\ 00}_{vv})_{(k_2=0)}\ \partial_r(\tilde{h}^{(1)\ 00}_{vv})_{(k_2=0)}}{\sqrt{\pi}} + \sum_{k_2lm>0}[...]_{\mathcal{AH}} \tag{20}$$

Remark

- Equations (19) and (20) differ by a slow-time derivative term and terms in the sum with $k_2, l, m > 0$
- We have a local solution for the event horizon

Numerical results

- The contribution is entirely given by the $\tilde{v}$-derivative terms
- The quadratic terms gives a contribution of order 10^{-21}
- The maximum difference is $\sim 10^{-8}\mathrm{M}_\odot$, assuming a mass ratio with $M \sim 10^5\mathrm{M}_\odot$ and $m \sim 10\mathrm{M}_\odot$

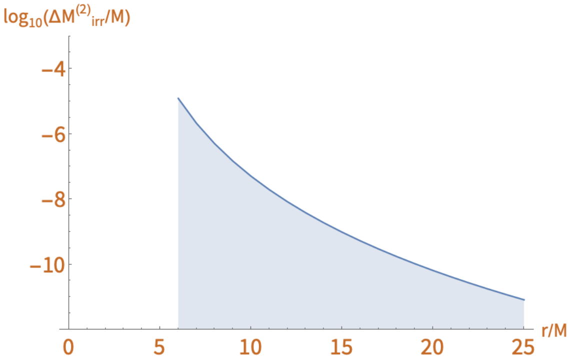


Figure: ΔM_{irr} vs. r . Data from Niels Warburton and Barry Wardell computations for quasi-circular orbits. Cut-off value $r \sim 6.01$

Conclusions

We obtained

- First order perturbation for event horizon and apparent horizon location
- Second order perturbation for event horizon and apparent horizon location (only for $l = m = 0$)
- First and second order perturbations for the area (for both the horizons)
- We checked the consistency of the equations with *Schwarzschild*, *Kerr* and *Vaidya* spacetime

Future prospects

- Compute the gauge transformations up to the second order
- Derive second order perturbations for the angular momentum
- Study and compare the results for different mass definitions