Progress towards self-consistent evolution of gravitational self-force around a Schwarzschild black hole

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Basic Goals

- Aims to develop time domain evolution code for gravitational self-force using the effective source approach
- Builds on Peter Diener's scalar evolution code
- Calculated in the Lorenz gauge
- Uses tortoise coordinates around the source
- Transitions to hyperboloidal coordinates in inner and outer regions

Derivation of Evolution equations

- Use first order perturbation equations for the trace-reversed metric in the Lorenz gauge:

$$\Box \bar{h}_{\alpha\beta} + 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\bar{h}_{\mu\nu} = -16\pi T_{\alpha\beta}$$

- Decompose results into multipole harmonics:

$$\Box^{2d}_{sc}\bar{h}^{(i)\ell m} + M^{(i)\ell}_{(j)}\bar{h}^{(j)\ell m} = -\frac{4\pi r f}{\mu a^{(i)\ell}}\pi T^{(i)\ell m}$$

where

$$\Box^{2d}_{sc} \equiv \partial_t^2 - \partial_{r_*}^2 + \frac{f}{4}\left(\frac{f'}{r} + \frac{\ell(\ell+1)}{r^2}\right)$$

- Expressions for the coupling matrix $M^{(i)\ell}_{(j)}$ were originally derived by Barack & Lousto [1].

Constraint Damping

Gundlach *et al* [2] provide a methodology to introduce constraint damping to the evolution equations. This damping requires adding term to evolution equations of the form

$$-\kappa(t_\alpha Z_\beta + t_\beta Z_\alpha),$$

where κ is a positive constant, t_α is a future-directed time-like vector field, and $Z_\alpha = \bar{h}_{\alpha\beta}{}^{;\beta}$ is the Lorenz gauge condition.

The damping requires careful handling at the horizon. I revisit the details of my choice of damping later.

Hyperboloidal Slicing

For the hyperboloidal layer from the tortoise coordinate region to $\mathscr{I}^+$, I construct the layer as done by Bernuzzi *et al* [3]. The following relations define the coordinate transformation $\{t, r\} \to \{\tau, \rho\}$ for the outer hyperboloidal layer.

Invariant Killing vector fields:	$\partial_t = \partial_\tau \to \tau = t - h(r_*)$
Invariant outgoing null rays:	$t - r_* = \tau - \rho \to \frac{d\rho}{dr_*} = 1 - H(\rho)$
Compactifying coordinates:	$r_* = \frac{\rho}{\Omega(\rho)}$ $\frac{dh}{dr_*} \equiv H(\rho) = 1 - \frac{\Omega^2}{\Omega - \rho\Omega'}$

In this compactification, $\Omega(\rho)$ is defined such that $\Omega(S^+) = 0 \Rightarrow r_*(S^+) = \infty$, where S^+ is some positive real number.

Hyperboloidal Slicing

For the hyperboloidal layer from the tortoise coordinate region to the horizon, I perform a similar construction as before. The primary difference is that I preserve the ingoing null rays instead of the outgoing rays. This amounts to changing the equation

$$t - r_* = \tau - \rho$$

to

$$t + r_* = \tau + \rho$$

In this compactification, $\Omega(\rho)$ is defined such that $\Omega(S^-) = 0 \Rightarrow r_*(S^-) = -\infty$, where S^- is some negative real number.

Tensor Spherical Harmonic Basis

For this work, I initially used the tensor spherical harmonics from Barack & Sago [4]. This basis introduces a factor of f in the $i = 3$ tensor spherical harmonic of the Barack & Lousto basis [1].

However, I encountered stability issues with the Barack-Sago basis. I resolve this issue by transforming the $h^{(i)}$ basis. I replace six of the $h^{(i)}$ with the linear combinations $h^{(1)} \pm h^{(2)}$, $h^{(4)} \pm h^{(5)}$, and $h^{(8)} \pm h^{(9)}$.

For both bases, the coupling matrix in the inner hyperboloidal layer has singular elements at the horizon. These can be canceled by adding the constraints to the evolution equations.

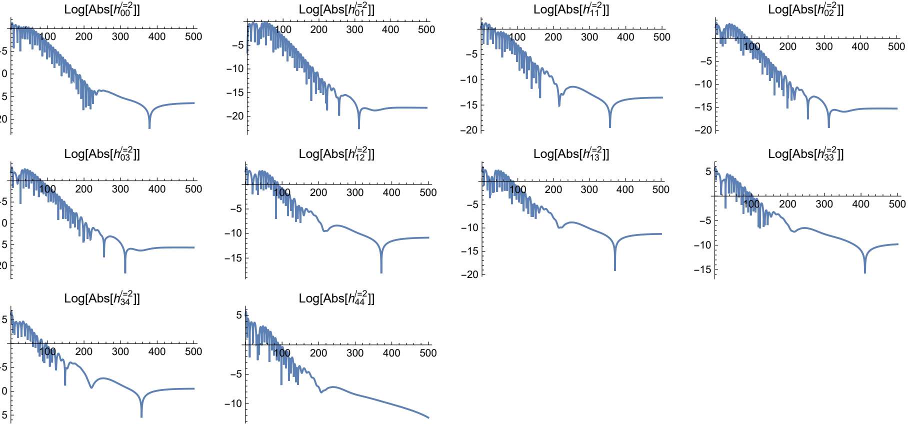
Evolution Stability

Evolution using the Barack-Sago variables was unstable, even with constraint damping. Evolution with the transformed basis also resulted in instabilities near the horizon, so I introduced an additional damping term in the inner hypoboloidal region which successfully stabilized evolution.

Constraint damping coefficient α for evolution equations in the three coordinate domains. Term with c coefficient is introduced for stability. Exact requirements for stability are unknown, but $c = 100$ is sufficient for performed simulations.

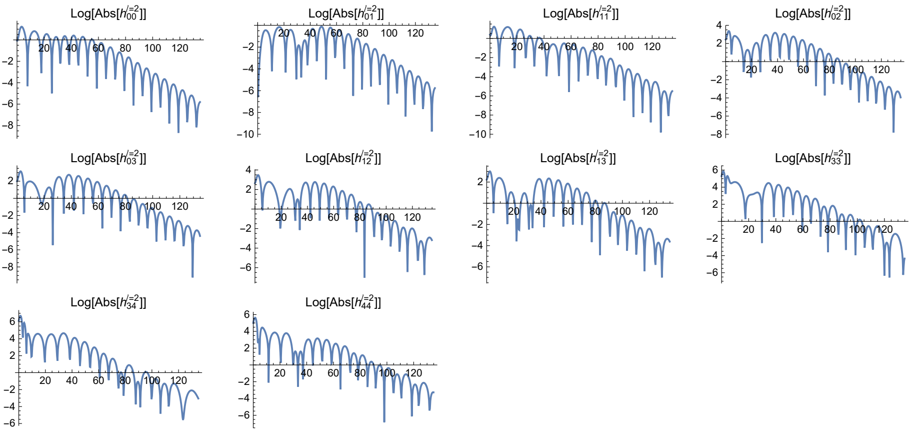
	$i = 1$	$i = 3$	$i \in 4, 8$
Inner Layer:	$-f'$	$-f(1+H)(1-cH)$	$-f'(1+H)(1-cH)$
Tortoise layer:	$-f'$	$-f$	$-f'$
Outer layer:	$-f'$	$-f(1-H)$	$-f'$

Reconstructed Perturbation



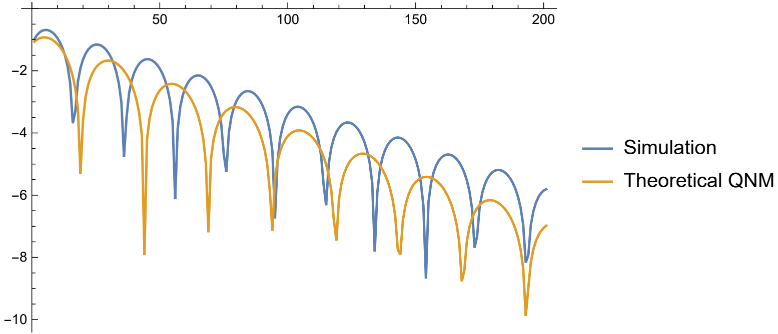
Plot of reconstructed metric perturbation for simulation of $\ell = 2$ modes. The m modes have been summed over for these plots.

Reconstructed Perturbation



Plots truncated to the only show active evolution. While the components in the $\{t, r_*\}$ space stabilize to damped sinusoidal behavior, the angular components have noticeable irregularities.

QNM Frequencies



Overlay of simulation data with the analytical solution for $h_{00}^{\ell=2}$. The analytical frequencies were generated with Leo Stein's python code in the Black Hole Perturbation Toolkit.

QNM Frequencies

- QNM frequencies clearly do not match analytical results
- I estimated numerical frequencies by plotting a damped sinusoid over the data and adjusting frequencies to match
- Data is given for two ℓ-modes in the table.

Comparison of numerical and analytical results for frequencies.

	Numerical		Analytic	
ℓ mode	ω_R	ω_I	ω_R	ω_I
2	0.474	0.077	0.374	0.0890
3	0.66	0.072	0.599	0.0927

Remaining Hurtles for evolution code

- Determine the source of the incorrect QNM frequencies
- Restore OpenMP parallelism that is present in the scalar evolution code
- Adapt existing code to interface with gravitational effective source code
- Refine the choice of damping if current method proves insufficient with effective source

References

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